On the Homologies of Multi-ary Relations

Horst Martini[1] and Petru Soltan[2]
[1]TU Chemnitz, Fakultät für Mathematik
D-09107 Chemnitz, GERMANY
[2]Faculty of Mathematics, Moldova State University
A. Mateevici str. 60, MD-2009 Kishinau,
REPUBLIC OF MOLDOVA

Abstract

We construct a complex K^n of m-ary relations, $1 \leq m \leq n+1$, in a finite set $X \neq \emptyset$, representing a model of an abstract cellular complex. For such a complex K^n we define the matrices of incidence and coincidence, the groups of homologies $\mathcal{H}_m(K^n)$ and cohomologies $\mathcal{H}^m(K^n)$ on the group of integers $\mathbf{Z}$, and the Euler characteristic. On a combinatorial basis we derive their main properties. In further publications we will derive more analogues of classical properties, and also applications with respect to the existence of fixed relations in the utilization of the isomorphisms will be investigated. In particular, we intend to complete the theory of hypergraphs with the help of such topological observations.

§ 1 The complex of multi-ary relations

Let $X = \{x_1, \cdots, x_r\}$ be a set of $r \geq 2$ elements, and let, furthermore, $X = X^1, X^2, \ldots, X^{n+1}, \ldots$ with $1 \leq n \leq r$ be a sequence of cartesian products (cf. [32]) of the set $X : X^{m+1} = X^m \cdot X$ with $m = 1, \ldots, n$. Any nonempty subset $R^m \subset X^m, m \geq 1$, is said to be an *m-ary relation* of elements from X (see [32] and [34]). The set $R^1 \subset X^1$ defines a subset of elements from X. The m-ary relation R^m consists of a family of sequences with m elements from X in a given order. For example, $(x_{i_1}, x_{i_2}, \ldots, x_{i_m})$ is a sequence from R^m in which the elements can be repeated. If there is no repetition in the sequence $(x_{i_1}, x_{i_2}, \ldots, x_{i_m})$, then any order of elements

JCMCC 45 (2003), pp. 219-243

from this sequence $x_{j_1}, x_{j_2}, \ldots, x_{j_l}$, $l < m$, which preserves the given order, is called *hereditary*.

Now we consider a finite subset of relations $R^1, R^2, \ldots, R^{n+1}$ of the infinite set mentioned above, and we require that this subset satisfies the following conditions:

I. $R^1 = X^1 = X$;

II. $R^{n+1} = \emptyset$;

III. If $R^m = \emptyset$, then any sequence $(x_{i_1}, x_{i_2}, \ldots, x_{i_m}) \in R^m$ contains the pairwise distinct elements of X, $m = 2, 3, \ldots, n+1$.

IV. Any subsequence $(x_{j_1}, x_{j_2}, \ldots, x_{j_l})$, $1 \leq l < m \leq n+1$, of the sequence $(x_{i_1}, x_{i_2}, \ldots, x_{i_m})$, which represents a hereditary sequence, is contained in R^l.

Definition 1. A family of relations $R^1, R^2, \ldots, R^{n+1}$ which satisfies the conditions I – IV is called a *finite complex of multi-ary relations* and denoted by $\mathcal{R}^{n+1} = (R^1, R^2, \ldots, R^{n+1})$. □

Moreover, since we will not investigate the infinite family of multi-ary relations, we can simply speak about the *complex of relations* $\mathcal{R}^{n+1}$.

According to the conditions I, II, IV it turns out that any set R^m of the complex of relations $\mathcal{R}^{n+1}$ is nonempty.

Considering the complex of relations $\mathcal{R}^2 = (R^1, R^2)$, it is obvious that this complex represents an oriented graph, cf. [7], [12], and [60]. Therefore it is natural to call the complex of relations $\mathcal{R}^{n+1}$ also an *oriented hyperpgraph*. This notion is frequently used in the literature, but it represents a structure distinct from the known notion of a hypergraph (see [5], [6], [40], [44], [57], and [60]). Below we will describe a procedure showing a possibility of obtaining the notion of a hypergraph from the notion of an oriented hypergraph.

