Upper line-distinguishing and upper harmonious chromatic numbers of cycles

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Abstract

A k-line-distinguishing coloring of a graph $G = (V, E)$ is a partition of V into k sets $V_1, \ldots, V_k$ such that $q(\langle V_i \rangle) \leq 1$ for $i = 1, \ldots, k$ and $q(V_i, V_j) \leq 1$ for $1 \leq i < j \leq k$. If the color classes in a line-distinguishing coloring is also independent, then it is called a harmonious coloring. A coloring is minimal if, when two color classes are combined, we no longer have a coloring of the given type. The upper harmonious chromatic number, $H(G)$, is defined as the maximum cardinality of a minimal harmonious coloring of a graph G, while the upper line-distinguishing chromatic number, $H'(G)$, is defined as the maximum cardinality of a minimal line-distinguishing coloring of a graph G. We determine $H'(C_n)$ and $H(C_n)$ for a cycle C_n.

Keywords. cycles, upper harmonious chromatic number, upper line-distinguishing chromatic number

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1 Introduction

Graph theory terminology not presented here may be found in [1]. Let $G = (V, E)$ be a graph with n vertices. If $A \subseteq V$ and $B \subseteq V$, we will use $q(A, B)$ to denote the number of edges between the sets A and B. Let $S \subseteq V$. The set S is *independent* if for distinct $u, v \in S$, $uv \notin E$, while S is a *packing* if every two vertices in S are at distance at least 3 apart in G. The subgraph induced by S is denoted by $\langle S \rangle$. The distance $d(v, S)$ from a vertex v to the set S is defined as the minimum distance from v to a vertex of S.

A *k-coloring* of G is a partition Π of V into k sets, $V_1, V_2, \ldots, V_k$. A *proper k-coloring* is a k-coloring such that each V_i is independent. A k-coloring is a *complete coloring* if for every i, j, $1 \le i < j \le k$, $q(V_i, V_j) \ge 1$.

The *chromatic number* $\chi(G)$ is defined as $\min\{k \mid G$ has a proper k-coloring$\}$, while the *achromatic number* $\psi(G)$ is defined as $\max\{k \mid G$ has a proper complete k-coloring$\}$.

A *k-line-distinguishing coloring* of G is a partition of V into k sets $V_1, \ldots, V_k$ such that $q(\langle V_i \rangle) \le 1$ for $i = 1, \ldots, k$ and $q(V_i, V_j) \le 1$ for $1 \le i < j \le k$.

If a line-distinguishing coloring is also a proper coloring, then it is called a *harmonious coloring*. In other words, the partition $\{V_1, V_2, \ldots, V_k\}$ is a harmonious coloring of G if and only if $q(\langle V_i \rangle) = 0$ for $i = 1, \ldots, k$ and $q(V_i, V_j) \le 1$, $1 \le i < j \le k$.

The *line-distinguishing coloring number* $h'(G)$ is defined as $\min\{k \mid G$ has a k-line-distinguishing coloring$\}$, while the *harmonious coloring number* $h(G)$ is defined as $\min\{k \mid G$ has a k-harmonious coloring$\}$.

The achromatic number was first introduced and studied by Harary, Hedetniemi and Prins [6]. The line-distinguishing number, $h'(G)$, was introduced independently by Frank, Harary and Plantholt [7] and Hopcroft and Krishnamoorthy [8] even though the latter authors called it the harmonious coloring number. Harmonious colorings were introduced by Miller and Pritikin in [9] and further investigated in [4] and [5].

Consider a partition $\Pi = \{V_1, V_2, \ldots, V_k\}$ of V according to some specified properties P and Q. This means that $\langle V_i \rangle$ has property P for $i = 1, \ldots, k$ and the bipartite graph (V_i, V_j) has property Q for distinct $i, j \in \{1, \ldots, k\}$. The partition is *minimal* with respect to properties P and Q if any partition Π' obtained from Π by combining color classes V_i and V_j no longer satisfies properties P and Q. The smallest and largest cardinality of minimal partitions with respect to properties P and Q give rise to two parameters associated with a graph. For example, the chromatic and achromatic numbers are, respectively, the minimum and maximum cardinality of a minimal partition where the property P specifies that the induced subgraph of each set in the partition contains no edge.

