Weighted Domination

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Abstract

A *weighted graph* (G, w) is a graph $G = (V, E)$ together with a positive weight-function on its vertices $w : V \to \mathbf{R}^{>0}$. The *weighted domination number* $\gamma_w(G)$ of (G, w) is the minimum weight $w(D) = \sum_{v \in D} w(v)$ of a vertex set $D \subseteq V$ with $N[D] = V$, i.e. a dominating set of G.

For this natural generalization of the well-known domination number we study some of the classical questions of domination theory. We characterize all extremal graphs for the simple Ore-like bound $\gamma_w(G) \leq \frac{1}{2}w(V)$ and prove Nordhaus-Gaddum-type inequalities for the weighted domination number.

Keywords: Domination; Weighted domination; Nordhaus-Gaddum-type results

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1 Introduction

All graphs will be simple, undirected and finite. The order $n(G)$ of graph $G = (V(G), E(G))$ with vertex set $V = V(G)$ and edge set $E = E(G)$ is $|V|$. The (closed) neighbourhood of a vertex $v \in V$ of G is denoted by $N(v) = N(v, G)$ ($N[v] = N[v, G]$). For a set $X \subseteq V$ let $N(X) = N(X, G) = \cup_{v \in X} N(v, G)$ and $N[X] = N[X, G] = \cup_{v \in X} N[v, G]$. The degree of a vertex $v \in V$ is denoted by $d(v) = d(v, G)$. The complement of a graph G is denoted by $\bar{G}$.

A graph $G = (V, E)$ together with a positive, real-valued weight-function $w : V \to \mathbf{R}^{>0}$ is called a *weighted graph* and is denoted by (G, w). If (G, w) is such that $\sum_{v \in V} w(v) = |V|$, then (G, w) is called a *normed weighted graph.*

Let (G, w) be a weighted graph. For a set $V' \subseteq V(G)$ or a subgraph H of G let $w(V') = \sum_{v \in V'} w(v)$ and $w(H) = w(V(H))$.

A set $D \subseteq V(G)$ is a *dominating set* of the graph G, if $N[D] = V(G)$. The *domination number* $\gamma(G)$ of a graph G is the minimum cardinality of a dominating of G. This parameter has received an ever-growing attention during the last three decades and a quite exhaustive overview of the related research can be found in the two books [5] and [6].

In this paper we will study a very natural generalization of the classical domination number for weighted graphs. Instead of considering a dominating set of small cardinality we will consider dominating sets of small weight and study some of the classical questions in domination theory for this new concept. The proofs of our results often display some surprising and interesting differences compared with the classical proofs. This sort of generalization already appeared several times in the construction of algorithms (see for instance Kratsch [8]) but has received little attention from a theoretical point of view.

Some totally different domination concepts also involving weights on the vertices of the underlying graphs have been studied under the names of *signed domination*, *minus domination* or *majority domination* etc.

Now we come to the precise definition. A dominating set D of the weighted graph (G, w) of minimum weight $w(D)$ is a *minimum weighted dominating set* and its weight is the *weighted domination number* $\gamma_w(G)$.

Let (G, w) be a weighted graph without isolated vertices. Since the complement of a minimum weighted dominating set of G is again a dominating set, we obtain

Observation 1.1 *If (G, w) is a weighted graph without isolated vertices, then $\gamma_w(G) \leq \frac{w(G)}{2}$.*

This generalizes Ore's classical bound $\gamma(G) \leq \frac{n}{2}$ for graphs G of order n without isolated vertices [9]. In several proofs of results for the classical domination number γ the existence of a minimum dominating set with so-called *exterior private neighbours* - which was observed by Bollobás and Cockayne [2] - is a valuable tool. Unfortunately, such a statement is no longer true for the weighted domination number as can be seen by the graph in Figure 1 whose unique minimum weighted dominating set is $\{u, v\}$.

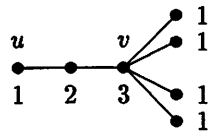

Figure 1.

As in the case of Ore's bound several results for γ easily generalize to γ_w.