Definition 2. Let there be given two complexes of relations $\mathcal{R}^{m+1} = (R_1^1, R_1^2, \ldots, R_1^{m+1})$ and $\mathcal{R}^{n+1} = (R^1, R^2, \ldots, R^{m+1})$ with $m \leq n$. If for any integer l, $1 \leq l \leq m+1$, $R_1^l \subset R^l$ holds, then $\mathcal{R}^{m+1}$ is said to be a *subcomplex* of the complex $\mathcal{R}^{n+1}$ ($\mathcal{R}^{m+1} \subset \mathcal{R}^{n+1}$). In the case that $R_1^l = R^l$, $1 \leq l \leq m+1$, the complex $\mathcal{R}^{m+1}$ is called the *skeleton* of degree $m+1$ of the complex $\mathcal{R}^{n+1}$. □

Clearly, the skeleton of degree 2 of the complex of relations $\mathcal{R}^{n+1}, n > 1$, is an oriented graph. Indeed, this complex can be represented as a pair $G = (X, R^2)$, i.e., as a set X of points (vertices) with the binary relation R^2 which does not contain the elements (x, x) (usually called loops), cf. [5], [7], and [40].

Definition 3. The complex of relations $\mathcal{R}^{n+1} = (R^1, R^2, \ldots, R^{n+1})$ is called *connected* if for any two elements $x_i, x_j \in R^1$ there exists a set $x_i = x_{t_1}, x_{t_2}, \ldots, x_{t_s} = x_j$ of elements of R^1 such that one of the pairs $(x_{t_r}, x_{t_{r+1}}), (x_{t_{r+1}}, x_{t_r})$ belongs to R^2, $r = 1, 2, \ldots, s-1$. The set of pairs of elements $x_i = x_{t_1}, x_{t_2}, \ldots, x_{t_s} = x_j$ is called a *chain of dimension 1* joining the vertices x_i and x_j, which themselves are said to be the *extremities* of this chain. Denote the chain of dimension 1 with extremities x_i, x_j by $L'(x_i, x_j)$. (Later on, $L'(x_i, x_j)$ will be represented also in an algebraic form.) □

Definition 4. Let there be given two complexes of relations $\mathcal{R}^{n_1+1} = (R_1^1, R_1^2, \ldots, R_1^{n_1+1})$ and $\mathcal{R}^{n_2+1} = (R_2^1, R_2^2, \ldots, R^{n_2+1})$. The *union* of the complexes $\mathcal{R}^{n_1+1}$, $\mathcal{R}^{n_2+1}$ is the complex $\mathcal{R}^{n+1} = \mathcal{R}^{n_1+1} \cup \mathcal{R}^{n_2+1} = (R_1^1 \cup R_2^1, R_1^2 \cup R_2^2, \ldots)$, where $n = \max\{n_1, n_2\}$. The *intersection* of the complexes $\mathcal{R}^{n_1+1}, \mathcal{R}^{n_2+1}$ is the complex $\mathcal{R}^{m+1} = \mathcal{R}^{n_1+1} \cap \mathcal{R}^{n_2+1}$, where $n = \min\{n_1, n_2\}$. □

The following assertion is a generalization of a theorem formulated in [27] and [28], and it refers to the majority of theorems and corollaries given below.

Theorem 1. *The complex $\mathcal{R}^{n+1}$ is connected if and only if there are no two complexes $\mathcal{R}^{n_1+1}, \mathcal{R}^{n_2+1}$ satisfying the equality $\mathcal{R}^{n+1} = \mathcal{R}^{n_1+1} \cup \mathcal{R}^{n_2+2}$, where $\mathcal{R}^{n_1+1} \cap \mathcal{R}^{n_2+1} \neq \emptyset$.*

Proof.

1. Let $\mathcal{R}^{n+1} = (R^1, R_2, \ldots, R^{n+1})$ be a connected complex of relations, and x be an arbitrary vertex of it. Consider the set of all vertices $x' \in \mathcal{R}^{n+1}$ for which there exists a chain $L'(x, x')$, and denote this set of vertices by R'_x. It is obvious that if an element x_{i_0} of a sequence $(x_{i_0}, \ldots, x_{i_1}, \ldots, x_{i_m}) \in R^{m+1}$ belongs to R'_x, then any other element x_{i_k}, $1 \leq k \leq m$, belongs to R'_x (condition IV). So the complex $\mathcal{R}^{n_1+1} = (R_1^1, R_1^2, \ldots, R_1^{n_1+1})$, for which any sequence $(x'_{i_0}, x'_{i_1}, \ldots, x'_{i_m}) \in \mathcal{R}_1^{m_1+1}$, $1 \leq m_1 \leq n_1$, contains an element x' ($x' = x_{i_0}$, say) such that there exists the chain $L'(x, x')$, forms a subcomplex of $\mathcal{R}^{n+1}$ ($\mathcal{R}^{n_1+1} \subset \mathcal{R}^{n+1}$).