Let P be the property "contains no edges" and Q be the property "contains at most one edge". If $\Pi = \{V_1, \ldots, V_k\}$ is a partition according to the properties P and Q, then Π is a harmonious coloring of G. If we change property P to "contains at most one edge", then Π becomes a line-distinguishing coloring of G. Before proceeding further, we state a characterization of minimal harmonious and minimal line-distinguishing colorings of a graph, as given in [2].

Lemma 1 (Chen et al. [2]) *A harmonious coloring* $\{V_1, \ldots, V_k\}$ *is minimal if and only for distinct* $i, j \in \{1, \ldots, k\}$

(1) $q(V_i, V_j) = 1$, *or*

(2) *if* $V_i \cup V_j$ *is independent, there is an* $r \in \{1, \ldots, k\} - \{i, j\}$ *such that* $q(V_i, V_r) = 1$ *and* $q(V_r, V_j) = 1$.

Lemma 2 (Chen et al. [2]) *A line-distinguishing coloring* $\{V_1, \ldots, V_k\}$ *is minimal if and only for distinct* $i, j \in \{1, \ldots, k\}$

(1) $q(\langle V_i \cup V_j \rangle) > 1$, *or*

(2) *if* $q(\langle V_i \cup V_j \rangle) \leq 1$, *there is an* $r \in \{1, \ldots, k\} - \{i, j\}$ *such that* $q(V_i, V_r) = 1$ *and* $q(V_r, V_j) = 1$.

The *upper harmonious chromatic number*, $H(G)$, is defined as the maximum cardinality of a minimal harmonious coloring of a graph G, while the *upper line-distinguishing chromatic number*, $H'(G)$, is defined as the maximum cardinality of a minimal line-distinguishing coloring of a graph G. These parameters were first introduced and studied in [2]. In particular, it was shown that the decision problems corresponding to the computation of $H(G)$ and $H'(G)$ for a general graph G are NP-complete, that the two parameters are incomparable, even for trees, and, lastly, $H(P_n)$ and $H'(P_n)$ were determined for the path P_n of order n.

In this paper, we determine $H'(C_n)$ and $H(C_n)$ for a cycle C_n on n vertices.

2 The value of $H'(C_n)$

In this section we determine the upper line-distinguishing chromatic number of a cycle. We start with the following result.

Lemma 3 *If* $n = 3k + r$ *is a positive integer with* $r \in \{0, 1, 2\}$, $m \geq 2k + 2$ *and* $V_1, \ldots, V_m$ *is a coloring of* C_n, *then there are at least two color classes of cardinality one.*

Proof. Let ℓ be the number of color classes of cardinality one. Then $n - \ell \geq 2(m - \ell)$, so that $n \geq 2m - \ell \geq 2(2k + 2) - \ell$. Thus, $(3k + r) \geq 4k + 4 - \ell$, so that $\ell \geq k + 4 - r \geq 2$. □

Theorem 4 *For* $n \geq 3$, $H'(C_n) = 2\lfloor \frac{n}{3} \rfloor + 1$.

