A Note on an Upper Bound for the Constraints of Some Balanced Arrays

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Abstract

In this paper we derive an inequality on the existence of bi-level balanced arrays (B-arrays) of strength eight by using a result involving central moments from statistics, and by counting in two ways the number of coincidences of various columns with a specific column. We discuss the use of this inequality in obtaining the maximum number of constraints for these arrays, and present some illustrative examples.

1. Introduction and Preliminaries.

For ease of reference, we recall here some basic definitions and results concerning balanced arrays (B-arrays). An array T with S levels (symbols), m constraints (rows), and N runs (columns) is merely a matrix of size $(m \times N)$ with S elements (say, 0,1,2,..,S-1). Furthermore, T is called a B-array of strength t ($t \leq m$) if in every $(t \times N)$ submatrix T^* of T, we have the following condition satisfied for all permutations P and for all t-vectors $\underline{\alpha}$ in T^*:

$\lambda(\underline{\alpha};T^*) = \lambda(P(\underline{\alpha});T^*)$ where $P(\underline{\alpha})$ is a vector obtained by permuting the elements of any t-vector $\underline{\alpha}$ in T^*, and $\lambda(\underline{\alpha};T^*)$ represents the frequency with which $\underline{\alpha}$ appears in T^*. In this paper we confine ourselves to B-arrays with $t = 8$ and $S = 2$ (say, 0 and 1). For this special case, we let $W(\underline{\alpha})$ represent the weight of the vector α (the weight of a vector is the number of ones in it). It is quite obvious that $W(\underline{\alpha}) = W(P(\underline{\alpha}))$.

If $W(\underline{\alpha}) = i$ $(0 \leq i \leq 8)$, then the above condition is reduced to $\lambda(\underline{\alpha};T^*) = \lambda(P(\underline{\alpha});T^*) = \mu_i$ (say) $i = 0, 1, 2, .., 8$. The symbols m,

$N = (\sum_{i=0}^{8} \binom{8}{i} \mu_i)$, $t = 8$ and $\mu_i (i = 0, 1, .., 8)$ are called the parameters of the array and sometimes we denote it by

$BA\{m, N, S(=2), t(=8); \underline{\mu}' = (\mu_0, \mu_1, .., \mu_8)\}$.

Next, we provide an example of a B-array T of strength $t = 2$.

Example, let us consider the following array.

$$T = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Here, $m = 7$, $N = 9$, $s = 2$, strength $t = 2$, and $\underline{\mu}' = (\mu_0, \mu_1, \mu_2) = (3, 2, 2)$. Take any (2×9) submatrix T^* of T (say, the first two rows of the B-array T). For this T^* we find μ_0 = the frequency of the vector $\binom{0}{0} = 3$, μ_1 = the frequency of the vector $\binom{1}{0}$ = the frequency of its permutation which is $\binom{0}{1} = 2$, and μ_2 = the frequency of the vector $\binom{1}{1} = 2$. It can be checked easilty that $(\mu_0, \mu_1, \mu_2) = (3, 2, 2)$ for every (2×9) submatrix T^* of T.

It is quite obvious that B-arrays may not exist for a given $\underline{\mu}'$ and m $(m > t = 8)$. The construction of such arrays with the maximum m for a given $\underline{\mu}'$ is very important both in statistical design of experiments and combinatorics. Orthogonal arrays (O-arrays) and the incidence matrix of a balanced incomplete block design ($B\ I\ B$ design) are special cases of B-arrays, and these arrays have been extensively used in the construction of balanced fractional factorial designs which permit us to estimate, under certain conditions, all the effects of interest to us. To learn more about the importance of B-arrays to design of experiments and combinatorics, the interested reader may consult the list of references, by no means an exhaustive list, at the end of this paper, and also further references listed therein. The problem of the existence of B-arrays for a given $\underline{\mu}'$ and $m > 8$, and, as a consequence, to obtain an upper bound on m is a nontrivial matter. Such problems for O-arrays and B-arrays have been investigated, among others, by Bose and Bush [1], Chopra and/or Dios [3, 4, 5], Rafter and Seiden [9], Saha et al [11], Seiden and Zemach [12], Yamamoto, et al [15], etc.

2. Upper Bounds for Constraints of Balanced Arrays.

First of all, we state some results for later use in deriving the necessary existence conditions for B-arrays with $t = 8$.

Lemma 2.1. A B-array T with an arbitrary $\underline{\mu}'$ and with $m = t = 8$ always exists.

Lemma 2.2. A B-array T of strength $t = 8$ and with $\underline{\mu}'=(\mu_0,\mu_1,.., \mu_8)$ is also of strength t' where $0 \leq t' \leq 8$. The index set of T, considered as an array of strength t', is given by

$$\underline{\mu}'(t') = \{\mu_j(t') \mid j = 0, 1, .., t'; \text{ with } \mu_j(t') = \sum_{i=0}^{t-t'} \binom{t-t'}{i} \mu_{i+j}\}.$$

Remark 1: It is quite obvious that each $\mu_j(t')$ is a linear function of the μ_i 's, $t' = 0$ corresponds to N (the total number of columns in T) and $t' = 8$ corresponds to the index set $\underline{\mu}'$.

