Generalized Steiner systems $GS_4(2, 4, v, 4)^*$

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Abstract

Generalized Steiner systems $GS_d(t, k, v, g)$ were first introduced by Etzion and used to construct optimal constant weight codes over an alphabet of size $g+1$ with minimum Hamming distance d, in which each codeword has length v and weight k. It was proved that the necessary conditions for the existence of a $GS_4(2, 4, v, g)$ are also sufficient for $g = 2, 3$ and 6. In this paper, a general result on the existence of a $GS_4(2, 4, v, g)$ is presented. By using this result, we prove that the necessary conditions $v \equiv 1 \pmod 3$ and $v \geq 7$ are also sufficient for the existence of a $GS_4(2, 4, v, 4)$.

Keywords: generalized Steiner systems, constant weight codes, singular indirect product.

1 Introduction

The H-design was first introduced by Hanani [7] as a generalization of Steiner systems (the notion of H-design is due to Mills [8]). An $H(v, g, k, t)$ design is a triple $(\mathcal{X}, \mathcal{G}, \mathcal{B})$, where $\mathcal{X}$ is a set of points whose cardinality is vg, and $\mathcal{G} = \{G_1, \cdots, G_v\}$ is a partition of $\mathcal{X}$ into v sets of cardinality g; the members of $\mathcal{G}$ are called *groups*. A *transverse* of $\mathcal{G}$ is a subset of $\mathcal{X}$ that meets each group in at most one point. The set $\mathcal{B}$ contains k-element transverse of $\mathcal{G}$, called *blocks*, with the property that each t-element transverse

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† Research supported in part by YNSFC Grant 10001026.

of $\mathcal{G}$ is contained in precisely one block. When $t=2$, an $H(v,g,k,2)$ is just a group divisible design of group type g^v and denoted by k-GDD(g^v).

As stated in Etzion [6] and Yin et al. [14], an optimal $(g+1)$-ary (v,k,d) constant weight code (CWC) over Z_{g+1} can be constructed from a given $H(v,g,k,t)$ $(I_v \times I_g, \{\{i\} \times I_g \mid i \in I_v\}, \mathcal{B})$, where $I_m = \{1,2,\cdots,m\}$ and d is the minimum Hamming distance of the resulting code. For each block $\{(i_1, a_1), (i_2, a_2), \cdots, (i_k, a_k)\} \in \mathcal{B}$, we form a codeword of length v by putting a_j in position i_j, $1 \leq j \leq k$, and zeros elsewhere. For convenience, when two codewords obtained from blocks B_1 and B_2 have distance d, we simply say that B_1 and B_2 have distance d.

In the code which is related to an $H(v,g,k,t)$, we want that the minimum Hamming distance d is as large as possible. It is not difficult to see that $k-t+1 \leq d \leq 2(k-t)+1$. In [6], an $H(v,g,k,t)$ which forms a code with minimum Hamming distance $2(k-t)+1$ was called a *generalized Steiner system* GS(t,k,v,g). An H(v,g,k,t) which forms a code with minimum Hamming distance d was denoted by GS$_d(t,k,v,g)$.

Much work has been done for the existence of GS(t,k,v,g) when $t=2$ and $k=3$ (see [6], [2], [10], [9], [3], [4], [11]). For $k=4$, it was proved that the necessary conditions for the existence of a GS$_4(2,4,v,g)$ are also sufficient for $g=2,3,6$ (see [12]). There are also some partial result on GS$(2,4,v,2)$(see [13]) and some product constructions stated in [6].

The following necessary conditions are stated in [12].

Lemma 1.1 *If there exists a $GS_4(2,4,v,g)$, then*

(1) $\binom{v-2}{2} \geq g$; *and*

(2) $v \equiv 1,4 \ (mod\ 12)$, *if* $g \equiv 1,5 \ (mod\ 6)$,

$v \equiv 1 \ (mod\ 3)$, *if* $g \equiv 2,4 \ (mod\ 6)$,

$v \equiv 0,1 \ (mod\ 4)$, *if* $g \equiv 3 \ (mod\ 6)$.

Since the existence of GS$_4(2,4,v,g)$ is completely solved for $g=2,3,6$ (see [12]), then in this paper we suppose that $g \notin \{2,3,6\}$. A general result on the existence of GS$_4(2,4,v,g)$ is presented. Using this result, we prove that the necessary conditions $v \equiv 1 \ (mod\ 3)$ and $v \geq 7$ are also sufficient for the existence of a GS$_4(2,4,v,4)$.