Proof. Let $k = \lfloor \frac{n}{3} \rfloor$. We first show that $H'(C_n) \leq 2k + 1$. Suppose, to the contrary, $H'(C_n) = m \geq 2k + 2$ and let $V_1, \ldots, V_m$ be a minimal line-distinguishing coloring of C_n. Without loss of generality assume $V_1 = \{w\}$ with $N(w) = \{x, y\}$ (cf. Lemma 3). Furthermore, assume $x \in V_2$ and $y \in V_3$. Let $A = V_2 \cup V_3$ and let $B = V - V_1 - A$. Since we cannot combine color classes V_1 and V_i, $i = 4, \ldots, m$, some vertex in V_i must be adjacent to some vertex in A. Then, since B is the union of at least $2k - 1$ color classes, each joined to A, it follows that $|B| \geq 2k - 1$ and $q(A, B) \geq 2k - 1$. Thus, $|A| = 3k + r - 1 - |B| \leq 3k + r - 1 - (2k - 1) = k + r$ and

$$2q(\langle A \rangle) + 2k + 1 \leq 2q(\langle A \rangle) + 2 + q(A, B) = \sum_{v \in A} \deg(v) = 2|A| \leq 2k + 2r. \qquad (1)$$

Thus, $2q(\langle A \rangle) \leq (2k + 2r) - (2k + 1) = 2r - 1 \leq 3$, so that $q(\langle A \rangle) \leq 1$. If A is independent, then we may combine V_1 and V_2 to obtain a line-distinguishing coloring of C_n, contrary to minimality. Thus, $q(\langle A \rangle) = 1$. If $r = 0$ or $r = 1$, Equation (1) leads to a contradiction. Thus, $r = 2$.

By Equation (1), $q(A, B) \leq 2k$ and $|A| \geq (2k + 3)/2$. Thus, $|A| = k + 2$ and $|B| = 2k - 1$. Let $B = \{v_1, \ldots, v_{2k-1}\}$. Since B is the union of at least $2k - 1$ color classes, each joined to A, $\{v_i\}$ is a color class and v_i is adjacent to at least one vertex of A ($i = 1, \ldots, 2k - 1$). Thus, $q(\langle B \rangle) \leq k - 1$ and $q(A, B) \geq 2k$. Hence, $q(\langle B \rangle) = k - 1$ and $\langle B \rangle \cong K_1 \cup (k - 1)K_2$. Without loss of generality, assume that $\deg_{\langle B \rangle}(v_1) = 0$.

First consider the case when $k = 1$. Since we cannot combine color classes V_1 and V_2, some vertex of V_2 is adjacent to some vertex of V_3. Let $\{z\} = A - \{x, y\}$. Without loss of generality assume $z \in V_2$. Then y is adjacent to z and v_1 is adjacent to x and z, which is a contradiction. Thus, $k \geq 2$.

Suppose the neighbor of v_2 in A is in color class V_2. Since we cannot combine color classes $\{v_2\}$ and $\{v_i\}$ ($i = 3, \ldots, 2k - 1$), the neighbor of v_i in A is also in V_2. Furthermore, since we cannot combine color classes $\{v_2\}$ and V_3, some vertex of V_3 is adjacent to some vertex of V_2. We conclude that V_2 and V_3 are independent sets. Since $|N(V_2)| = 2k + 1$, $|V_2| \geq k + 1$. Thus, $V_3 = \{y\}$, and, consequently, y is adjacent to some vertex of V_2, the vertex w and the vertex v_1, which is a contradiction. Thus, $H'(C_n) \leq 2k + 1$.

Let $n = 3k + r$, where k and r are nonnegative integers and denote the consecutive vertices of the cycle by $v_1, \ldots, v_n$.

If $r = 0$, then the partition $\{\{v_1, v_4, \ldots, v_{3k-2}\}, \{v_2\}, \{v_3\}, \ldots, \{v_{3k-1}\}, \{v_{3k}\}\}$ is a minimal line-distinguishing coloring of C_n, whence $H'(C_n) \geq 2k + 1$.

If $r = 1$, then the partition $\{\{v_1, v_4, \ldots, v_{3k-2}, v_{3k-1}\}, \{v_2\}, \{v_3\}, \ldots, \{v_{3k-1}\}, \{v_{3k}\}\}$ is a minimal line-distinguishing coloring of C_n, whence $H'(C_n) \geq 2k + 1$.