Definition 2.1. Two columns of a B-array T with m constraints are said to have j coincidences $(0 \leq j \leq m)$ if the j rows of these two columns have the same symbols.

Lemma 2.3. Consider a B-array T $(m \times N)$ with the index set $\underline{\mu}'=(\mu_0,\mu_1,.., \mu_8)$. If l is the weight of some column (say, the first one) of T and x_j is the number of columns of T (other than the first one) having exactly j coincidences with the first one, then the following nine equalities hold:

$$I_k = \sum_{j=0}^{m} j^k x_j = \sum_{i=1}^{k-1} (-1)^{(k-1)-i}\, b_i(k) I_i + k! B_k, \text{ where}$$

$$B_k = \sum_{i=0}^{k} \binom{l}{i} \binom{m-l}{k-i} (\mu_i\ (k) - 1),\ k = 0, 1, 2, .., 8. \qquad (2.1)$$

$\mu_i\ (k) = N$ for $k = 0$ and $\mu_i\ (k) = \mu_i$ for $k = 8$, $b_i(k)$ are known constants obtained while deriving (2.1), $I_0 = N - 1$, and 0^0 is defined to be equal to 1.

Proof outline: One can obtain (2.1) by considering successively T as an array of strength k $(\leq 8))$ and counting in two ways (i.e. through columns, and through using the fact that T is a B-array of strength k $(\leq 8))$ the total number of column vectors which are identical with the corresponding vectors in the first column. For clarity, we derive one such result for $k = 8$ (say). Consider any eight rows of T. If the (8×1) vector in the first column of T is of weight 0, it then appears $(\mu_0 - 1)$ times amongst the remaining columns of T. Similarly if it is of weight i $(i = 1, 2, ..., 8)$, it will appear $(\mu_i - 1)$ more times amongst the remaining columns. Since the weight of the first column is l, the total number of ways to select eight rows so that there is an (8×1) vector of weight i in the first column is clearly $\binom{l}{i}\binom{m-l}{8-i}$ where $\binom{a}{b} = 0$ if $a < b$. Let B_8 denote the total number of 8-tuples appearing in columns other than the first which are identical with the corresponding 8-tuples of the first column. Then $B_8 = \sum_{i=0}^{8} \binom{l}{i}\binom{m-l}{8-i}(\mu_i - 1)$. Next, any column having j $(j \geq 8)$ coincidences with the first column will contribute $\binom{j}{8}$ to B_8. Thus we have $B_8 = \sum_{j=8}^{m} \binom{j}{8} x_j = \sum_{j=0}^{m} \binom{j}{8} x_j$. Therefore, $\sum_{j=0}^{m} \binom{j}{8} x_j = \sum_{i=0}^{8} \binom{l}{i}\binom{m-l}{8-i}(\mu_i - 1)$. Similarly results for other values of k can be derived. Further simplification will lead us to (2.1).

Remark 2: It is quite obvious that (2.1) expresses the moments of order k of the coinidences in T with a certain column in terms of the parameters of the array T and l. Next we state, without proof, the following result from Lakshmanamurti [7].

Result: Let $Z_i (i = 1, 2, .., n)$ be reals such that $\sum Z_i = 0$ and $\sum Z_i^2 = n$. Let $\alpha_m = \frac{1}{n} \sum Z_i^m$. Then we have

$$\alpha_8 \geq \alpha_4^2 + \alpha_5^2 \tag{2.2}$$

Next we obtain a necessary condition for the existence of a balanced array of strength $t = 8$ by using (2.1) and (2.2).

Theorem 2.1. Consider a B-array T of size $(m \times N)$ with the index set $\underline{\mu}' = (\mu_0, \mu_1, .., \mu_8)$. If l is the weight of some column (say, the first column) of T, then the following result holds:

$$L_2 L_8 \geq L_2 L_4^2 + L_5^2 \tag{2.3}$$

where L_i's $(i = 2, 4, 5, \text{ and } 8)$ are given by

$$L_2 = (N - 1)\, I_2 - I_1^2,$$

$L_4 = (N-1)^3 I_4 - 4(N-1)^2 I_3 I_1 + 6(N-1) I_2 I_1^2$,
$L_5 = (N-1)^4 I_5 - 5(N-1)^3 I_4 I_1 + 10(N-1)^2 I_3 I_1, -10(N-1) I_2 I_1^3 + 4 I_1^5$,
and $L_8 = (N-1)^7\, I_8 - 8(N-1)^6 I_7 I_1 + 28(N-1)^5 I_6 I_1^2 - 56(N-1)^4 I_5 I_1^3$
$+70(N-1)^3 I_4 I_1^4 - 56(N-1)^2 I_3 I_1^5 + 28(N-1) I_2 I_1^6 - 7 I_1^8$ where I_k 's are defined in (2.1).