Let

$T_g = \{v\text{: there exists a } GS_4(2,4,v,g)\}$.

$B_g = \{n : n \text{ satisfying the necessary conditions of a } GS_4(2,4,n,g)\}$,

$$M_g = \begin{cases} \{n : n \in B_g, u_0 \leq n \leq f(g)(h(g)+2)+1\}, & \text{if } g \neq 10, \\ \{n : n \equiv 1 \ (mod\ 3), 7 \leq n \leq 235\}, & \text{if } g = 10. \end{cases}$$

where, u_0=min $\{n : n \in B_g\}$,

$$f(g)=\begin{cases}13, & \text{if } g\not\equiv 3 \ (mod\ 6),\\ 16, & \text{if } g\equiv 3 \ (mod\ 6).\end{cases}$$

$h(g)$=max $\{n_0, k(g)\}$,

$$k(g)=\begin{cases}40, & \text{if } g\equiv 1,5 \ (mod\ 6),\\ 13, & \text{if } g\equiv 0,2,4 \ (mod\ 6),\\ 16, & \text{if } g\equiv 3 \ (mod\ 6).\end{cases}$$

n_0=min $\{n : n \geq g, n \in B_g\}$, if $g \equiv 3$ $(mod\ 6)$, we need that $n_0 \equiv 0$ $(mod\ 4)$.

We state the main results of this paper below.

Theorem 1.2 *Suppose that $g \notin \{2,3,6\}$. If $M_g \subset T_g$, then $B_g = T_g$, i.e., the necessary conditions for the existence of a $GS_4(2,4,v,g)$ are also sufficient.*

Theorem 1.3 *The necessary conditions $v \equiv 1$ $(mod\ 3)$ and $v \geq 7$ are also sufficient for the existence of a $GS_4(2,4,v,4)$.*

For general background on designs, see [1], [5].

2 The existence of $GS_4(2,4,v,g)$

In order to prove Theorem 1.2, we need some lemmas, which were stated in [12]. We first give the following conception.

A *holey group divisible design*, K−HGDD, is a four-tuple $(\mathcal{V},\mathcal{G},\mathcal{H},\mathcal{B})$, where $\mathcal{V}$ is a set of points, $\mathcal{G}$ is a partition of $\mathcal{V}$ into subsets called *groups*, $\mathcal{H} \subset \mathcal{G}$, $\mathcal{B}$ is a set of *blocks* such that a group and a block contain at most one common point and every pair of points from distinct groups, not both in $\mathcal{H}$, occurs in a unique block in $\mathcal{B}$, where $|B| \in K$ for any $B \in \mathcal{B}$. A k−HGDD$(g^{(v,u)})$ denotes a K-HGDD with v groups of size g in $\mathcal{G}$, u groups in $\mathcal{H}$ and $K = \{k\}$. Similarly to the way a (v,k,d) CWC is constructed from an an $H(v,g,k,t)$, we can also construct a (v,k,d) CWC from an k-HGDD$(g^{(v,u)})$. The distance of two blocks in a k-HGDD$(g^{(v,u)})$, is the Hamming distance of the two codewords obtained from the two blocks. A *holey generalized Steiner system*, $HGS_d(2,k,(v,u),g)$, is a k−HGDD$(g^{(v,u)})$ with the property that the minimum Hamming distance of related CWC is d. For convenience, we also say that the design has minimum Hamming distance d.

It is easy to see that if $u = 0$ or $u = 1$, then an $HGS_d(2,k,(v,u),g)$ is just a GS$_d(2,k,v,g)$.

Lemma 2.1 *Let m, t, u, n and a be integers such that $0 \le a \le u$, $n \ge 2a$, $1 \le t \le n$, and $(n, a) \ne (5, 1)$. Suppose the following designs exist: (1) a 4−GDD(g^m) with the property that all its blocks can be partitioned into t sets $S_0, S_1, \cdots, S_{t-1}$, such that the minimum distance in S_r, $0 \le r \le t-1$, is 4. (2) an $HGS_4(2, 4, (n+u, u), g)$. Then there exists an $HGS_4(2, 4, (e, f), g)$, where $f = (m-1)a + u$ and $e = mn + f$. Further, if the following design exists a $GS_4(2, 4, f, g)$, then there exists a $GS_4(2, 4, e, g)$.*