If $r = 2$, then the partition $\{\{v_1, v_4, \ldots, v_{3k-2}\} \cup \{v_{3k+2}\}, \{v_2\}, \{v_3\}, \ldots, \{v_{3k-4}\}, \{v_{3k-3}\}, \{v_{3k-1}\}, \{v_{3k}, v_{3k+1}\}\}$ is a minimal line-distinguishing coloring of C_n, whence $H'(C_n) \geq 2k + 1$. The result follows. □

3 The value of $H(C_n)$

In this section we determine the upper harmonious chromatic number of a cycle.

Theorem 5 *For $n \geq 3$,*

$$H(C_n) = \begin{cases} 2\lceil \frac{n}{3} \rceil & \text{if } n \in \{4,8\} \\ 2\lceil \frac{n}{3} \rceil + 1 & \text{if } n = 5 \\ 2\lfloor \frac{n}{3} \rfloor + 1 & \text{otherwise} \end{cases}$$

Proof. We first determine the values $H(C_4), H(C_5)$ and $H(C_8)$. By putting each vertex of C_4 in its own color class, we obtain a minimal harmonious coloring of C_4, so that $H(C_4) = 4$. Similarly, $H(C_5) = 5$.

If $v_1, \ldots, v_8$ are consecutive vertices of C_8, then $\{\{v_1\}, \{v_2, v_5\}, \{v_4, v_8\}, \{v_3\}, \{v_6\}, \{v_7\}\}$ is a minimal harmonious coloring of C_8, so that $H(C_8) \geq 6$. We now show that $H(C_8) \leq 6$. Suppose, to the contrary, that $H(C_8) = m \geq 7$ and let $V_1, \ldots, V_m$ be a minimal harmonious coloring of C_8. Without loss of generality assume $V_1 = \{w\}$ with $N(w) = \{x, y\}$ (cf. Lemma 3). Furthermore, assume $x \in V_2$ and $y \in V_3$. Let $A = V_2 \cup V_3$ and let $B = V - V_1 - A$. Since we cannot combine color classes V_1 and $V_i, i = 4, \ldots, m$, some vertex in V_i must be adjacent to some vertex in A. Since $m - 3 \geq 4$, we have $q(A, B) \geq 4$, $|B| \geq 4$ and $|A| \leq 3$. Thus,

$$2q(\langle A \rangle) + 6 \leq 2q(\langle A \rangle) + 2 + q(A, B) = \sum_{v \in A} \deg(v) \leq 2|A| \leq 6. \quad (2)$$

Thus, $2q(\langle A \rangle) \leq 0$, so that A is an independent set. By Equation (2), $|A| = 3$, so that $|B| = 4$. Let $\{z\} = A - \{x, y\}$ and let $N(z) = \{v_2, v_3\}$. Suppose v_1 is adjacent to x and v_4 is adjacent to y. Without loss of generality assume v_1v_2 and v_3v_4 are edges. Moreover, we may without loss of generality assume that $z \in V_3$. But then we may combine color classes $\{v_2\}$ and $\{v_4\}$ to obtain a harmonious coloring of C_8, which is a contradiction. Thus, $H(C_8) = 6$.

In what follows, suppose $n = 3k + r \geq 3$ and $n \notin \{4, 5, 8\}$ is an integer with $r \in \{0, 1, 2\}$. We first show that $H(C_n) \leq 2k + 1$. Suppose, to the contrary, $H(C_n) = m \geq 2k + 2$ and let $V_1, \ldots, V_m$ be a minimal harmonious coloring of C_n.