Observation 1.2 *If* (G, w) *is a normed weighted graph of order* n *and* $\Delta_w = \max\left\{\frac{w(N(v))}{w(v)} | v \in V(G)\right\}$, *then*

$$\gamma_w(G) \geq \frac{n}{\Delta_w + 1}.$$

Proof: If D is a minimum weighted dominating set of G, then $V(G) = D \cup \bigcup_{v \in D} N(v)$ and $w(D) = \gamma_w(G)$. Since (G, w) is a normed weighted graph, it follows

$$\begin{aligned} n = w(V(G)) &\leq \sum_{v \in D} w(v) + \sum_{v \in D} w(N(v)) \\ &= \sum_{v \in D} w(v)\left(1 + \frac{w(N(v))}{w(v)}\right) \\ &\leq (1 + \Delta_w) \sum_{v \in D} w(v) \\ &= (1 + \Delta_w)\gamma_w(G). \ \square \end{aligned}$$

This generalizes the bound $\gamma(G) \geq \frac{n}{\Delta+1}$ for graphs of order n and maximum degree Δ.

We cite further examples without their simple proofs. If the weights of the vertices of a weighted graph are chosen uniformly at random from the interval $(0, 1)$ and normed afterwards to produce a normed weighted graph, then the expected cardinality of a dominating set is also its expected weight. This implies that

$$\gamma_w(G) \leq \frac{\ln(\delta+1)+1}{\delta+1}n$$

for a normed weighted graph (G, w) of order n with minimum degree δ which generalizes a probabilistic bound on γ in Alon and Spencer's book [1].

Similarly, proofs for theorems dealing with γ sometimes even imply results for γ_w. The proof by Jaeger and Payan [7] (see also [5], p. 238) of the bound $\gamma(G)\gamma(\bar{G}) \leq n$, for instance, immediately implies for a normed weighted graph (G, w) that $\gamma(G)\gamma_w(\bar{G}) \leq n$.

2 Connected graphs G with $\gamma_w(G) = \frac{w(G)}{2}$

In this section we will characterize the connected weighted graphs (G, w) for which $\gamma_w(G) = \frac{w(G)}{2}$. The alignment of our statements closely follows Fink, Jacobson, Kinch and Roberts [4]. Nevertheless, the proofs present some essential differences to the work in [4].

An *endvertex* is a vertex of degree at most 1. Just for convenience in this section let $N_{end}(v) = N_{end}(v, G)$ denote the set of endvertices in the neighbourhood of a vertex $v \in V(G)$.

A weighted tree (T, w) is said to *fulfill condition* $(*)$ if and only if either every non-endvertex v of T satisfies $w(v) = w(N_{end}(v))$ or $T = K_2$ and the two vertices of T have the same weight.

Theorem 2.1 *A weighted tree (T, w) of order at least 2 satisfies $\gamma_w(T) = \frac{w(T)}{2}$ if and only if (T, w) fulfills condition $(*)$.*

Proof: Since the statement is trivial for $T = K_2$, we assume that T is of order at least 3. If (T, w) fulfills $(*)$, then for every non-endvertex v the set $N_{end}(v)$ is not empty and a dominating set of T either contains v or all elements of $N_{end}(v)$. This implies that the set of non-endvertices is a minimum weighted dominating set with a total weight of $\frac{w(T)}{2}$.

Now we assume the existence of a weighted tree (T, w) with $\gamma_w(T) = \frac{w(T)}{2}$ such that (T, w) does not fulfill $(*)$. We assume that (T, w) has minimum order given this condition.

CASE 1: There is a non-endvertex v with $w(v) < w(N_{end}(v))$.

The components $T_1, ..., T_k$ of $T - [\{v\} \cup N_{end}(v)]$ all have order at least 2. Hence $\gamma_w(T_i) \leq \frac{w(T_i)}{2}$ for $1 \leq i \leq k$. Let D_i be a minimum weighted dominating set of T_i. Now $D = \{v\} \cup \bigcup_{i=1}^{k} D_i$ is a dominating set of T with

$$\begin{aligned} w(D) &= w(v) + \sum_{i=1}^{k} w(D_i) \\ &< \frac{1}{2}[w(v) + w(N_{end}(v))] + \sum_{i=1}^{k} \frac{w(T_i)}{2} = \frac{w(T)}{2} \end{aligned}$$

which is a contradiction.

CASE 2: There exists a non-endvertex v with $w(v) > w(N_{end}(v))$ and $N_{end}(v) \neq \emptyset$.