Now let $\mathcal{R}^{n_2+1} = (R_2^1, R_2^2, \dots, R_2^{n_2+1})$, where $R_2^{m_2} = R^m \backslash R^{m_1}$, $m_2 = 1, 2, \dots, n_2$. It is obvious that the following equalities hold: $R_2^1 = \emptyset$, $R_2^2 = \emptyset, \dots, R_2^{m_2+1} = \emptyset$; otherwise we obtain a construction with the connection of $\mathcal{R}^{n+1}$. Since x is an arbitrary vertex of the subcomplexes $\mathcal{R}^{n_1+1}$ and $\mathcal{R}^{n_2+1}$ with the property $\mathcal{R}^{n_1+1} = \mathcal{R}^{n_1+1} \cup \mathcal{R}^{n_2+1}$, the intersection $\mathcal{R}^{n_1+1} \cap \mathcal{R}^{n_2+1} \neq \emptyset$ does not exist.

2. Assume the contrary, i.e., let $\mathcal{R}^{n+1}$ be a non-connected complex. In the same way as above one can construct the complexes $\mathcal{R}^{n_1+1} = (R_1^1, R_1^2, \dots, R_1^{n_1+1})$ and $\mathcal{R}^{n_2+1} = (R_2^1, R_2^2, \dots, R_2^{n_2+1})$, where $R_2^1 = \emptyset, R_2^2 = \emptyset, \dots, R_2^{n_2+1} = \emptyset$ and for which $\mathcal{R}^{n+1} = \mathcal{R}^{n_1+1} \cup \mathcal{R}^{n_2+1}$ and $\mathcal{R}^{n_1+1} \cap \mathcal{R}^{n_2+1} \neq \emptyset$, a contradiction. ■

Now let $\overline{x}$ be a fixed element of X. Consider the subset $\overline{X} \subset X$ of all elements $x' \in X$ (including $\overline{x}$) for which there exists a chain $L'(\overline{x}, x')$. Let $\overline{X}^1 = \overline{X}, \overline{X}^2, \dots, \overline{X}_m, \dots$ be the sequence of natural powers of $\overline{X}$ and consider the intersections $\overline{R}^m := \overline{X}^m \cap R^m$, $m = 1, 2, \dots, n+1$. Let n_1 be the maximal index such that $R^{n_1+1} \neq \emptyset$.

Definition 5. The subcomplex of relations $\mathcal{R}_{\overline{x}}^{n_1+1} = (\overline{R}^1, \overline{R}^2, \dots, \overline{R}^{n_1+1})$ is called a *connected component* of the complex of relations $\mathcal{R}^{n+1}$. □

It is obvious that this complex does not depend on $\overline{x}$. Therefore, in the notation of the conneced component, the index $\overline{x}$ will be omitted, i.e., we simply write $\mathcal{R}^{n_1+1}$.

Theorem 2. *Let $\mathcal{R}^{n_1+1}, \mathcal{R}^{n_2+1}, \dots, \mathcal{R}^{n_q+1}$ be a family of all connected and pairwise distinct components of the complex of relation $\mathcal{R}^{n+1}$. Then the following equality holds:*

$$\mathcal{R}^{n+1} = \mathcal{R}^{n_1+1} \cup \mathcal{R}^{n_2+1} \cup \dots \cup \mathcal{R}^{n_q+1}, \tag{1}$$

where $\mathcal{R}^{n_i+1} \cap \mathcal{R}^{n_j+1} = \emptyset$ for $i \neq j$ and $i, j = 1, 2, \dots, q$.