Without loss of generality assume $V_1 = \{w\}$ with $N(w) = \{x, y\}$ (cf. Lemma 3). Furthermore, assume $x \in V_2$ and $y \in V_3$. Let $A = V_2 \cup V_3$ and let $B = V - V_1 - A$. Since we cannot combine color classes V_1 and $V_i, i = 4, \ldots, m$, some vertex in V_i must be adjacent to some vertex in A. Then, since B is the union of at least $2k - 1$ color classes, each joined to A, it follows that $|B| \geq 2k - 1$ and $q(A, B) \geq 2k - 1$. Thus, $|A| = 3k + r - 1 - |B| \leq 3k + r - 1 - (2k - 1) = k + r$ and

$$2q(\langle A \rangle) + 2k + 1 < 2q(\langle A \rangle) + 2 + q(A, B) = \sum_{v \in A} \deg(v) \leq 2|A| = 2k + 2r. \quad (3)$$

Hence, $0 \leq 2q(\langle A\rangle) \leq (2k+2r) - (2k+1) = 2r-1$, so that $r \in \{1,2\}$. By the definition of a harmonious coloring, $q(\langle A\rangle) \leq 1$.

Case 1. $r = 1$.

Equation (3) shows that A is independent. $|A| \geq \lceil (2k+1)/2 \rceil = k+1$. Thus, $|A| = k+1$ and $|B| = 2k-1$. Let $B = \{v_1, \ldots, v_{2k-1}\}$. Since m is the union of at least $2k-1$ color classes, each joined to A, $\{v_i\}$ is a color class and v_i is adjacent to at least one vertex of A $(i = 1, \ldots, 2k-1)$. Thus, $q(\langle B\rangle) \leq k-1$ and $q(A, B) \geq 2k$. By Equation (3), $q(A, B) \leq 2k$, so that $q(\langle B\rangle) = k-1$ and $\langle B\rangle \cong K_1 \cup (k-1)K_2$. Without loss of generality, assume that $\deg_{\langle B\rangle}(v_1) = 0$ and assume $v_i v_{i+1}$, $i = 2, 4, \ldots, 2k-2$, is an edge. Note that $k \geq 2$.

If $k = 2$, then not both v_2 and v_3 are adjacent to $z \in A - \{x, y\}$. Without loss of generality assume v_2 is adjacent to x. Then v_1 is adjacent to z and y, whence $z \in V_2$. We conclude that v_3 is adjacent to z. But then we may combine color classes $\{v_2\}$ and V_3 to obtain a harmonious coloring, contrary to minimality.

We henceforth assume that $k \geq 3$. Suppose the neighbor of v_2 in A is in color class V_2. Since we cannot combine color classes $\{v_2\}$ and $\{v_i\}$ $(i = 4, \ldots, 2k-1)$, the neighbor of v_i in A is also in V_2. Furthermore, since we cannot combine $\{v_1\}$ with $\{v_3\}$, the neighbor of v_3 in A is also in V_2. Thus, none of the vertices in $\{v_2, \ldots, v_{2k-1}\}$ are adjacent to vertices in V_3. But then we may combine color classes V_3 and $\{v_2\}$ to obtain a harmonious coloring of C_n, contrary to minimality.

Case 2. $r = 2$.

Note that $k \geq 3$.

Case 2.1 $q(\langle A\rangle) = 1$. By Equation (3), $2k+3 \leq 2|A| \leq 2k+4$, so that $|A| = k+2$. Hence, $|B| = 2k-1$. As before, each $\{v_i\}$ is a color class and is adjacent to at least one vertex of A $(i = 1, \ldots, 2k-1)$. Thus, $q(\langle B\rangle) \leq k-1$ and $q(A, B) \geq 2k$. Hence, $q(\langle B\rangle) = k-1$ and $q(A, B) = 2k$. Let $B = \{v_1, \ldots, v_{2k-1}\}$. Without loss of generality, assume that $\deg_{\langle B\rangle}(v_1) = 0$ and assume $v_i v_{i+1}$, $i = 2, 4, \ldots, 2k-2$, is an edge. Suppose the neighbor of v_2 in A is in color class V_2. Since we cannot combine color classes $\{v_2\}$ and $\{v_i\}$ $(i = 4, \ldots, 2k-1)$, the neighbor of v_i in A is also in V_2. Furthermore, since we cannot combine $\{v_1\}$ with $\{v_3\}$, the neighbor of v_3 in A is also in V_2. Since $|N(V_2)| = 2k+1$, $|V_2| \geq k+1$. But then $V_3 = \{y\}$, and it follows that y is adjacent to some vertex of V_2, the vertex w and the vertex v_1, which is a contradiction.