We obtain a similar contradiction as in CASE 1 considering the dominating set $N_{end}(v) \cup \bigcup_{i=1}^{k} D_i$.

CASE 3: There is a non-endvertex v with $N_{end}(v) = \emptyset$.

Let u be an arbitrary neighbour of v. Let $T'_1, ..., T'_l$ be the components that arise from T if we delete all edges incident with v except uv. (Note that all these components have order at least 2.)

If there is at least one index $1 \leq i \leq l$ such that $\gamma_w(T'_i) < \frac{w(T'_i)}{2}$, then the union of minimum weighted dominating sets of $T'_1, ..., T'_l$ is a dominating set D' of T with $w(D') < \frac{w(T)}{2}$, which is a contradiction.

Hence, $\gamma_w(T'_i) = \frac{w(T'_i)}{2}$ for all $1 \leq i \leq l$. By the choice of T, we deduce that T'_i fulfills $(*)$ for $1 \leq i \leq l$. Since v is now an endvertex adjacent to u, we conclude that $w(u) = w(v) + w(N_{end}(u))$. Therefore, $w(u) > w(N_{end}(u))$. If $N_{end}(u) \neq \emptyset$, then we obtain a contradiction as in CASE 2. Consequently, $N_{end}(u) = \emptyset$.

Let $vx_1x_2...x_ky$ be a shortest path in T from v to a vertex y which is a neighbour of an endvertex. If we repeatedly apply the above argument along this path, then we obtain that $N_{end}(y) = \emptyset$ which clearly contradicts the choice of y. This completes the proof. □

Lemma 2.2 *Let (G, w) be a connected weighted graph with $\gamma_w(G) = \frac{w(G)}{2}$ and let T be a spanning tree of G. Then every endvertex of T is an endvertex of G.*

Proof: Since the statement is trivial for $|V(G)| \leq 3$, we assume that $|V(G)| \geq 4$. Since for any spanning tree T' of G

$$\frac{w(T')}{2} \geq \gamma_w(T') \geq \gamma_w(G) = \frac{w(G)}{2} = \frac{w(T')}{2},$$

we have, by Theorem 2.1, that T' fulfills $(*)$. Now let T be a spanning tree of G. We assume that T has an endvertex u that is no endvertex of G. Let v be the neighbour of u in T.

CASE 1: u is adjacent to a non-endvertex v' of T.

The spanning tree $T' = T - vu + v'u$ of G does not fulfill (*), as

$$\begin{aligned} w(v') &= w(N_{end}(v',T)) < w(N_{end}(v',T')) \\ &= w(N_{end}(v',T)) + w(u). \end{aligned}$$

This is a contradiction.

CASE 2: u is adjacent to an endvertex $u' \in N_{end}(v,T)$.

The spanning tree $T' = T - vu + u'u$ of G does not fulfill (*), as $d(v,T') \geq 2$ and

$$\begin{aligned} w(v) &= w(N_{end}(v,T)) > w(N_{end}(v,T')) \\ &= w(N_{end}(v,T)) - w(u) - w(u'). \end{aligned}$$

This is a contradiction.

CASE 3: u is adjacent to an endvertex $u' \notin N_{end}(v,T)$.

Let v' be the unique neighbour of u' in T. Note that v' is no endvertex. The spanning tree $T' = T - vu + u'u$ of G does not fulfill (*), as

$$\begin{aligned} w(v') &= w(N_{end}(v',T)) \\ &= w(N_{end}(v',T')) + w(u') \\ &> w(N_{end}(v',T')). \end{aligned}$$

This is a contradiction and the proof is complete. □

A connected weighted graph (G, w) is said to *fulfill condition* (**) if and only if either

(i) $G = K_2$ and the two vertices of G have the same weight or

(ii) $G = C_4$ and the four vertices of G have the same weight or

(iii) every non-endvertex v of G satisfies $w(v) = w(N_{end}(v))$.

Note that (iii) implies that the weighted graph (G, w) has the following structure. Let $v_1, ..., v_k$ be the non-endvertices of G. Then for $1 \leq i \leq k$ we have $N_{end}(v_i) \neq \emptyset$ and $w(v_i) = w(N_{end}(v_i))$. Furthermore, $G[\{v_1, ..., v_k\}]$ is connected.