Lemma 2.2 *Let m, t, u, n be integers such that $u = 0$, or 1, $1 \le t \le n$, $n \notin \{2, 6\}$. Suppose the following designs exist: (1) a 4−GDD(g^m) with the property that all its blocks can be partitioned into t sets $S_0, S_1, \cdots, S_{t-1}$, such that the minimum distance in S_r, $0 \le r \le t-1$, is 4; (2) a $GS_4(2, 4, n+u, g)$. Then there exist both a $GS_4(2, 4, mn+u, g)$ and an $HGS_4(2, 4, (mn+u, n+u), g)$.*

Lemma 2.3 *Let m, n, u be integers such that $u = 0$, or 1, $n \notin \{2, 6\}$. Suppose there exist both a $GS_4(2, 4, m, g)$ and a $GS_4(2, 4, n+u, g)$. Then there exist both a $GS_4(2, 4, mn+u, g)$ and an $HGS_4(2, 4, (mn+u, n+u), g)$.*

We first suppose that $g \ne 10$, and hence $N(g) \ge 3$. So, there exists an RTD($4, g$), which is also a 4-RGDD(g^4). Such an RGDD has g parallel classes. It is clear that the minimum distance of each parallel class is 4. So, we have the following result.

Lemma 2.4 *If $g \notin \{2, 3, 6, 10\}$, then there exists a 4-GDD(g^4) whose blocks can be partitioned into g sets, such that the minimum distance of each is 4.*

Lemma 2.5 *Suppose p, a are integers, $p \notin \{2, 6\}$, $p \ge g$ and $0 \le a \le p$. If both p and $p + 3a$ are in T_g, then $13p + 3a \in T_g$.*

Proof Take $m = 4$, $n = p$, $u = 0$ and $t = g$ in Lemma 2.3 to obtain an HGS$_4(2, 4, (4p, p), g)$, the input designs are from Lemma 2.4 and the assumption. The result is obtained by taking $m = 4$, $n = 3p$, $u = p$ and $t = g$ in Lemma 2.1. □

To deal with the case of $g \equiv 3$ (*mod* 6), we need some further results. Suppose $g \equiv 3$ (*mod* 6), and $g \ne 3$, let

$$p(g) = \begin{cases} \frac{g}{3}, & \text{if } g \ne 9, \\ 9 & \text{if } g = 9. \end{cases}$$

It is clear that $p(g) \le g$.

Lemma 2.6 *If $g = 6s + 3$, and $g \ne 3$, then there exists a 4-GDD(g^5) with the property that the blocks of the GDD can be partitioned into $p(g)$ sets, such that the minimum distance of each is 4.*

Proof There exists a $GS_4(2,4,5,3)$ from [12]. Suppose that $(\mathcal{V}_1,\mathcal{G}_1,\mathcal{B}_1)$ is a $GS_4(2,4,5,3)$. First we consider the case $g \neq 9$. In this case, we have that $N(2s+1) > 3$, and hence there exists an $RTD(4,2s+1)$, which is also a 4-RGDD($(2s+1)^4$). For each block $B \in \mathcal{B}_1$, let $\mathcal{A}_B$ denote the blocks of the 4-RGDD($(2s+1)^4$) on point set $(B \times Z_{2s+1})$, and group $\mathcal{G}_B = \{\{b\} \times Z_{2s+1} : b \in B\}$. Let S^i_B denote the i-th parallel class of the RGDD, $1 \leq i \leq 2s+1$. Let $\mathcal{V} = \mathcal{V}_1 \times Z_{2s+1}$, $\mathcal{G} = \{G \times Z_{2s+1} : G \in \mathcal{G}_1\}$. $\mathcal{A} = \bigcup_{B\in\mathcal{B}_1} \mathcal{A}_B$. It is well known that $(\mathcal{V},\mathcal{G},\mathcal{A})$ is a 4-GDD(g^5). Let $S_i = \bigcup_{B\in\mathcal{B}_1} S^i_B$, $1 \leq i \leq 2s+1$. It is evident that $\mathcal{A} = \bigcup_{i=1}^{2s+1} S_i$. Since the starting GDD is a $GS_4(2,4,5,3)$ and the minimum distance of each parallel class is 4, it is not difficult to check the minimum distance of each S_i is 4.

