On partition of a Boolean lattice into chains of equal sizes

Muktar Elzobi and Zbigniew Lonc
Department of Mathematics and Information Sciences
Warsaw University of Technology
00-661 Warsaw, Poland

ABSTRACT. In this paper we show that for every sufficiently large integer n and every positive integer $c \leq \left\lfloor \frac{1}{6}(\log\log n)^{\frac{1}{2}} \right\rfloor$, a Boolean lattice with n atoms can be partitioned into chains of cardinality c, except for at most $c-1$ elements which also form a chain.

1 Introduction

In 1985 Sands [7] gave a conjecture that for n sufficiently large given k, there exists a partition of a Boolean lattice with n atoms $\mathbf{B}_n$ into chains of cardinality 2^k. The conjecture of Sands holds for $k = 1$. The required partition is $\{\{A, A \cup \{n\}\} : A \subseteq [n-1]\}$. For $k = 2$ it was settled by Griggs et al. [4] who proved that $\mathbf{B}_n$ can be partitioned into chains of cardinality 4 if and only if $n \geq 9$. Griggs [3] posed a stronger conjecture proved by Lonc [6].

There exists an integer $n_0 = n_0(c)$ such that for $n \geq n_0$ the Boolean lattice $\mathbf{B}_n$ can be partitioned into chains of size c except for at most $c-1$ elements which also form a chain.

Notice that if there is a partition of $\mathbf{B}_n$ into chains of size c except for at most $c-1$ elements which also form a chain then by Dilworth's Theorem [1] the number of chains in this partition is not smaller than the maximum size of an antichain in $\mathbf{B}_n$. Hence by Sperner's Theorem [8]

$$\left\lceil \frac{2^n}{c} \right\rceil \geq \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

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Using Stirling's formula

$$c \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} = \sqrt{\frac{\pi}{2}}\sqrt{n}(1 + o(1)),$$

where $o(1)$ is a function of n tending to 0 as n approaches ∞.

Füredi [2] conjectured that there exists a partition of $\mathbf{B}_n$ into chains of size $c = \sqrt{\frac{\pi}{2}}\sqrt{n}(1+o(1))$, except for at most $c-1$ elements which also form a chain.

The purpose of this paper is to examine the method of Lonc [6] of partitioning of $\mathbf{B}_n$ into chains. He assumed in his reasoning that c is a constant with respect to n. In this paper we shall assume that c is an increasing function of n and check what we can prove using the method of Lonc [6]. Our main result says that if n is sufficiently large then $\mathbf{B}_n$ can be partitioned into chains of size $c = \left\lfloor \frac{1}{6}(\log\log n)^{\frac{1}{2}} \right\rfloor$, except for possibly $c-1$ elements which also form a chain. It is still far from $c = \sqrt{\frac{\pi}{2}}\sqrt{n}(1 + o(1))$ expected by Füredi but improves the original result of Lonc [6].

Since the method of partitioning $\mathbf{B}_n$ into chains is described in Lonc [6], we skip proofs which can be found there. Many of our proofs are similar to the proofs given in [6], however we must give them here to demonstrate how we can use them to show stronger results.

In this paper we denote by $\mathcal{L} = \{L_0, L_1, \ldots, L_n\}$ (resp. by $\mathcal{C} = \{C_1, C_2, \ldots, C_\ell\}$, where $\ell = \binom{n}{\lfloor \frac{n}{2} \rfloor})$ a partition of $\mathbf{B}_n$ into levels (resp. into symmetric chains).

Let us sketch the idea of the reasoning. A central role is played by a notion of so called pseudofence. This is a relatively large subset of $\mathbf{B}_n$ which can be partitioned into chains of size c except for at most $c-1$ elements which also form a chain (Lemma 2.2). For n sufficiently large it is proved (Lemma 2.6) that $\mathbf{B}_n$ contains a pseudofence $PF \subseteq L_p \cup L_{p+1} \cup \cdots \cup L_s$, where p and s, $p < s < \lfloor \frac{n}{2} \rfloor$, are some integers "asymptotically" close to $\frac{n}{2}$. Then, each chain $\left(C_i \cap \bigcup_{j=p}^{n} L_j\right) - PF$ is partitioned into chains of size c except for at most $c-1$ elements which form a chain C_i^* such that its minimum element is a member of $\bigcup_{j=s+1}^{\lfloor \frac{n}{2} \rfloor} L_j$. Denote by $\mathcal{F}_1$ the set of chains of size c obtained this way. Each chain C_i^* is completed to a chain of size c using some elements of $\bigcup_{i=0}^{p-1} L_i$. Denote the set of these c-element chains by $\mathcal{F}_2$. It turns out that the set obtained from $\mathbf{B}_n$ by deleting the chains of $\mathcal{F}_1$ and $\mathcal{F}_2$ is a pseudofence PF' containing PF. It completes the reasoning because the pseudofence has the required partition into chains.