Proof. The proof of this statement is quite easy. First, determine the inclusions $\mathcal{R}^{n+1} \subset \mathcal{R}^{n_1+1} \cup \mathcal{R}^{n_2+1} \cup \dots \cup \mathcal{R}^{n_q+1}$ and $\mathcal{R}^{n_1+1} \cap \mathcal{R}^{n_2+1} \cap \dots \cap \mathcal{R}^{n_q+1} \subset \mathcal{R}^{n+1}$, whence equality (1) is obtained. Second, verify the equality $\mathcal{R}^{n_i+1} \cap \mathcal{R}^{n_j+1} = \emptyset$ for $i \neq j$ and $i, j = 1, 2, \dots, q$. And third, by using condition II one gets $n = \max\{n_1, n_2, \dots, n_q\}$. ■

Definition 6. The complex of relations $\mathcal{R}^{n+1} = (R^1, R^2, \dots, R^{n+1})$ is said to be *locally complete* (see [44]) if for any $m \in \{1, 2, \dots, n\}$ and any sequence $(x_{i_0}, x_{i_1}, \dots, x_{i_m}) \in R^{m+1}$ the relation R^{m+1} contains all sequences obtained after all permutations of indices $i_0, i_1, \dots, i_m$. □

The following two *examples of locally complete complexes* might be instructive.

1. The symmetric graph $\mathcal{R}^2 = (R^1, R^2)$ without loops, where R^2 is a binary and symmetric relation, is a locally complete complex, cf [4], [40], and [45].
2. According to condition III, the complex of relations $\mathcal{R}^{n+1} = (X_0^1 = X^1, X_0^2, \ldots, X_0^{n+1})$, where X_0^{m+1} represents the set of all sequences of X^{m+1} without repetitions of elements from X, is a locally complete complex.

Following the purpose announced in the title of this paper (and by analogy to what is known from the classical literature about combinatorial and algebraic topology) the conditons I – IV suggest to use new notions and notation, keeping the equivalence of exposition of the complex relations and the properties we are interested in. Having this in mind, we give the following definitions.

Definition 7. Any sequence $(x_{i_0}, \ldots, c_{i_1}, \ldots, x_{i_m}) \in R^{m+1}$ is said to be an *abstract simplex of dimension* m and denoted by

$$S_i^m = (x_{i_0}, x_{i_1}, \ldots, x_{i_m}) \in R^{m+1}, \ \ m = \dim S_i^m .$$

Any hereditary sequence $(x_{j_0}, x_{j_1}, \ldots, x_{j_l}) \in R^{l+1}$ which can be considered as a subsequence of the given sequence S_i^m is called a *face of dimension* l of the simplex S_i^m and denoted by $S_j^l = (x_{j_0}, x_{j_1}, \ldots, x_{j_l})$, $S_j^l \subset S_i^m$. A face of dimension 0 is also called a *vertex* of S_i^m, $0 \leq m \leq n$. □

Thus the number of distinct abstract simplices of dimension m, that can be constructed on the set of $m+1$ pairwise different elements from X, is determined by the number of permutations of these elements of a simplex. Hence one can imagine the distinct abstract simplices of dimension $m+1$, which are strained on the elements $x_1, x_2, \ldots, x_{m+1}$ from X, as membranes.

So we can represent the complex of relations $\mathcal{R}^{n+1} = (R^1, R^2, \ldots, R^{n+1})$ in the following way: $\mathbb{S}^0 = R^1, \mathbb{S}^1 = R^2, \ldots, \mathbb{S}^n = R^{n+1}$ and $(\mathbb{S}^0, \mathbb{S}^1, \ldots, \mathbb{S}^n) = K^n$, keeping for K^n the name "complex", where $n = \dim K^n$ is the dimension of K^n. The following statement is obvious.

Theorem 3. *The complex of relations K^n is not a simplicial abstract complex since many simplices can be strained on the same set of vertices.*

Definition 8. Let $S_i^m \in \mathbb{S}^m$ be an abstract simplex of dimension m, $i = 1, 2, \ldots, \alpha_m$, and st S_i^m be the set of all simplices of dimension $m+1$ from $\mathbb{S}^m$ having S_i^m as common face. This set st S_i^m is called the *star* of the simplex S_i^m, $m = 1, 2, \ldots, n$ (cf. [16] and [27]). □

Depending on what is needed, the complex of relations K^n will be represented in one of the equivalent forms: $K^n = (R^1, R^2, \ldots, R^{n+1})$, $K^n = (\mathbb{S}^0, \mathbb{S}^1, \ldots, \mathbb{S}^n)$.