For $g = 9$, note that there exists a 4-GDD(3^4), which has 9 blocks. Take each block as a partial parallel class in the above process, the result is then obtained. This completes the proof. □

Lemma 2.7 *Suppose that $p \equiv 0 \pmod 4$ is an integer, $\delta = 0$ or 1 and $0 \leq a \leq p+\delta$ is an integer. If both $p+\delta$ and $p+4a+\delta$ are in T_g, then $16p+4a+\delta \in T_g$.*

Proof We can apply Lemma 2.3 with $m = 4$, $n = p$, $u = \delta$ and $t = g$ to obtain an $HGS_4(2,4,(4p+\delta,p+\delta),g)$. Take $m = 5$, $n = 3p$, $u = p+\delta$ and $t = p(g)$ in Lemma 2.1, we can obtain the result, the input designs are from Lemma 2.6 and the assumption. □

For convenience, let $[x,y]^c_b$ denote the set of integers v, such that $x \leq v \leq y$, and $v \equiv c \pmod b$. $[x,y]^{c,f}_b$ denote the set of integers v, such that $x \leq v \leq y$ and $v \equiv c,\ f \pmod b$.

Lemma 2.8 *Suppose that $g \notin \{2,3,6,10\}$. If $M_g \subset T_g$, then $B_g = T_g$.*

Proof For $g \equiv 0 \pmod 6$, Lemma 2.5 guarantees that for any $p \geq g$, $[p,4p]^p_3 \subset T_g$ implies $[13p,16p]^p_3 \subset T_g$. In Lemma 2.5, take $p = n, n+1, n+2$ respectively, we obtain that $[n,4(n+2)] \subset T_g$ implies $[13(n+2),16n] \subset T_g$. Note that $f(g) = 13$, it is not difficult to see that $\bigcup_{p\geq h(g)} [13(p+2),16p] = [f(g)(h(g)+2),\infty)$. We will prove that for any $v \in B_g$, $v \in T_g$. If $v \in M_g$, then the result comes from assumption. Otherwise, there exists a $p \geq h(g)$ such that $v \in [13(p+2),16p]$. From the definition of $h(g)$, we have that $h(g) \geq g$. So, if $[p,4(p+2)] \subset M_g$, then from the above we have that $v \in T_g$. If there exists a $v' \in [p,4(p+2)]$ such that $v' \notin M_g$, then we can repeat the above process to obtain a new p'. It is evident that $v' < v$. After certain steps, we have that $[p', 4(p'+2)] \subset M_g$. This makes $[p, 4(p+2)] \subset M_g$, thus $v \in T_g$. So, the result is true for $g \equiv 0 \pmod 6$.

For $g \equiv 1, 5 \pmod 6$, notice that if $p \in B_g$, then $p \equiv 1, 4 \pmod{12}$ and $p + 3a \equiv 1, 4 \pmod{12}$ when $a \equiv 0, 1, 3 \pmod 4$. From Lemma 2.5, we can obtain that for any $p \geq g$ in B_g, $[p, 4p]_{12}^{1,4} \subset T_g$ implies $[13p, 16p]_{12}^{1,4} \subset T_g$. Note that

$$\bigcup_{\substack{p \geq h(g) \\ p \equiv 1,4 \pmod{12}}} [13p, 16p]_{12}^{1,4} = [f(g)h(g), \infty)_{12}^{1,4},$$

the rest part is similar to the case of $g \equiv 0 \pmod 6$.

For $g \equiv 2, 4 \pmod 6$, if $p \in B_g$, then $p \equiv 1 \pmod 3$. From Lemma 2.5, we have that for any $p \geq g$ in B_g, $[p, 4p]_3^1 \subset T_g$ implies $[13p, 16p]_3^1 \subset T_g$. Note that

$$\bigcup_{\substack{p \geq h(g) \\ p \equiv 1 \pmod 3}} [13p, 16p]_3^1 = [f(g)h(g), \infty)_3^1,$$

the rest part is similar to the above.

Finally, for $g \equiv 3 \pmod 6$, we suppose that $p \equiv 0 \pmod 4$. From Lemma 2.7, we can obtain that for any $p \geq g$ and $p + \delta \in B_g$, $[p+\delta, 4(p+\delta)]_4^\delta \subset T_g$ implies $[16p + \delta, 20p + 5\delta]_4^\delta \subset T_g$. So, we have that for any $p \geq g$ in B_g, $[p, 4(p+1)]_4^{0,1} \subset T_g$ implies $[16p+1, 20p]_4^{0,1} \subset T_g$. Note that

$$\bigcup_{\substack{p \geq h(g) \\ p \equiv 0 \pmod 4}} [16p+1, 20p]_4^{0,1} = [f(g)h(g)+1, \infty)_4^{0,1},$$

we can obtain the result similar to the above. The proof is complete. □

For $g = 10$, we also need a lemma on the partition of blocks of a 4-GDD(10^4). A *transversal* in a Latin square of side n is a set of n cells, one from each row and column containing each of the n symbols exactly once. A *partial transversal* of length k in a Latin square of side n is a set of k cells, each from a different row and each from a different column such that no two containing the same symbols. It is well known that $(k-2)MOLS(n)$s is equivalent to a TD(k, n). It is not difficult to see that a common transversal (partial transversal) of the $(k-2)MOLS(n)$s gives a parallel class (partial parallel class) of the corresponding TD(k, n). So, we have the following result since that a TD$(4, n)$ is a 4-GDD(n^4).