Let us define a generalization of the Newton coefficient. For any real number x and a nonnegative integer k, let

$$\binom{x}{k} = \frac{x(x-1)\cdots(x-k+1)}{k!}.$$

One can easily check that for every positive integer m there exists a unique $x \geq k$ such that $m = \binom{x}{k}$.

In our considerations we shall use the following weaker version of the famous Kruskal–Katona Theorem due to Lovasz. We shall call it the KKL-theorem.

Theorem (KKL). *Let $\mathcal{A}$ be a family of k-element sets and $|\mathcal{A}| = \binom{x}{k}$, $x \geq k$. Then for any $s \leq k$ the sets of $\mathcal{A}$ contain at least $\binom{x}{s}$ different s-element sets.*

Let, for $a \in B_n$, $C(a)$ (resp. $L(a)$) denote the element of $\mathcal{C}$ (resp. $\mathcal{L}$) containing a. For an ordered set $\mathcal{P}$, let $\max \mathcal{P}$ (resp. $\min \mathcal{P}$) be the greatest (resp. the least) element of $\mathcal{P}$, if it exists. For a graph G and a set Z of vertices in G, we mean by $\Gamma_G(Z)$ the set of neighbors of vertices in Z. Let $\deg_G(x)$ denote the degree of a vertex x in G.

2 Results

We shall use the following technical lemma several times.

Lemma 2.1.

i) $\left(\frac{1+\frac{a}{x}}{1-\frac{a}{x}}\right)^x > e^{2a}$ *for* $x > a > 0$

ii) $\left(\frac{1+\frac{a}{x}}{1-\frac{a}{x}}\right)^x \leq 3^{2a}$ *for* $x \geq 2a > 0$.

Proof: The proof of this lemma follows by the observation that the function $f(x) = \left(\frac{1+\frac{a}{x}}{1-\frac{a}{x}}\right)^x$ is decreasing in the interval (a, ∞). □

Definition. *Let $A_1, \ldots, A_{\binom{n}{k}}$, $B_1, \ldots, B_{\binom{n}{k}-1}$ be pairwise disjoint chains of size at least c in $\mathbf{B}_n$. By a pseudofence of rank c based on the level L_k we mean the ordered set induced by $A_1 \cup \cdots \cup A_{\binom{n}{k}} \cup B_1 \cup \cdots \cup B_{\binom{n}{k}-1}$ if the following conditions are satisfied*

i) $\max A_i < \min B_i$ *and* $\max A_{i+1} < \min B_i$, *for* $i = 1, \ldots, \binom{n}{k} - 1$,

ii) $\{\max A_1, \ldots, \max A_{\binom{n}{k}}\} = L_k$ *and*

iii) $A_i \subseteq C(\max A_i)$ *for every* i *such that* $|C(\max A_i) \cap \bigcup_{i=0}^{k} L_i| \geq c$

(see Figure 1).

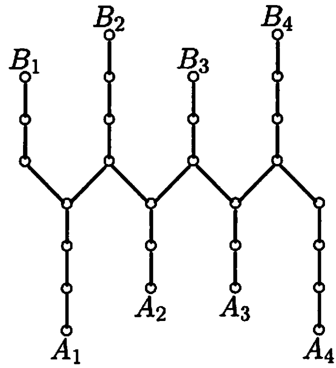

Figure 1. A pseudofence of rank 3

Lonc [6] proved the following lemma which plays a central role in the reasoning that follows. Since the proof of the lemma is short and simple we give it beneath.

Lemma 2.2. *A pseudofence of rank c based on the level L_k can be partitioned into chains of size c except for at most $c-1$ elements which also form a chain.*

Proof: Denote by $A_1, \ldots, A_{\binom{n}{k}}, B_1, \ldots, B_{\binom{n}{k}-1}$ the chains occurring in the definition of pseudofence PF of rank c based on the level L_k. Construct a sequence σ, whose terms are, in turn, the elements of $A_1, B_1, A_2, B_2, \ldots, A_{\binom{n}{k}-1}, B_{\binom{n}{k}-1}, A_{\binom{n}{k}}$. Since, for $i = 1, \ldots, \binom{n}{k}-1$, $A_i \cup B_i$ and $B_i \cup A_{i+1}$ are chains, every set of c consecutive terms in σ is a chain. Thus, partitioning σ into subsequences of consecutive elements of length c except for at most one of a shorter length, we get the required chain partition of our pseudofence. □

Lemma 2.3. *There is a constant n_0 such that if $n \geq n_0$ then for every integer k, $1 \leq k \leq \frac{1}{3}\log n$, and every integer function $h(n)$ such that $\frac{n}{2} - \sqrt{n}\log n + 2^k \leq h(n) \leq \frac{n}{2}$ the Boolean lattice $\mathbf{B}_n$ contains a set of pairwise disjoint chains $D_1, D_2, \ldots, D_t$ of size k, where $t = \binom{n}{h(n)}$, satisfying the conditions*

i) $\bigcup_{i=1}^{t} D_i \subseteq \bigcup_{i=0}^{2^k-2} L_{h(n)-i}$ *and*

ii) $\bigcup_{i=1}^{t} \max D_i = L_{h(n)}$.