Theorem 2.3 *A connected weighted graph (G, w) satisfies $\gamma_w(G) = \frac{w(G)}{2}$ if and only if (G, w) fulfills condition (**).*

Proof: Since the statement is easily verified for $|V(G)| \leq 4$, we assume that $|V(G)| \geq 5$. Let T be any spanning tree of G. Theorem 2.1 and Lemma 2.2 imply, that T fulfills condition (*) and all endvertices of T are endvertices of G. This easily implies that G fulfills condition (**). □

Theorem 2.3 implies the following well-known result.

Corollary 2.4 (Payan, Xuong [10]; Fink, Jacobson, Kinch and Roberts [4]) *Let G be a connected graph of order n. Then $\gamma(G) = \frac{n}{2}$ if and only if $G = K_2$, $G = C_4$ or every non-endvertex is adjacent to exactly one endvertex.*

3 Nordhaus-Gaddum-type results

For a given graph-theoretical parameter ν a *Nordhaus-Gaddum-type result* is a bound on the sum or the product of $\nu(G)$ and $\nu(\bar{G})$ for a graph G. In this section we will prove some results of this type for the weighted domination number.

Lemma 3.1 *Let (G, w) be a normed weighted graph of order n. Then*

$$\gamma_w(G) + \gamma_w(\bar{G}) \leq n + \min\{w(v) | v \in V(G)\}.$$

Proof: Let $w(v_0) = \min\{w(v) | v \in V(G)\}$. Since $V(G) \setminus N(v_0)$ is a dominating set of G and $\{v_0\} \cup N(v_0)$ is a dominating set of $\bar{G}$, we have

$$\begin{aligned} \gamma_w(G) + \gamma_w(\bar{G}) &\leq w(V(G) \setminus N(v_0)) + w(\{v_0\} \cup N(v_0)) \\ &= w(V(G)) + w(v_0) = n + w(v_0) \end{aligned}$$

and the proof is complete. □

The bound in Lemma 3.1 is best possible as can be seen by a star $K_{1,n-1}$ in which the centre vertex has weight $\frac{n}{2}$ and all other vertices share the remaining weight of $\frac{n}{2}$. Clearly, for such a star $\gamma_w(K_{1,n-1}) = \frac{n}{2}$ and $\gamma_w(\bar{K}_{1,n-1}) = \frac{n}{2} + \min\{w(v) | v \in V(G)\}$. Note that by Observation 1.1, the bound in Lemma 3.1 is only interesting if either G or $\bar{G}$ has isolated vertices, since otherwise $\gamma_w(G) + \gamma_w(\bar{G}) \leq n$.

Theorem 3.2 *Let $k \geq 1$ and let (G, w) be a normed weighted graph of order $n \geq k + 2$ such that $\bar{G}$ has k isolated vertices. Then*

$$\gamma_w(G) \cdot \gamma_w(\bar{G}) \leq \frac{n^2}{(k+1)^2} \left[k + \frac{1}{n-k}\right]. \tag{1}$$

Equality holds in (1) if and only if every vertex in the set X of isolated vertices of $\bar{G}$ has weight $\frac{n}{k+1}$, every vertex in $V(G) \setminus X$ has weight $\frac{n}{(k+1)(n-k)}$ and $V(G) \setminus X$ is an independent set in G.

Proof: Clearly, if (G, w) has the described structure, then equality holds in (1). Hence it remains to prove (1) and the 'only if'-part of the statement.

Let $X = \{v_1, ..., v_k\}$, $w^* = \min\{w(v) | v \in V(G)\}$, and, in addition, $w^*_X = \min\{w(v_i) | 1 \le i \le k\}$. Since the sets $\{v_1\}$, $\{v_2\}$, ..., $\{v_k\}$ and $V(G) \setminus X$ are dominating sets of G, we have

$$\gamma_w(G) \le \min\left\{w^*_X, \frac{n}{k+1}\right\} \le \min\left\{\frac{w(X)}{k}, \frac{n}{k+1}\right\}. \qquad (2)$$

By Lemma 3.1 and as $w^* \le \frac{n-w(X)}{n-k}$, we have

$$\begin{aligned}\gamma_w(G) \cdot \gamma_w(\bar{G}) &\le \gamma_w(G)[n + w^* - \gamma_w(G)] \\ &\le \gamma_w(G)\left[n + \frac{n - w(X)}{n-k} - \gamma_w(G)\right].\end{aligned}$$

We consider two cases. In what follows we will use the fact that the function $g(x) = x(a - x)$ for $x \in [0, x_1]$ with $x_1 \le \frac{a}{2}$ assumes its unique maximum value for $x = x_1$.