Lemma 2.9 *There exists a 4-GDD(10^4) with the property that the blocks of the design can be partitioned into 16 sets $S_0, \cdots, S_{15}$, such that the minimum distance of S_r, $0 \leq r \leq 15$, is 4.*

Proof We need only to find $2MOLS(10)$s, such that the $2MOLS(10)$s have 16 common partial transversals which partition the 100 positions of the Latin squares. The following SOLS(10) was stated in [5, *Chapter* IV, *p.*444], and it is generated pseudo-cyclically.

SOLS(10) on symbol set $Z_9 \cup \{x\}$.

$$L = \begin{array}{|c|c|c|c|c|c|c|c|c|c|}
\hline
0 & 2 & 8 & 6 & x & 7 & 1 & 5 & 4 & 3 \\ \hline
5 & 1 & 3 & 0 & 7 & x & 8 & 2 & 6 & 4 \\ \hline
7 & 6 & 2 & 4 & 1 & 8 & x & 0 & 3 & 5 \\ \hline
4 & 8 & 7 & 3 & 5 & 2 & 0 & x & 1 & 6 \\ \hline
2 & 5 & 0 & 8 & 4 & 6 & 3 & 1 & x & 7 \\ \hline
x & 3 & 6 & 1 & 0 & 5 & 7 & 4 & 2 & 8 \\ \hline
3 & x & 4 & 7 & 2 & 1 & 6 & 8 & 5 & 0 \\ \hline
6 & 4 & x & 5 & 8 & 3 & 2 & 7 & 0 & 1 \\ \hline
1 & 7 & 5 & x & 6 & 0 & 4 & 3 & 8 & 2 \\ \hline
8 & 0 & 5 & 2 & 3 & 4 & 5 & 6 & 7 & x \\ \hline
\end{array}$$

In the notion of (i, j), i denotes the i-th row of the Latin square, and j denotes the j-th column. We suppose both the rows and the columns of the SOLS are indexed by $\{0, 1, \cdots, 8, x\}$. Let $D_0 = \{(i, j) : i = j,\ i, j \in Z_9 \cup \{x\}\}$.
For $k = 1, 2, 3, 6, 7, 8$, define

$$D_k = \{(i, j) : i - j = k, \quad i, j \in Z_9\},$$

Let

$$P_0 = \{(0, 4), (1, 6), (2, x), (x, 1)\},$$

$$P_i = \{(0 + i, 5 + i), (1 + i, 6 + i), (2 + i, x), (x, 1 + i)\}, 1 \le i \le 8.$$

Then D_k ($k = 0,\ 1,\ 2,\ 3,\ 6,\ 7,\ 8$) and $P_i (0 \le i \le 8)$ are 16 common partial transversals of L and its transpose, which partition the 100 positions of L, thus we complete the proof of this lemma. □

We are now in a position to prove Theorem 1.2.

Proof of Theorem 1.2 For $g \notin \{2, 3, 6, 10\}$, the result comes from Lemma 2.8. For $g = 10$, from Lemma 2.9, there exists a 4-GDD(10^4) whose blocks can be partitioned into 16 sets, such that the minimum distance of each is 4. So, if we take $n_0 = 16$, then Lemma 2.8 still works. In this case, we have that $f(10) = 13$, $h(10) = 16$, we obtain the result. This completes the proof. □

In the next section, we will use the above result to deal with the existence of $GS_4(2, 4, v, 4)$.

3 The existence of $GS_4(2,4,v,4)$

For $g = 4$, it is easy to see that $u_0 = n_0 = 7$, $h(4) = 13$, and $f(4) = 13$. So, $M_4 = [7, 193]_3^1$.