Proof: Notice that $\sqrt{n} \geq 4\log n$ for $n \geq 2^{12}$, so by Lemma 2.1 ii)

$$1 \geq \left(\frac{\frac{n}{2} - \sqrt{n}\log n}{\frac{n}{2} + \sqrt{n}\log n}\right)^{\frac{1}{2}n^{1/3}} = \left(\frac{1 - \frac{2\log n}{\sqrt{n}}}{1 + \frac{2\log n}{\sqrt{n}}}\right)^{\sqrt{n}\cdot\frac{1}{2}\frac{1}{n^{1/6}}}$$
$$\geq \left(\frac{1}{3}\right)^{4\log n\cdot\frac{1}{2}n^{-1/6}} = \left(\frac{1}{3}\right)^{\frac{2\log n}{n^{1/6}}}.$$

Since $\lim_{n\to\infty} \frac{2\log n}{n^{1/6}} = 0$, $\lim_{n\to\infty} \left(\frac{1}{3}\right)^{\frac{2\log n}{n^{1/6}}} = 1$ and consequently $\lim_{n\to\infty} \left(\frac{\frac{n}{2}-\sqrt{n}\log n}{\frac{n}{2}+\sqrt{n}\log n}\right)^{\frac{1}{2}n^{1/3}} = 1$. Let n_0 be the smallest integer n such that $n \geq 2^{12}$ and $\left(\frac{\frac{n}{2}-\sqrt{n}\log n}{\frac{n}{2}+\sqrt{n}\log n}\right)^{\frac{1}{2}n^{1/3}} \geq \frac{1}{2}$.

We shall show the lemma by induction on k. (We shall write h instead of $h(n)$ to shorten notation.)

For $k = 1$ the lemma holds trivially.

Suppose we have already proved the lemma for integers smaller than k. We shall show it for $k \geq 2$.

Notice that $h - 2^{k-1} \geq \frac{n}{2} - \sqrt{n}\log n + 2^{k-1} \geq 0$, for $n \geq n_0$. Let $G = (X, Y; E)$ be a bipartite graph such that $X = L_h$, $Y = L_{h-1} \cup L_{h-2^{k-1}}$ and a pair UV, $U \in X$, $V \in Y$ be an edge in G if $V \leq U$ in B_n. We show that G contains a matching covering all vertices of X. To this end we apply Hall's theorem.

Let $Z \subseteq X$ and $|Z| = \binom{x}{h}$, where $h \leq x \leq n$. Applying the KKL-theorem we get

$$|\Gamma_G(Z)| \geq \binom{x}{h-1} + \binom{x}{h-2^{k-1}} \tag{2.1}$$

If $\frac{1}{2}(x+1) \leq h$ then $\binom{x}{h-1} \geq \binom{x}{h}$ so $|\Gamma_G(Z)| \geq |Z|$. Otherwise, if $\frac{1}{2}(x+1) > h$ then $\binom{x}{h} > \binom{x}{h-1} \geq \binom{x}{h-2^{k-1}}$ and by (2.1)

$$\frac{|\Gamma_G(Z)|}{|Z|} \geq \frac{2\binom{x}{h-2^{k-1}}}{\binom{x}{h}} = 2\frac{(h-2^{k-1}+1)(h-2^{k-1}+2)\cdots h}{(x-h+1)(x-h+2)\cdots(x-h+2^{k-1})}$$
$$\geq 2\frac{(h-2^{k-1}+1)(h-2^{k-1}+2)\cdots h}{(n-h+1)(n-h+2)\cdots(n-h+2^{k-1})}$$
$$\geq 2\left(\frac{h-2^{k-1}}{n-h+2^{k-1}}\right)^{2^{k-1}}.$$

Since the function $f(h) = \frac{h-A}{B-h}$ is increasing for $0 \le h < B$ and all constants A, B, $A \le B$ and since $2^{k-1} \le \frac{1}{2}n^{1/3}$, we get for $n \ge n_0$

$$\begin{aligned}\frac{|\Gamma_G(Z)|}{|Z|} &\ge 2\left(\frac{\frac{n}{2} - \sqrt{n}\log n + 2^k - 2^{k-1}}{\frac{n}{2} + \sqrt{n}\log n - 2^k + 2^{k-1}}\right)^{2^{k-1}} \\ &\ge 2\left(\frac{\frac{n}{2} - \sqrt{n}\log n}{\frac{n}{2} + \sqrt{n}\log n}\right)^{2^{k-1}} \ge 2\left(\frac{\frac{n}{2} - \sqrt{n}\log n}{\frac{n}{2} + \sqrt{n}\log n}\right)^{\frac{1}{2}n^{1/3}} \\ &\ge 2\cdot\frac{1}{2} = 1.\end{aligned}$$

Hence $|\Gamma_G(Z)| \ge |Z|$.