Case 2.2 A is independent. By Equation (3), $k+1 \leq |A| \leq k+2$.

Case 2.2.1 $|A| = k+1$. Then $|B| = 2k$. Recall from the second paragraph of the proof that $q(A, B) \geq 2k-1$. Since $m-3 \geq 2k-1$, each $\{v_i\}, i = 1, \ldots, 2k$, forms a color class or there is a color class, say C, containing exactly two vertices, while each remaining vertex of B forms a color class. Let $B = \{v_1, \ldots, v_{2k}\}$.

Case 2.2.1.1 Each vertex of B is adjacent to at least one vertex of A.

Let ℓ be the number of vertices of B that is adjacent to exactly two vertices of A. Then $3k + 2 = q(C_n) = 2 + q(A, B) + q(\langle B\rangle) = 2 + 2\ell + (2k - \ell) + k - \frac{\ell}{2} = 3k + \frac{\ell}{2} + 2$, so that $\ell = 0$. Without loss of generality, assume v_iv_{i+1}, $i = 1, 3, \dots, 2k - 1$, is an edge. Suppose the neighbor of v_1 in A is in color class V_2. Since we cannot combine color classes $\{v_1\}$ and $\{v_i\}$ $(i = 3, \dots, 2k)$, the neighbor of v_i in A is also in V_2. Since we cannot combine $\{v_3\}$ with $\{v_2\}$, the neighbor of v_2 in A is also in V_2. Since $|N(V_2)| = 2k + 1$, $|V_2| \geq k + 1$, which is a contradiction.

Case 2.2.1.2 Some vertex of B is not adjacent to A, say v_1. This vertex is in C. Then all vertices, except v_1, are adjacent to at least one vertex of A. Let ℓ be the number of vertices of B that is adjacent to exactly two vertices of A. Then $3k + 2 = q(C_n) = 2 + q(A, B) + q(\langle B\rangle) = 2 + 2\ell + (2k - 1 - \ell) + 2 + (k - \frac{3}{2} - \frac{\ell}{2}) = 3k + \frac{\ell}{2} + \frac{3}{2}$, so that $\ell = 1$. Without loss of generality, assume v_1 is adjacent to v_2 and to v_3 and that v_4 is isolated in $\langle B\rangle$. Moreover, suppose $v_iv_{i+1}, i = 5, 7, \dots, 2k - 1$, is an edge of C_n.

Suppose v_4 is the other vertex in C. Without loss of generality, suppose the neighbor of v_3 in A is in V_2. But then $\{v_3\}$ is a color class of size one which is adjacent to a vertex in C and a vertex in V_2. Let $A' = V_2 \cup C$ and let $B' = V - \{v_3\} - A'$. Note that $q(\langle A'\rangle) = 1$ and we obtain a contradiction as in Case 2.1.

Thus $C = \{v_1, v_i\}$ for some $i \in \{5, \dots, 2k\}$. Without loss of generality assume that $C = \{v_1, v_5\}$. Suppose the neighbor of v_3 in A is in V_2. If the neighbor of v_5 in A is also in V_2, we obtain a contradiction as in the previous paragraph. Thus, the neighbor of v_5 in A is in V_3. By the same reasoning, the neighbor of v_2 in A is also in V_2. Since we cannot combine $\{v_6\}$ and V_2, the neighbor of v_6 in A is in V_2. If $k = 3$, then one of the vertices in V_3 has degree one, which is a contradiction.

Thus, $k \geq 4$. Suppose the neighbor of v_7 in A is in V_2. We may then combine color classes C and $\{v_7\}$ to obtain a harmonious coloring of C_n, which is contrary to minimality. If the neighbor of v_7 in A is in V_3, we may combine color classes $\{v_6\}$ and $\{v_7\}$ to obtain a harmonious coloring of C_n, which is contrary to minimality.