For $v = 7$, to construct a $GS_4(2,4,v,4)$ in Z_{28}, it suffices to find a set of generalized base blocks, $\mathcal{A} = \{B_1, \cdots, B_8\}$, such that $(\mathcal{V}, \mathcal{G}, \mathcal{B})$ forms a $GS_4(2,4,v,4)$, where $\mathcal{V} = Z_{28}$, $\mathcal{G} = \{G_0, G_1, \cdots, G_6\}, G_i = \{i + 7j : 0 \le j \le 3\}, 0 \le i \le 6$, and $\mathcal{B} = \{B + 4j : B \in \mathcal{A},\ 0 \le j \le 6\}$. For convenience, we write $\mathcal{A} = \bigcup_{i=0}^{3} \{\{i, x, y, z\} : \{x, y, z\} \in S_i\}$. So, for $\mathcal{A}$ we need only display the corresponding $S_i, 0 \le i \le 3$.

Lemma 3.1 *There exists a $GS_4(2,4,7,4)$.*

Proof With the aid of a computer, we have found a generalized base blocks, which are listed below.

$v = 7$
$S_0 : \{4,5,13\}, \{12,18,22\}, \{15,19,20\}$; $S_1 : \{17,19,27\}, \{14,25,26\}, \{10,12,23\}$; $S_2 : \{3,25,28\}$; $S_3 : \{14,19,22\}$.
This completes the proof. □

For $v \neq 7$, to construct a $GS_4(2,4,v,4)$ in Z_{4v}, it suffices to find a set of base blocks, $\mathcal{A} = \{B_1, \cdots, B_s\}$, where $s = \frac{v-1}{3}$ such that $(\mathcal{V}, \mathcal{G}, \mathcal{B})$ forms a $GS_4(2,4,v,4)$, where $\mathcal{V} = Z_{4v}$, $G = \{G_0, G_1, \cdots, G_v\}, G_i = \{i + vj : 0 \le j \le 3\}, 0 \le i \le v-1$, and $\mathcal{B}$ is obtained by developing $\mathcal{A}$ (*mod* $4v$).

Lemma 3.2 *There exists a $GS_4(2,4,v,4)$ for $v \in \{10, 13, 16, 19, 22, 25, 31, 34, 43, 46, 55, 58, 67, 79, 82, 115\}$*

Proof For each v, with the aid of a computer, we have found a set of base blocks, which are listed below.

$v = 10$
$\mathcal{A}$: $\{0,5,11,19\}, \{0,2,17,24\}, \{0,1,4,13\}$.

$v = 13$
$\mathcal{A}$: $\{0,38,43,46\}, \{0,2,17,36\}, \{0,1,30,42\}, \{0,4,24,31\}$.

$v = 16$
$\mathcal{A}$: $\{0,7,15,20\}, \{0,18,30,39\}, \{0,1,38,62\}, \{0,11,33,47\}, \{0,4,10,45\}$.

$v = 19$
$\mathcal{A}$: $\{0,3,12,47\}, \{0,10,37,53\}, \{0,1,22,46\}, \{0,8,36,50\}, \{0,7,20,25\}, \{0,2,6,17\}$.

$v = 22$
$\mathcal{A}$: $\{0,61,76,79\}, \{0,11,43,82\}, \{0,1,60,81\}, \{0,4,35,69\}, \{0,30,50,63\}, \{0,2,26,74\}, \{0,5,41,51\}$.

$v = 25$

$\mathcal{A}$: {0,67,77,80}, {0,11,53,74}, {0,18,48,86}, {0,9,15,69}, {0,2,59,83}, {0,39,51,55}, {0,28,92,93}, {0,5,34,78}.

$v = 31$

$\mathcal{A}$: {0,39,48,77}, {0,52,54,74}, {0,80,87,105}, {0,3,67,91}, {0,11,107,123}, {0,8,69,90}, {0,26,49,120}, {0,51,56,66}, {0,65,79,92}, {0,6,46,89}.

$v = 34$

$\mathcal{A}$: {0,23,77,115}, {0,49,80,106}, {0,63,121,128}, {0,85,91,94}, {0,4,24,74}, {0,13,25,35}, {0,16,97,134}, {0,11,40,83}, {0,19,52,119}, {0,1,61,89}, {0,5,32,46}.

$v = 43$

$\mathcal{A}$: {0,93,101,159}, {0,49,117,140}, {0,24,146,152}, {0,30,78,151}, {0,18,47,64}, {0,27,96,127}, {0,110,167,171}, {0,15,34,75}, {0,3,119,133}, {0,25,88,162}, {0,36,118,156}, {0,102,139,161}, {0,67,132,144}, {0,80,163,165}.

$v = 46$

$\mathcal{A}$: {0,118,177,183}, {0,44,58,171}, {0,4,90,141}, {0,38,150,173}, {0,77,85,104}, {0,18,20,30}, {0,31,73,114}, {0,16,64,116}, {0,24,78,145}, {0,36,129,151}, {0,25,53,128}, {0,87,102,163}, {0,5,37,40}, {0,45,95,155}, {0,9,26,88}.