By Hall's theorem the required matching exists, equivalently, there are 2-element chains $E_1, E_2, \ldots, E_t$ such that $\bigcup_{i=1}^{t} E_i \subseteq L_h \cup L_{h-1} \cup L_{h-2^{k-1}}$ and $\bigcup_{i=1}^{t} \max E_i = L_h$. Denote by x_i and y_i, $y_i \le x_i$, the elements of E_i, for $i = 1, 2, \ldots, t$. Let $I = \{i\colon y_i \in L_{h-1}\}$ and $J = \{i\colon y_i \in L_{h-2^{k-1}}\}$. By the induction hypothesis (since $h - 1 \ge \frac{n}{2} - \sqrt{n}\log n + 2^{k-1}$), for $n \ge n_0$, there exists a set $\{D_i'\colon i \in I\}$ of pairwise disjoint $(k-1)$-element chains such that $\bigcup_{i\in I} D_i' \subseteq \bigcup_{i=0}^{2^{k-1}-2} L_{h-1-i}$ and $\max D_i' = y_i$ for every $i \in I$. Similarly (since $h - 2^{k-1} \ge \frac{n}{2} - \sqrt{n}\log n + 2^k - 2^{k-1} = \frac{n}{2} - \sqrt{n}\log n + 2^{k-1}$), for $n \ge n_0$, there is a set $\{D_i'' : i \in J\}$ of pairwise disjoint $(k-1)$-element chains such that $\bigcup_{i\in J} D_i'' \subseteq \bigcup_{i=0}^{2^{k-1}-2} L_{h-2^{k-1}-i} = \bigcup_{i=2^{k-1}}^{2^k-2} L_{h-i}$ and $\max D_i'' = y_i$ for every $i \in J$.

The chains $D_i = E_i \cup D_i'$ for $i \in I$ and $D_i = E_i \cup D_i''$ for $i \in J$ are pairwise disjoint and satisfy the conditions (i) and (ii). Hence the lemma follows by the principle of mathematical induction. □

To construct a pseudofence in $\mathbf{B}_n$ Lonc [6] constructed a sequence $C_{n,k} = (a_1, a_2, \ldots, a_{\binom{n}{k}})$ which is a permutation of the set of all k-element subsets of the set $[n]$ and such that the union of every two consecutive terms in the sequence $C_{n,k}$ is a $(k+1)$-element set.

Let $b_i' = a_i \cup a_{i+1}$, for $i = 1, 2, \ldots, \binom{n}{k} - 1$ and denote by $C'_{n,k}$ the sequence $(b_1', b_2', \ldots, b'_{\binom{n}{k}-1})$ of $(k+1)$-element subsets of $[n]$. Denote by $S_n^k(l)$ the set of $(k+1)$-element subsets of $[n]$ occurring exactly l times in the sequence $C'_{n,k}$.

The following lemma was shown in [6].

Lemma 2.4.

$$|S_n^k(l)| = \begin{cases} \binom{n-l-3}{k-l} + \binom{n-l-2}{k-l} & \text{for } 2 \le l \le k-2 \\ 2(n-l-2) & \text{for } 2 \le l = k-1 \\ 1 & \text{for } l = k \\ 0 & \text{for } l > k. \end{cases}$$

The next lemma follows from the KKL-theorem and was shown in [6] too.

Lemma 2.5. *Let $L_0, L_1, \ldots, L_n$ be the levels in $\mathbf{B}_n$. Let $T \subseteq L_k$ and $|T| = \binom{x}{n-k}$, where $x \geq n-k$ is a real number. If $m \geq k$ then the number of elements $w \in L_m$ such that $v \leq w$ for some $v \in T$ is at least $\binom{x}{n-m}$.* □

We shall use the following notation in the sequel:

$$\begin{aligned} p &= \left\lfloor \frac{n}{2} - (c+1)\sqrt{n} - 2^{2c-1} + 2 \right\rfloor, \\ q &= \left\lfloor \frac{n}{2} - (c+1)\sqrt{n} \right\rfloor, \\ r &= \left\lfloor \frac{n}{2} - c\sqrt{n} \right\rfloor \text{ and} \\ s &= \left\lfloor \frac{n}{2} - c \right\rfloor. \end{aligned}$$

Moreover, denote $S_n^k = \bigcup_{l=2}^k S_n^k(l)$.

The following lemma is a strengthening of a similar lemma proved in [6].