Further on, it is convenient to represent the family of simplices $\mathbb{S}^0, \mathbb{S}^1, \ldots, \mathbb{S}^n$ in the following way:

$$\begin{array}{rcl} \mathbb{S}^0 & = & \{S_1^0, S_2^0, \ldots, S_{\alpha_0}^0\}, \\ \mathbb{S}^1 & = & \{S_1^1, S_2^1, \ldots, S_{\alpha_1}^1\}, \\ & & \ldots \\ \mathbb{S}^m & = & \{S_1^m, S_2^m, \ldots, S_{\alpha_m}^m\}, \\ & & \ldots \\ \mathbb{S}^n & = & \{S_1^n, S_2^n, \ldots, S_{\alpha_n}^n\}, \end{array}$$

where $\alpha_0, \alpha_1, \ldots, \alpha_n$ are the respective cardinalities of these families.

Definition 9 ([1], [2], [4], [25]). Given the complex of relations $K^n = (\mathbb{S}^0, \mathbb{S}^1, \ldots, \mathbb{S}^n)$, the function of integer values

$$\mathcal{X}(K^n) = \sum_{i=0}^{n} (-1)^i \alpha_i$$

is called the *Euler characteristic* of this complex. □

The following corollaries are obvious.

Corollary 1. *If K^{n_1} and K^{n_2} are two subcomplexes of the complex of relations K^n, then*

$$\mathcal{X}(K^{n_1} \cup K^{n_2}) = \mathcal{X}(K^{n_1}) + \mathcal{X}(K^{n_2}) - \mathcal{X}(K^{n_1} \cap K^{n_2}).$$

Corollary 2. *If $K^{n_1}, K^{n_2}, \ldots, K^{n_q}$ represent all connected components of K^n, then*

$$\mathcal{X}(K^n) = \sum_{i=0}^{q} \mathcal{X}(K^i). \tag{2}$$

§ 2 The orientation of simplices and the incidence matrices

Let $K^n = (\mathbb{S}^0, \mathbb{S}^1, \ldots, \mathbb{S}^n)$ be a complex of relations, and $S_i^m \subset \mathbb{S}^m$, $i = 1, 2, \ldots, \alpha_m$, be an arbitrary simplex represented by the respective sequence $S_i^m = (x_{i_0}, x_{i_1}, \ldots, x_{i_m})$, $m = 0, 1, \ldots, n$. For the sequence of indices $(i_0, i_1, \ldots, i_m)$, one might consider the number of transpositions (cf. [32]) of the sequence $(0, 1, \ldots, m)$ and denote this number by $t(i_0, i_1, \ldots, i_m)$.

Definition 10. If for $S_i^m \in \mathbb{S}^m$ the number $t(i_0, i_1, \ldots, i_m)$ is even, then the simplex S_i^m is said to be *positively oriented* and denoted by $+S_i^m$. Otherwise, if this number is odd, then the simplex S_i^m is *negatively oriented* and denoted by $-S_i^m$. □

Now let $S_j^{m-1} \in \mathbb{S}^{m-1}$ be a simplex of dimension $m-1$ such that $S_j^{m-1} \subset S_i^m$, i.e., S_j^{m-1} is a face of S_i^m. Assume that the simplices S_i^m and S_j^{m-1} have the representations

$$
\begin{aligned}
S_i^m &= (x_{i_0}, x_{i_1}, \ldots, x_{i_{k-1}}, x_{i_k}, x_{i_{k+1}}, \ldots, x_m), \\
S_j^{m-1} &= (x_{i_0}, x_{i_1}, \ldots, x_{i_{k-1}}, \hat{x}_{i_k}, x_{i_{k+1}}, \ldots, x_m),
\end{aligned}
$$

where the symbol ˆ signifies the absence of x_{i_k}.

Under these conditions the simplices S_i^m and $(-1)^k S_j^{m-1}$ are called *coherent*, and S_i^m and $(-1)^{k+1} S_j^{m-1}$ are said to be *non-coherent*.

We will not extend the notions of coherence and non-coherence to simplices whose difference of dimensions is larger than 1.

Definition 11 ([2], [12], [25], [48]). For the pairs of arbitrary simplices $S_j^m \in \mathbb{S}^m, S_j^{m-1} \in \mathbb{S}^m$ we will introduce the following symbols:

1. $[S_j^m : S_i^{m-1}] = \varepsilon_j^i(m, \Delta)$ is called the *coefficient of* Δ*-incidence* of S_j^m and S_i^{m-1} in the given order, where

$$
\varepsilon_j^i(m, \Delta) = \begin{cases} +1 & \text{if} \quad S_j^m \text{ and } S_i^{m-1} \text{ are coherent ,} \\ -1 & \text{if} \quad S_j^m \text{ and } S_i^{m-1} \text{ are non-coherent ,} \\ \ \ 0 & \text{if} \quad S_i^{m-1} \not\subset S_j^m \text{ are coherent ,} \end{cases}
$$

with $m = 1, 2, \ldots, n$ and $i = 1, 2, \ldots, \alpha_{m-1}$ as well as $j = 1, 2, \ldots, \alpha_m$.