$v = 55$

$\mathcal{A}$: {0,25,124,141}, {0,5,19,179}, {0,22,45,176}, {0,86,168,181}, {0,77,145,194}, {0,29,83,180}, {0,43,91,150}, {0,98,106,156}, {0,18,33,42}, {0,2,80,87}, {0,1,12,93}, {0,31,37,157}, {0,47,51,67}, {0,3,115,147}, {0,148,158,186}, {0,65,101,136}, {0,102,163,190}, {0,21,74,130}.

$v = 58$

$\mathcal{A}$: {0,23,87,97}, {0,29,53,54}, {0,114,146,176}, {0,60,121,197}, {0,16,63,191}, {0,2,9,93}, {0,48,142,161}, {0,33,99,110}, {0,12,43,79}, {0,108,112,130}, {0,59,147,193}, {0,5,13,83}, {0,28,55,215}, {0,44,117,167}, {0,69,89,194}, {0,75,81,96}, {0,3,106,198}, {0,150,192,218}, {0,49,100,180}.

$v = 67$

$\mathcal{A}$: {0,185,251,258}, {0,106,128,257}, {0,12,64,103}, {0,19,50,186}, {0,36,68,227}, {0,8,71,179}, {0,65,80,241}, {0,93,137,180}, {0,69,217,242}, {0,90,149,248}, {0,5,47,189}, {0,18,141,164}, {0,96,152,154}, {0,53,183,196}, {0,3,147,262}, {0,34,94,264}, {0,105,133,166}, {0,1,49,239}, {0,40,233,254}, {0,37,113,187}, {0,111,156,211}, {0,16,198,222}.

$v = 79$

$\mathcal{A}$: {0,80,153,258}, {0,52,69,199}, {0,41,286,302}, {0,113,181,306}, {0,1,61,161}, {0,145,194,219}, {0,53,87,283}, {0,3,95,143}, {0,64,284,304}, {0,168,177,289}, {0,84,246,311}, {0,137,279,287}, {0,40,90,278}, {0,4,51,118}, {0,149,180,215}, {0,11,223,314}, {0,22,141,222}, {0,46,72,280}, {0,184,272,295}, {0,99,127,309}, {0,6,109,260}, {0,77,187,262}, {0,133,218,257}, {0,42,157,301}, {0,19,43,233}, {0,146,164,209}.

$v = 82$

$\mathcal{A}$: {0,107,223,290}, {0,84,182,207}, {0,16,28,94}, {0,102,206,286}, {0,15,34,214}, {0,8,117,306}, {0,35,93,162}, {0,60,171,203}, {0,40,41,196}, {0,63,113,178}, {0,55,141,275}, {0,88,225,310}, {0,6,233,301}, {0,37,51,61}, {0,152,163,321}, {0,9,204,281}, {0,136,138,264}, {0,39,218,247}, {0,4,197,284}, {0,46,49,315}, {0,89,236,256}, {0,5,36,79}, {0,142,232,302}, {0,45,118,175}, {0,54,228,311},

$\{0,21,112,209\},\{0,23,75,252\}$.

$v=115$

$\mathcal{A}$: $\{0,218,274,383\},\{0,42,103,335\},\{0,91,181,221\},\{0,147,202,416\},\{0,14,29,146\},$
$\{0,203,213,298\},\{0,13,318,407\},\{0,52,98,250\},\{0,36,179,391\},\{0,192,254,337\},$
$\{0,136,243,244\},\{0,3,280,396\},\{0,300,333,384\},\{0,140,392,412\},\{0,49,80,289\},$
$\{0,284,342,385\},\{0,2,229,363\},\{0,113,286,372\},\{0,4,193,419\},\{0,260,266,397\},$
$\{0,153,316,436\},\{0,74,378,428\},\{0,8,339,449\},\{0,59,312,334\},\{0,27,119,309\},$
$\{0,219,276,388\},\{0,65,159,237\},\{0,285,364,435\},\{0,34,299,434\},\{0,12,93,425\},$
$\{0,17,158,373\},\{0,37,236,358\},\{0,5,278,442\},\{0,114,336,406\},\{0,197,204,225\},$
$\{0,157,290,306\},\{0,255,421,451\},\{0,100,138,211\}$. □

Lemma 3.3 *There exists a* $GS_4(2,4,v,4)$ *for* $v\in\{28,37,40,49,52,61,64,70,73,76,85,88,118,121,124,127\}$

Proof With suitable m and n, Lemma 2.3 can be used to obtain a $GS_4(2,4,mn+u,4)$ for $u=0,1$. This takes care of $v\in\{49,70,85,118,121,127\}$. For other v, we can write $v=4n+u$, where $u=0$ or 1 and $n+u\in T_g$. Take $m=4$, $t=4$ in Lemma 2.2, there exists a $GS_4(2,4,4n+u,4)$, the 4-GDD(4^4) comes from Lemma 2.4. This completes the proof. □

We are now in a position to prove Theorem 1.3.