Lemma 2.6. *For n sufficiently large and $2 \leq c \leq \frac{1}{6}\log n$ there is a pseudofence of rank c based on the level L_q of $\mathbf{B}_n$ contained in $\bigcup_{i=p}^s L_i$.*

Proof: Denote by a_i (resp. b_i') the i-th term of the sequence $C_{n,q}$ (resp. $C_{n,q}'$). Let n be sufficiently large to satisfy the inequality $p > 0$. Define $G = G_n = (X, Y; F)$ to be a bipartite graph such that $X = \{i \colon b_i' \in S_n^q\}$, $Y = \bigcup_{j=q+2}^r L_j$ and $F = \{(i, b) \in X \times Y \colon b_i' \leq b\}$. We shall prove existence of a matching in G covering all vertices of X, for n sufficiently large and $c \leq \frac{1}{6}\log n$. We shall check Hall's condition

$$|Z| \leq |\Gamma_G(Z)| \text{ for every } Z \subseteq X. \tag{2.2}$$

Proceeding exactly like in the proof of Lemma 6 in [6] we conclude that in order to prove (2.2) it suffices to show that the following inequality

$$\frac{4n[(n-2r)r^{r-q-2} + (n-2r+1)\Phi_1(x_1) + \Phi_2(x_2)]}{(n-q-1)(n-r)^{r-q-1}} \leq 1 \tag{2.3}$$

holds for n sufficiently large and $c \leq \frac{1}{6}\log n$.

In the above inequality

$$\Phi_\alpha(x) = x^\alpha (r-x)^{r-q-2}$$

and $x_\alpha = \frac{\alpha r}{r-q-2+\alpha}$, for $\alpha = 1, 2$.

Notice that

$$\begin{aligned}
\Phi_1(x_1) &= \frac{r}{r-q-1}\left(r - \frac{r}{r-q-1}\right)^{r-q-2} \\
&= \frac{r}{r-q-1} r^{r-q-2}\left(1 - \frac{1}{r-q-1}\right)^{r-q-2} \\
&< \frac{1}{r-q-1} r^{r-q-1}
\end{aligned}$$

and

$$\begin{aligned}
\Phi_2(x_2) &= \left(\frac{2r}{r-q}\right)^2\left(r - \frac{2r}{r-q}\right)^{r-q-2} \\
&= \frac{4r^2}{(r-q)^2} r^{r-q-2}\left(1 - \frac{2}{r-q}\right)^{r-q-2} \\
&< \frac{4r}{(r-q)^2} r^{r-q-1}
\end{aligned}$$

Hence

$$\begin{aligned}
I &= \frac{4n[(n-2r)r^{r-q-2} + (n-2r+1)\Phi_1(x_1) + \Phi_2(x_2)]}{(n-q-1)(n-r)^{r-q-1}} \\
&< \frac{4n}{(n-q-1)(n-r)^{r-q-1}} \\
&\quad \left[(n-2r)r^{r-q-2} + (n-2r+1)\frac{1}{r-q-1} r^{r-q-1} + \frac{4r}{(r-q)^2} r^{r-q-1}\right] \\
&= \frac{4n}{n-q-1}\frac{r^{r-q-1}}{(n-r)^{r-q-1}}\left[\frac{n-2r}{r} + \frac{n-2r+1}{r-q-1} + \frac{4r}{(r-q)^2}\right].
\end{aligned}$$

Clearly, $r = \lfloor \frac{n}{2} - c\sqrt{n} \rfloor \geq \lfloor \frac{n}{2} - \frac{1}{6}\log n\sqrt{n} \rfloor > \frac{n}{3}$, for sufficiently large n. Hence

$$\frac{n-2r}{r} < 1.$$

Moreover, for sufficiently large n

$$\begin{aligned}
r-q-1 &= \left\lfloor \frac{n}{2} - c\sqrt{n} \right\rfloor - \left\lfloor \frac{n}{2} - (c+1)\sqrt{n} \right\rfloor - 1 \\
&\geq \frac{n}{2} - c\sqrt{n} - 1 - \left(\frac{n}{2} - (c+1)\sqrt{n}\right) - 1 = \sqrt{n} - 2, \\
2r &= 2\left\lfloor \frac{n}{2} - c\sqrt{n} \right\rfloor \geq 2\left(\frac{n}{2} - \frac{3}{2}c\sqrt{n}\right) + 1 = n - 3c\sqrt{n} + 1
\end{aligned}$$

and

$$3\sqrt{n} \leq 4\sqrt{n} - 8.$$

Consequently

$$\frac{n-2r+1}{r-q-1} \le \frac{n-2r+1}{\sqrt{n}-2} \le \frac{n-(n-3c\sqrt{n})}{\sqrt{n}-2} = \frac{3c\sqrt{n}}{\sqrt{n}-2} \le 4c.$$

Further,

$$\frac{4r}{(r-q)^2} \le \frac{4\left(\frac{n}{2}-c\sqrt{n}\right)}{\left(\sqrt{n}-1\right)^2} < \frac{2n}{\left(\sqrt{n}-1\right)^2} \le 3$$

for sufficiently large n.