Case 2.2.2 $|A| = k + 2$. Then $|B| = 2k - 1$. Let $B = \{v_1, \dots, v_{2k-1}\}$. As before, each $\{v_i\}, i = 1, \dots, 2k - 1$, is a color class and is adjacent to at least one vertex of A. Let ℓ be the number of vertices of B that is adjacent to two vertices of A. Then $3k + 2 = q(C_n) = 2 + q(A, B) + q(\langle B\rangle) = 2 + 2\ell + 2k - 1 - \ell + k - \frac{1}{2} - \frac{\ell}{2} = \frac{1}{2} + \frac{\ell}{2} + 3k$, so that $\ell = 3$. Thus, $\langle B\rangle \cong \overline{K}_3 + (k - 2)K_2$. Suppose v_1, v_2 and v_3 are the isolated vertices of $\langle B\rangle$. Let $v_iv_{i+1}, i = 4, 6, \dots, 2k - 2$, be edges of C_n.

First consider the case when $k = 3$. If $V_2 = \{x\}$, then x is adjacent to w, v_1, v_2 and v_3, which is a contradiction. Thus, $|V_2| \geq 2$, and, by symmetry, $|V_3| \geq 2$. Without loss of generality assume $|V_2| = 2$ and $|V_3| = 3$. Let $V_2 = \{x, z\}$. Without loss of generality, assume that v_1 is adjacent to x and that v_2 and v_3 are both adjacent to z. Notice that $N(\{v_1, v_5\}) \cap V_2 = \emptyset$. But then we may combine V_2 and $\{v_5\}$ to obtain a harmonious coloring of C_n, which is a contradiction.

Henceforth we assume $k \geq 1$. Suppose the neighbor of v_1 in A is in color class V_2. Since we cannot combine color classes $\{v_1\}$ and $\{v_i\}$ $(i = 6, \ldots, 2k-1)$, the neighbor of v_i in A is also in V_2. Since we cannot combine $\{v_6\}$ with $\{v_5\}$, the neighbor of v_5 in A is also in V_2. Thus, none of the vertices in $\{v_1, \ldots, v_{2k-1}\}$ are adjacent to vertices of V_3. But then we may combine color classes $\{v_1\}$ and V_3 to obtain a harmonious coloring, contrary to minimality.

Let $n = 3k+r$, where k and r are nonnegative integers and denote the consecutive vertices of the cycle by $v_1, \ldots, v_n$.

If $r = 0$, then the partition $\{\{v_1, v_4, \ldots, v_{3k-2}\}, \{v_2\}, \{v_3\}, \ldots, \{v_{3k-1}\}, \{v_{3k}\}\}$ is a minimal harmonious coloring of C_n, whence $H'(C_n) \geq 2k+1$.

If $r = 1$, then the partition $\{\{v_1, v_4\} \cup \{v_8, v_{11}, \ldots, v_{3k-1}\}, \{v_2\}, \{v_3, v_6\}, \{v_5\}, \{v_7\}, \{v_9\}, \{v_{10}\}, \ldots, \{v_{3k}\}, \{v_{3k-1}\}\}$ is a minimal harmonious coloring of C_n, whence $H(C_n) \geq 2k+1$.

If $r = 2$, then the partition $\{\{v_1, v_4, \ldots, v_{3k-2}\}, \{v_2\}, \{v_3\}, \ldots, \{v_{3k-10}\}, \{v_{3k-9}\}, \{v_{3k-7}\}, \{v_{3k-6}, v_{3k-1}\}, \{v_{3k-4}\}, \{v_{3k-3}, v_{3k}\}, \{v_{3k-1}\}, \{v_{3k+2}\}\}$ is a minimal harmonious coloring of C_n, whence $H(C_n) \geq 2k+1$. The result follows. □

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