Proof outline: In order to use (2.2) we observe that Z_i's need not be distinct and may occur with different frequencies. If f_i is the frequency of $Z_i (i = 1, 2, 3, ...n)$ such that $\sum_i f_i Z_i = 0$ and $\sum f_i Z_i^2 = N$ where $N = \sum f_i$, then $\alpha_8 \geq \alpha_4^2 + \alpha_5^2$ where $\alpha_m = \frac{1}{N}\sum f_i Z_i^m$. In order to use this version of (2.2) we need to transform our data appropriately. Let us set $M = \dfrac{\sum j x_j}{N-1} = \dfrac{I_1}{N-1}$, and $s^2 = \dfrac{1}{N-1}\sum_{j=0}^{m}(j-M)^2 x_j = \dfrac{1}{N-1}\left[I_2 - (N-1)I_1^2\right]$. It is quite obvious that $\sum \left(\frac{j-M}{s}\right) x_j = 0$ and $\sum \left(\frac{j-M}{s}\right)^2 x_j = N-1$. Clearly here x_j are playing the role of frequencies f_j, and $\left(\frac{j-M}{s}\right)$ the role of Z_j. Using this transformed data, we set $\alpha_k = \dfrac{1}{N-1}\sum \left(\frac{j-M}{s}\right)^k x_j$. Substituting α_4, α_5, and α_8 in (2.2), we obtain

$$\frac{1}{N-1}\sum \left(\tfrac{j-M}{s}\right)^8 x_j \geq \left[\frac{1}{N-1}\sum \left(\tfrac{j-M}{s}\right)^5 x_j\right]^2 + \left[\frac{1}{N-1}\sum \left(\tfrac{j-M}{s}\right)^4 x_j\right]^2$$

which gives us

$$(N-1)\, s^2 \sum (j-M)^8 x_j \geq \left[\sum (j-M)^5 x_j\right]^2 + s^2 \left[\sum (j-M)^4 x_j\right]^2.$$

Expanding $\sum (j-M)^k x_j$ ($k = 4, 5$, and 8), and using $M = \dfrac{I_1}{N-1}$, and $s^2 = \dfrac{1}{N-1}\left[I_2 - (N-1)I_1^2\right]$ we obtain the desired inequality after some simplification.

Remark 3: It is obvious that (2.3) is a polynomial function of l, m, and μ_i 's. For a given l and $\underline{\mu}'$, we can check if (2.3) is satisfied for each $m (> 8)$. If the first contradiction occurs at $m = m^* + 1$ (say) then m^* is an upper bound for the B-array under investigation. Thus (2.3) is a useful necessary condition for the existence of a B-array for a given $\underline{\mu}', l$, and m.

Remark 4: A computer program was prepared to check (2.3) for a B-array with a given $\underline{\mu}'$, and l. Since computations involved exceedingly large numbers, we will restrict our illustrative examples to B-arrays with $l = 0$ and small values of μ_i 's.

Remark 5: For ease of calculations and computations, we list here the values of $b_i(k), 1 \leq i \leq k-1$, which appear in (2.1). The values of $b_i(k)$ are listed starting with $i = 1$ and ending with $i = k-1$. These are: (for $k = 8$; values are 5040, 13068, 13132, 6769, 1960, 322, and 28), (for $k = 7$; values are 720, 1764, 1624, 735, 175, and 21), (for $k = 6$; values are 120, 274, 225, 85, 15), (for $k = 5$; values are 24, 50, 35, 10), (for $k = 4$; values are 6, 11, 6), (for $k = 3$; values are 2,3), and (for $k = 2$; value is 1).

Next we present some illustrative examples.

Example 1. Consider the array with $\underline{\mu}'= (2,2,2,2,3,3,3,2,2)$. Taking $l = 0$ and using (2.3), the contradiction occurred for the first time at $m = 13$. Therefore an upper bound for m is 12. Using the condition given by Dios and Chopra [6], the contradiction occurs at $m = 18$. Therefore we had an upper bound of m =17. Therefore (2.3) given here provides a significant improvement for an upper bound.

Example 2. The upper bounds on m for the arrays (1,3,6,4,1,7,5,1,2), (1,3,2,2,1,5,5,2,2), and (1,4,3,3,2,8,4,1,1) are 11, 10, and 9 respectively as given by Dios and Chopra in [6] whereas the corresponding upper bounds obtained using (2.3) are 10, 9, and 9.

ACKNOWLEDGMENTS

The authors would like to thank the referee for suggestions resulting in the improvement of the presentation of this paper.

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