Finally,

$$\frac{4n}{n-q-1} \le \frac{4n}{\frac{n}{2}} = 8$$

because

$$n-q-1 \ge n-\left(\frac{n}{2}-(c+1)\sqrt{n}\right)-1 = \frac{n}{2}+(c+1)\sqrt{n}-1 \ge \frac{n}{2}.$$

Hence

$$I < 8\left(\frac{r}{n-r}\right)^{r-q-1}[1+4c+3] = 32(c+1)\left(\frac{r}{n-r}\right)^{r-q-1}.$$

Notice that $r-q = \lfloor \frac{n}{2}-c\sqrt{n}\rfloor - \lfloor \frac{n}{2}-(c+1)\sqrt{n}\rfloor = \frac{n}{2}-c\sqrt{n}-\varepsilon_1 - \left(\frac{n}{2}-(c+1)\sqrt{n}-\varepsilon_2\right) = \sqrt{n}-\varepsilon_1+\varepsilon_2 = \sqrt{n}+\varepsilon$, where ε_1, ε_2 and ε are real numbers such that $0 \le \varepsilon_1 < 1$, $0 \le \varepsilon_2 < 1$ and $-1 < \varepsilon = \varepsilon_2 - \varepsilon_1 < 1$. Thus

$$\begin{aligned}\left(\frac{r}{n-r}\right)^{r-q-1} &= \left(\frac{\lfloor \frac{n}{2}-c\sqrt{n}\rfloor}{n-\lfloor \frac{n}{2}-c\sqrt{n}\rfloor}\right)^{\sqrt{n}+\varepsilon-1} \\ &\le \left(\frac{\frac{n}{2}-c\sqrt{n}}{n-\left(\frac{n}{2}-c\sqrt{n}\right)}\right)^{\sqrt{n}+\varepsilon-1} = \left(\frac{\frac{n}{2}-c\sqrt{n}}{\frac{n}{2}+c\sqrt{n}}\right)^{\sqrt{n}+\varepsilon-1} \\ &= \left(\frac{1-\frac{2c}{\sqrt{n}}}{1+\frac{2c}{\sqrt{n}}}\right)^{\sqrt{n}+\varepsilon-1} = \left(\frac{1-\frac{2c}{\sqrt{n}}}{1+\frac{2c}{\sqrt{n}}}\right)^{\sqrt{n}}\left(\frac{1-\frac{2c}{\sqrt{n}}}{1+\frac{2c}{\sqrt{n}}}\right)^{\varepsilon-1}.\end{aligned}$$

By Lemma 2.1 i) and the inequality $\frac{2c}{\sqrt{n}} \le \frac{1}{3}\frac{\log n}{\sqrt{n}} \le \frac{1}{3}$ we get

$$\begin{aligned}\left(\frac{r}{n-r}\right)^{r-q-1} &< \frac{1}{e^{4c}}\left(\frac{1-\frac{2c}{\sqrt{n}}}{1+\frac{2c}{\sqrt{n}}}\right)^{\varepsilon-1} = e^{-4c}\left(\frac{1+\frac{2c}{\sqrt{n}}}{1-\frac{2c}{\sqrt{n}}}\right)^{1-\varepsilon} \\ &< e^{-4c}\left(\frac{1+\frac{2c}{\sqrt{n}}}{1-\frac{2c}{\sqrt{n}}}\right)^{2} \le e^{-4c}\left(\frac{1+\frac{1}{3}}{1-\frac{1}{3}}\right)^{2} = 4e^{-4c}.\end{aligned}$$

Therefore

$$I < 32(c+1)4e^{-4c} < 1 \text{ for } c \geq 2.$$

So (2.3) holds and so does (2.2).

By Hall's theorem there exists a matching in G covering all vertices of X. Denote by b_i the vertex in Y matched with i, for each $i \in X$. Moreover, let $b_i = b_i' \in S_n^q(1)$ for every $i \in \{1, 2, \ldots, \binom{n}{q} - 1\} - X$. By the definition of the sequences $C_{n,q}$, $C'_{n,q}$ and the graph G, $a_i \leq b_i' \leq b_i$, $a_{i+1} \leq b_i' \leq b_i$ and $b_i \in \bigcup_{j=q+1}^{r} L_j$ for $i = 1, 2, \ldots, \binom{n}{q} - 1$.

For $i = 1, \ldots, \binom{n}{q} - 1$, define $B_i = C(b_i) \cap \bigcup_{j=0}^{c-1} L_{t+j(r-q)}$ and $L_t = L(b_i)$. Notice that the sets B_i are c-element pairwise disjoint chains such that $\bigcup_{i=1}^{\binom{n}{q}-1} B_i \subseteq \bigcup_{j=q+1}^{cr-(c-1)q} L_j$ and $\min B_i = b_i$ for $i = 1, \ldots, \binom{n}{q} - 1$. Since

$$\begin{aligned} cr - (c-1)q &= c\left\lfloor \frac{n}{2} - c\sqrt{n} \right\rfloor - (c-1)\left\lfloor \frac{n}{2} - (c+1)\sqrt{n} \right\rfloor \\ &\leq c\left(\frac{n}{2} - c\sqrt{n}\right) - (c-1)\left(\frac{n}{2} - (c+1)\sqrt{n} - 1\right) \\ &= \frac{n}{2} - \sqrt{n} + c - 1 \leq \frac{n}{2} - \frac{1}{3}\log n + c - 1 \\ &\leq \frac{n}{2} - 2c + c - 1 = \frac{n}{2} - c - 1 \leq \left\lfloor \frac{n}{2} - c \right\rfloor = s, \end{aligned}$$

we get, $\bigcup_{i=1}^{\binom{n}{q}-1} B_i \leq \bigcup_{j=q+1}^{s} L_j$.