CASE 1: $\frac{w(X)}{k} \ge \frac{n}{k+1}$

We have

$$\begin{aligned}\gamma_w(G) \cdot \gamma_w(\bar{G}) &\le \gamma_w(G)[n + w^* - \gamma_w(G)] \\ &\le \gamma_w(G)\left[n + \frac{n - w(X)}{n-k} - \gamma_w(G)\right] \\ &\le \frac{n}{k+1}\left[n + \frac{n - w(X)}{n-k} - \frac{n}{k+1}\right] \\ &\le \frac{n}{k+1}\left[n + \frac{n - \frac{kn}{k+1}}{n-k} - \frac{n}{k+1}\right] \\ &= \frac{n^2}{(k+1)^2}\left[k + \frac{1}{n-k}\right].\end{aligned}$$

If equality holds in the above inequality sequence, then

(i) $w^* = \frac{n - \frac{kn}{k+1}}{n-k} = \frac{n}{(k+1)(n-k)}$ and

(ii) $\gamma_w(G) = \frac{n}{k+1}$.

Property (ii) implies that all of the sets $\{v_1\}$, $\{v_2\}$, ..., $\{v_k\}$ and $V(G) \setminus X$ have the same weight $\frac{n}{k+1}$. Hence $w(v_i) = \frac{n}{k+1}$ for $1 \le i \le k$. Property (i) and $w(V(G) \setminus X) = \frac{n}{k+1}$ imply that all vertices in $V(G) \setminus X$ have weight $\frac{n}{(k+1)(n-k)}$. If $V(G) \setminus X$ contains an edge xy, then $V(G) \setminus (X \cup \{x\})$ is a dominating set of G of weight $\frac{n}{k+1} - \frac{n}{(k+1)(n-k)}$ which is a contradiction to property (ii). Hence $V(G) \setminus X$ is independent in G and the proof of CASE 1 is complete.

CASE 2: $\frac{w(X)}{k} < \frac{n}{k+1}$

We proceed as above. Note that $\frac{kn}{k+1} \leq \frac{kn-k^2+k}{2}$ for $n \geq k+2$.

$$\begin{aligned}
\gamma_w(G)\cdot\gamma_w(\bar{G}) &\leq \gamma_w(G)\left[n+\frac{n-w(X)}{n-k}-\gamma_w(G)\right]\\
&\leq \frac{w(X)}{k}\left[n+\frac{n-w(X)}{n-k}-\frac{w(X)}{k}\right]\\
&= \frac{n}{k^2(n-k)}w(X)[(kn-k^2+k)-w(X)]\\
&< \frac{n}{k^2(n-k)}\frac{kn}{k+1}\left[(kn-k^2+k)-\frac{kn}{k+1}\right]\\
&= \frac{n^2}{(k+1)^2}\left[k+\frac{1}{n-k}\right].
\end{aligned}$$

Since equality cannot occur in this case, the proof is complete. □

Corollary 3.3 *Let (G, w) be a normed weighted graph of order $n \geq 2$. Then*

$$\gamma_w(G)\cdot\gamma_w(\bar{G}) \leq \frac{n^2}{4}\frac{n}{n-1}. \tag{3}$$

Equality holds in (3) if and only if G is a star with centre vertex v_0, v_0 has weight $\frac{n}{2}$ and every vertex in $V(G)\setminus\{v_0\}$ has weight $\frac{n}{2(n-1)}$.