Proof of Theorem 1.3 From Lemmas 3.1- 3.3, we need only to deal with $v\in[91,112]^1_3\cup[130,193]^1_3$.

For $v\in[91,112]^1_3$, apply Lemma 2.2 with $m=4$, $n=7$, $u=0$ and $t=4$ to obtain an $HGS_4(2,4,(28,7),4)$. Take $m=4$, $n=21$, $u=7$, $t=4$ in Lemma 2.1, to get a $GS_4(2,4,v,4)$, the input designs are from Lemma 3.1, Lemma 3.2 and Lemma 3.3.

For $v\in[130,160]^1_3$, take $m=4$, $n=10$, $u=0$ and $t=4$ in Lemma 2.2, one gets an $HGS_4(2,4,(40,10),4)$. Applying Lemma 2.1 with $m=4$, $n=30$, $u=10$, $t=4$, we obtain a $GS_4(2,4,v,4)$ since there exists a $GS_4(2,4,f,4)$ from Lemma 3.2 and Lemma 3.3, where $f=10+3a$ and $0\le a\le 10$.

Finally, for $v\in[163,193]^1_3$, take $m=4$, $n=12$, $u=1$ and $t=4$ in Lemma 2.2 to obtain an $HGS_4(2,4,(49,13),4)$. Applying Lemma 2.1 with $m=4$, $n=36$, $u=13$, $t=4$ and $2\le a\le 12$, we get a $GS_4(2,4,v,4)$, the input designs are from Lemma 3.2 and Lemma 3.3. This completes the proof. □

Acknowledgement The authors would like to thank Prof. Zhu Lie for his helpful discussions.

References

[1] T. Beth, D. Jungnickel and H. Lenz, *Design Theory*, Cambridge University Press, London, 1986.

[2] S. Blake-Wilson and K. Phelps, *Constant weight codes and group divisible design*, Designs, Codes and Cryptography, **16** (1999), 11-27.

[3] K. Chen, G. Ge and L. Zhu, *Generalized Steiner triple systems with group size five*, J. Combin. Designs, **7** (1999), 441-452.

[4] K. Chen, G. Ge and L. Zhu, *Starters and related codes*, J. Statist. Plan. Infer., **86** (2000), 379-395.

[5] C.J. Colbourn and J.H. Dinitz (eds.), *The CRC Handbook of Combinatorial Designs*, CRC Press, Boca Raton, 1996.

[6] T. Etzion, *Optimal constant weight codes over Z_k and generalized designs*, Discrete Math., **169** (1997), 55-82.

[7] H. Hanani, *On some tactical configurations*, Canad. J. Math., **15** (1963), 702-722.

[8] W.H. Mills, *On the covering of triples by quadruple*, In Proc. of the Fifth Southeastern Conference on Combinatorics Graph Theory and Algorithms, 1974, 573-581.

[9] K. Phelps and C. Yin, *Generalized Steiner systems with block three and group size four*, Ars Combin., **53** (1999), 133-146.

[10] K. Phelps and C. Yin, *Generalized Steiner systems with block three and group size* $g \equiv 3 \pmod 6$, J. Combin. Designs, **5** (1997), 417-432.

[11] D. Wu, G. Ge and L. Zhu, *Generalized Steiner triple systems with group size* $g = 7, 8$, Ars Combin., **57** (2000), 175-192.

[12] D. Wu, G. Ge and L. Zhu, *Generalized Steiner systems* $GS_4(2, 4, v, g)$ *for* $g = 2, 3, 6$, J. Combin. Designs, to appear.

[13] D. Wu and L. Zhu, *Generalized Steiner systems* $GS(2, 4, v, 2)$ *with* v *a prime power* $\equiv 7 \pmod{12}$, Designs, Codes and Cryptography, to appear.

[14] J. Yin, Y. Lu and J. Wang, *Maximum diatance holey packings and related codes*, Science in China (Series A), **42** (1999), 1262-1269.