Let J be the set of those i's for which $|C(a_i) \cap \bigcup_{j=0}^{q} L_j| \geq c$. Define, for each $i \in J$, $A_i = C(a_i) \cap \bigcup_{j=q-c+1}^{q} L_j$. By Lemma 2.3, applied for $k = 2c-1 \leq 2 \cdot \frac{1}{6}\log n - 1 \leq \frac{1}{3}\log n$ and $\frac{n}{2} \geq h(n) = q = \lfloor \frac{n}{2} - (c+1)\sqrt{n} \rfloor \geq \frac{n}{2} - \sqrt{n}\log n + n^{\frac{1}{3}} \geq \frac{n}{2} - \sqrt{n}\log n + 2^k$, for n sufficiently large, there is a set of pairwise disjoint $(2c-1)$-element chains D_i, $i \in \bar{J} = \{1, \ldots, \binom{n}{q}\} - J$, such that $\bigcup_{i \in J} D_i \subseteq \bigcup_{i=0}^{2^{2c-1}-2} L_{q-i}$ and $\max D_i = a_i$. Define, for every $i \in \bar{J}$, $A_i = D_i - \bigcup_{j=q-c+1}^{q-1} L_j$.

The posets induced by $\bigcup_{i=1}^{\binom{n}{q}} A_i \cup \bigcup_{i=1}^{\binom{n}{q}-1} B_i$ is a pseudofence of rank c based on the level L_q contained in $\bigcup_{i=p}^{s} L_i$. □

The following lemma was shown in Lonc [5].

Lemma 2.7. *Let G be a bipartite graph with vertex classes X and Y such that $\deg_G v = x$ for every $v \in X$ and $\deg_G v \leq y$ for every $v \in Y$. Then G has a factor whose every component is a star with either $\lfloor \frac{y}{x} \rfloor$ or $\lceil \frac{y}{x} \rceil$ leaves and with the center in a vertex of Y.*

In fact Lemma 2.7 was formulated in Lonc [5] with an assumption $y > x$ but the proof given there works without this assumption too.

The following theorem is our main result of this paper.

Theorem 2.8. *There exists an integer n_0 such that for $n \geq n_0$ and $c \leq \frac{1}{6}(\log\log n)^{\frac{1}{2}}$ the Boolean lattice $\mathbf{B}_n$ can be partitioned into chains of size c except for at most $c-1$ elements which also form a chain.*

Proof: Since the theorem is trivial for $c = 1$, we assume that $c \geq 2$. Denote by PF the pseudofence whose existence is guaranteed by Lemma 2.6 for n sufficiently large. Let $D = \bigcup_{i=p}^{n} L_i - PF$. Notice that each chain $C_i \cap D$, $C_i \in \mathcal{C}$, can be partitioned into a certain number of chains of size c and a chain C_i^* of size at most $c-1$ such that $\min C_i^* \in \bigcup_{j=s+1}^{\lfloor \frac{n}{2} \rfloor} L_j$, for each $C_i^* \neq \emptyset$. Denote by $\mathcal{F}_1$ the set of all chains of size c obtained this way.

For $j = s+1, s+2, \ldots, \lfloor \frac{n}{2} \rfloor$ define $Q_j = \{C_i^* \colon \min C_i^* \in L_j\}$. Let G_j be a bipartite graph $(Q_j, L_{p-1}; E)$, where $C_i^* a \in E$ if $a \leq \min C_i^*$. The degree of every vertex in Q_j is $\binom{j}{p-1}$ while the degrees of the vertices in L_{p-1} are not greater than $\binom{n-p+1}{j-p+1}$. By Lemma 2.7, there is a factor F_j in G_j such that $\deg_{F_j} C_i^* = 1$ for every $C_i^* \in Q_j$ and

$$\begin{aligned}
\deg_{F_j} a &\leq \left\lceil \binom{n-p+1}{j-p+1} \Big/ \binom{j}{p-1} \right\rceil \\
&= \left\lceil \frac{(n-j+1)(n-j+2)\cdots(n-p+1)}{p(p+1)\cdots j} \right\rceil \\
&\leq \left\lceil \left(\frac{n-p+1}{p} \right)^{j-p+1} \right\rceil \leq \left\lceil \left(\frac{n-p+1}{p} \right)^{n/2-p+1} \right\rceil
\end{aligned}$$

for every $a \in L_{p-1}$.