Proof: If neither G nor $\bar{G}$ have isolated vertices, then

$$\gamma_w(G)\cdot\gamma_w(\bar{G}) \leq \frac{n^2}{4} < \frac{n^2}{4}\frac{n}{n-1},$$

by Observation 1.1. Hence we assume, without loss of generality, that $\bar{G}$ has $k \geq 1$ isolated vertices. If $n-1 \leq k \leq n$, then $k = n$, i.e. $G = K_n$ and

$$\gamma_w(G)\cdot\gamma_w(\bar{G}) \leq 1\cdot n \leq \frac{n^2}{4}\frac{n}{n-1}$$

with equality if and only if $G = K_2$ and both vertices have weight 1. Thus, it remains that $1 \leq k \leq n-2$ and Theorem 3.2 applies. Since in this case the right side in (1) is maximum for $k = 1$ (note that $\frac{k}{(k+1)^2}$ is decreasing and $\frac{1}{4(n-1)} \geq \frac{1}{(k+1)^2(n-k)}$ for $1 \leq k \leq n-2$), the desired result follows from Theorem 3.2. □

Now we will establish a Nordhaus-Gaddum-type result for bipartite graphs.

Theorem 3.4 *Let (G, w) be a normed weighted bipartite graph of order n with no isolated vertex and partite sets A and B such that $|A|, |B| \geq k \geq 2$. Then*

$$\gamma_w(G) + \gamma_w(\bar{G}) \leq \frac{n}{2} + \frac{n}{2k} + \frac{n}{2(n-k)} \tag{4}$$

and

$$\gamma_w(G)\gamma_w(\bar{G}) \leq \frac{n^2}{4k}\left[1 + \frac{k}{n-k}\right]. \tag{5}$$

Proof: We will only prove (5), since the proof of (4) will then be immediate. We assume without loss of generality that $w(A) \geq w(B) = n - w(A)$. We have $k \leq |A|, |B| \leq n-k$. Since $V(G) \setminus A$ is a dominating set of G, we have $\gamma_w(G) \leq n - w(A)$. Since a set containing a vertex from A and a vertex from B is a dominating set of $\bar{G}$, we have

$$\gamma_w(\bar{G}) \leq \frac{w(A)}{|A|} + \frac{n - w(A)}{n - |A|}$$

and hence

$$\begin{aligned}
\gamma_w(G)\gamma_w(\bar{G}) &\leq (n - w(A))\left[\frac{w(A)}{|A|} + \frac{n - w(A)}{n - |A|}\right] \\
&\leq (n - w(A))\left[\frac{w(A)}{k} + \frac{n - w(A)}{n-k}\right] \\
&= -\frac{n-2k}{k(n-k)}w(A)^2 + \left[\frac{n(n-2k)}{k(n-k)} - \frac{n}{n-k}\right]w(A) \\
&+ \frac{n^2}{n-k}
\end{aligned}$$

The second inequality is equivalent to

$$w(A) \cdot n \cdot (|A| - k) \cdot (n - |A| - k) + (2w(A) - n) \cdot |A| \cdot k \cdot (|A| - k) \geq 0$$

which is true by the assumptions. If $n = 2k$, then the last expression in the above estimate of $\gamma_w(G)\gamma_w(\bar{G})$ is a decreasing linear function in $w(A)$ and if $n > 2k$, then it is a square function in $w(A)$ assuming its unique maximum value at $\frac{n}{2} - \frac{nk}{2(n-2k)}$. We have

$$w(A) \geq \frac{n}{2} \geq \frac{n}{2} - \frac{nk}{2(n-2k)}$$

and therefore

$$\gamma_w(G)\gamma_w(\bar{G}) \leq \left(n - \frac{n}{2}\right)\left[\frac{n}{2k} + \frac{n}{2(n-k)}\right] = \frac{n^2}{4k}\left[1 + \frac{k}{n-k}\right]$$

and the proof is complete. □

Corollary 3.5 *Let (G, w) be a normed weighted bipartite graph of order n such that neither G nor $\bar{G}$ has isolated vertices. Then*

$$\gamma_w(G) + \gamma_w(\bar{G}) \leq \frac{3n}{4} + \frac{n}{2(n-2)}$$

and

$$\gamma_w(G)\gamma_w(\bar{G}) \leq \frac{n^2}{8}\left[1 + \frac{2}{n-2}\right].$$

Proof: Each of the partite sets of G must have at least 2 elements. Since the right sides of (4) and (5) are maximum for $k = 2$, the desired bounds follow. □

Using Corollary 3.5, we have shown in [3] that the inequalities in Corollary 3.5 remain valid even for normed weighted triangle-free graphs.