Notice that $2c - 1 \leq 2\log\log n$ so, for n sufficiently large,

$$2^{2c-1} \leq 2^{2\log\log n} = \left(2^{\log\log n}\right)^2 = (\log n)^2 < (c-1)\sqrt{n}. \qquad (2.4)$$

Hence

$$\begin{aligned}
\left(\frac{n-p+1}{p} \right)^{\frac{n}{2}-p+1} &\leq \left(\frac{n - (\frac{n}{2} - (c+1)\sqrt{n} - 2^{2c-1} + 1) + 1}{\frac{n}{2} - (c+1)\sqrt{n} - 2^{2c-1}} \right)^{\frac{n}{2}-p+1} \\
&\leq \left(\frac{\frac{n}{2} + (c+1)\sqrt{n} + 2^{2c-1}}{\frac{n}{2} - (c+1)\sqrt{n} - 2^{2c-1}} \right)^{(c+1)\sqrt{n}+2^{2c-1}} \\
&< \left(\frac{\frac{n}{2} + 2c\sqrt{n}}{\frac{n}{2} - 2c\sqrt{n}} \right)^{2c\sqrt{n}} = \left(\frac{n + 4c\sqrt{n}}{n - 4c\sqrt{n}} \right)^{2c\sqrt{n}} \\
&= \left[\left(\frac{1 + \frac{4c}{\sqrt{n}}}{1 - \frac{4c}{\sqrt{n}}} \right)^{\sqrt{n}} \right]^{2c}.
\end{aligned}$$

By Lemma 2.1 ii) (since $\sqrt{n} \geq 2 \cdot 4c$)

$$\deg_{F_j} a \leq \left(3^{8c}\right)^{2c} = 3^{16c^2}.$$

Let F be the union of the graphs F_j, $j = s+1, s+2, \ldots, \lfloor \frac{n}{2} \rfloor$. Clearly, $\deg_F a \leq c \cdot 3^{16c^2}$ for every $a \in L_{p-1}$ and $\deg_F C_i^* = 1$ for every $C_i^* \in Q = \bigcup_{j=s+1}^{\lfloor \frac{n}{2} \rfloor} Q_j$.

Let $k = c(c-1)3^{16c^2}$. Notice that

$$k = c(c-1)3^{16c^2} \leq \frac{1}{36} \log\log n \cdot 3^{\frac{16}{36} \log\log n} = \frac{1}{36} \log\log n \cdot 2^{\frac{16}{36} \log_2 3 \log\log n}$$
$$= \frac{1}{36} \log\log n (\log n)^{\frac{16}{36} \log_2 3} \leq \frac{1}{36} \log\log n (\log n)^{\frac{3}{4}} \leq \frac{1}{3} \log n,$$

for n sufficiently large. Moreover, define $h(n) = p-1$. Then by (2.4)

$$\frac{n}{2} \geq h(n) = p - 1 \geq \frac{n}{2} - (c+1)\sqrt{n} - 2^{2c-1} \geq \frac{n}{2} - 2c\sqrt{n}$$
$$\geq \frac{n}{2} - 2 \cdot \frac{1}{6} (\log\log n)^{\frac{1}{2}} \sqrt{n} \geq \frac{n}{2} - \sqrt{n} \log n + n^{\frac{1}{3}}$$
$$= \frac{n}{2} - \sqrt{n} \log n + 2^{\frac{1}{3} \log n} \geq \frac{n}{2} - \sqrt{n} \log n + 2^k,$$

for n sufficiently large. By Lemma 2.3, for n sufficiently large, there is a set $\{D_a : a \in L_{p-1}\}$ of pairwise disjoint chains of size $k = c(c-1)3^{16c^2}$ such that $\bigcup_{a \in L_{p-1}} D_a \subseteq \bigcup_{i=0}^{2^k - 2} L_{p-1-i}$ and $\bigcup_{a \in L_{p-1}} \max D_a = L_{p-1}$. The set $\{\Gamma_F(a) : a \in L_{p-1}\}$ is a partition of Q (we allow empty classes in this partition). It is clear by the definition of F that for each $a \in L_{p-1}$ the chains C_i^* in $\Gamma_F(a)$ can be completed to pairwise disjoint c-element chains by adjoining some elements of D_a. Denote the set of all these c-element chains by $\mathcal{F}_2$.

Consider the pseudofence PF. Let $A_1, \ldots, A_{\binom{n}{q}}, B_1, \ldots, B_{\binom{n}{q}-1}$ be the chains inducing PF. Replace each chain $A_i \subseteq C(\max A_i)$ in PF by a chain $A_i' = A_i \cup \left[C(\max A_i) \cap \left(\bigcup_{i=0}^{p-1} L_i - \bigcup \mathcal{F}_2 \right) \right]$ and denote the resulting ordered set by PF'. Notice that PF' is still a pseudofence of rank c based on the level L_q. Moreover, $\mathbf{B}_n$ is a disjoint union of $\bigcup \mathcal{F}_1$, $\bigcup \mathcal{F}_2$.and PF'. By Lemma 2.2 the theorem follows. □

Note. We have learned recently that Hsu, Logan, Shahriari, and Towse made a significant progress toward proving Füredi's conjecture by showing that a Boolean lattice $\mathbf{B}_n$ can be partitioned into chains such that the size of the shortest of them is approximately $\frac{1}{2}\sqrt{n}$.

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