The bounds in Theorem 3.4 and Corollary 3.5 are asymptotically best possible as can be seen by the following example.

For $n \geq 2k$ the normed weighted bipartite graph (G, w) has vertex set

$$V(G) = \{v_0, v_1, ..., v_{k-1}, u_1, u_2, ..., u_{n-k}\}$$

and edge set

$$E(G) = \{v_0u_i, v_iu_i | 1 \leq i \leq k-1\} \cup \{v_0u_i | k \leq i \leq n-k\}.$$

The weight of the vertices $v_0, v_1, ..., v_{k-1}, u_1, u_2, ..., u_{k-1}$ is $\frac{n}{2k}$ and the weight of the vertices $u_k, u_{k+1}, ..., u_{n-k}$ is $\frac{n}{2k[n-(2k-1)]}$. (See Figure 2 for the case $k = 2$.)

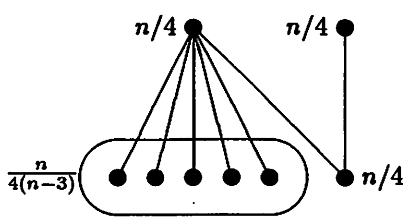

Figure 2.

It is easily verified that

$$\gamma_w(G) + \gamma_w(\bar{G}) = \frac{n}{2} + \frac{n}{2k} + \frac{n}{2k[n-(2k-1)]} \tag{6}$$

and

$$\gamma_w(G)\gamma_w(\bar{G}) = \frac{n^2}{4k}\left[1+\frac{1}{n-(2k-1)}\right]. \quad (7)$$

For $p \geq 3$ and normed weighted p-partite graphs G such that neither G nor $\bar{G}$ has isolated vertices it is not possible to obtain upper bounds on $\gamma_w(G)\gamma_w(\bar{G})$ and $\gamma_w(G)+\gamma_w(\bar{G})$ which are essentially better than $\frac{n^2}{4}$ and n, respectively. This can be seen by the following example.

For $n \geq 2p$ and $p \geq 3$ the normed weighted p-partite graphs G has vertex set

$$V(G) = \{u_{i,j} | 1 \leq i \leq p-1, j = 1,2\} \cup \{u_{p,1}, ..., u_{p,n-2(p-1)}\}.$$

The vertices in $V(G) \setminus \{u_{1,2}, u_{2,2}\}$ induce a complete p-partite graph with partite sets $\{u_{p,1}, ..., u_{p,n-2(p-1)}\}$, $\{u_{1,1}\}$, $\{u_{2,1}\}$, and $\{u_{i,1}, u_{i,2}\}$ for $3 \leq i \leq p-1$. Furthermore, $N(u_{1,2}) = \{u_{2,1}\}$ and $N(u_{2,2}) = \{u_{1,1}\}$. All vertices but $u_{1,1}$, $u_{1,2}$, $u_{2,1}$ and $u_{2,2}$ have a small weight $\epsilon > 0$ and the four remaining vertices have weight $\frac{n-\epsilon(n-4)}{4}$. It is easily verified that

$$\gamma_w(G)\gamma_w(\bar{G}) = \left(\frac{n-\epsilon(n-4)}{2}\right)^2 \sim \frac{n^2}{4}$$

and

$$\gamma_w(G)+\gamma_w(\bar{G}) = n-\epsilon(n-4) \sim n.$$

4 Conclusion

We have seen that the study of weighted domination allows at the same time to find natural generalizations of results for the classical domination and to discover some interesting new behaviours and features of domination. Therefore, we believe that further investigations are worthwhile and will lead to interesting results. In order to incite such work, we close with some conjectures.

Firstly, we believe that the bounds in Theorem 3.4 are not best possible and that they should be replaced by (6) and (7). We believe that we can prove the case $k = 2$ by a very clumsy and boring case analysis.

A second possible research problem would be to generalize the well-known Vizing bound (see [11])

$$|E(G)| \leq \frac{1}{2}(n(G)-\gamma(G))(n(G)-\gamma(G)+2) \quad (8)$$

for graphs G to the weighted case. Considering the proof of (8) in [5] it becomes apparent, that most of the discussion needed to prove (8) easily generalizes to weighted graphs.

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