2. $[S_j^m : S_l^{m+1}] = \varepsilon_j^l(m,\nabla)$ is said to be the *coefficient of* ∇*-incidence* of the simplices S_j^m and S_l^{m+1} in the given order, where

$$\varepsilon_j^i(m,\nabla) = \begin{cases} +1 & \text{if} \quad S_j^m \text{ and } S_l^{m+1} \text{ are coherent }, \\ -1 & \text{if} \quad S_j^m \text{ and } S_l^{m+1} \text{ are non-coherent }, \\ 0 & \text{if} \quad S_j^m \not\subset S_l^{m+1} \text{ are coherent }, \end{cases}$$

with $m = 0,1,\ldots,n-1$, $j = 1,2,\ldots,\alpha_m$ and $l = 1,2,\ldots,\alpha_{m+1}$.
□

Corollary 3. *The following equalities hold:*

$$\begin{array}{lll} \varepsilon_j^i(m,\Delta) = \varepsilon_i^j(m-1,\nabla), & \textit{where} & m = 1,2,\ldots,n, \\ \varepsilon_j^l(m,\nabla) = \varepsilon_l^j(m+1,\Delta), & \textit{where} & m = 0,1,\ldots,n-1 \end{array}$$

and $m = 0,1,\ldots,\alpha_{m-1}$, $j = 1,2,\ldots,\alpha_m$ *and* $l = 1,2,\ldots,\alpha_{m+1}$.

Remark 1. The symbols Δ and ∇, taken from [14] and [28], are suitable for the exposition here.

Remark 2. The coefficient of incidence $[S_j^m : S_k^{m-1}]$ for the simplices $S_j^m = (x_{j_0}, x_{j_1}, \ldots, x_{j_m})$ and $S_k^{m-1} = (x_{k_0}, x_{k_1}, \ldots, x_{k_{m-1}})$, where the sequence $x_{k_0}, x_{k_1}, \ldots, x_{k_{m+1}}$ consists of the elements of S_j^m but is not a hereditary sequence of the sequence $(x_{j_0}, x_{j_1}, \ldots, x_{j_m})$, equals $0 : S_k^{m-1}$, where the latter is a face of the simplex $S_k^m = (x_{k_0}, x_{k_1}, \ldots, x_{k_m})$.

Definition 12. Given the complex of relations $K^n = \{\mathbb{S}^0, \mathbb{S}^1, \ldots, \mathbb{S}^n\}$. The matrices

1. $I^m(\Delta) := (\varepsilon_j^i(m,\Delta))_{\alpha_m \cdot \alpha_{m-1}}, m = 1,2,\ldots,n$, where i and j indicate the order of respective lines and columns with $i = 1,2,\ldots,\alpha_{m-1}$ and $j = 1,2,\ldots,\alpha_m$, as well as
2. $I^m(\nabla) := (\varepsilon_j^l(m,\Delta))_{\alpha_m \cdot \alpha_{m+1}}, m = 0,1,\ldots,n-1$, where l and j indicate the order of respective lines and columns with $l = 1,2,\ldots,\alpha_m$ and $j = 1,2,\ldots,\alpha_{m+1}$

are called *matrix of* Δ*-incidence of dimension* m and *matrix of* ∇*-incidence of dimension* m of K^n, respectively (see [25] and [48]). □

This definition yields

Corollary 4. *For the complex of relations K^n the pairs of matrices $I^m(\Delta)$, $m = 1, 2, \dots, n$ and $I^m(\nabla)$, $m = 0, 1, \dots, n-1$, are described in the transposed form by $(I^m(\Delta))^* = I^{m-1}(\nabla)$ and $(I^m(\nabla))^* = I^{m+1}(\Delta)$, respectively.*

§ 3 The homologies of the complex of relations

Given the complex of relations $K^n = (\mathbb{S}^0, \mathbb{S}^1, \dots, \mathbb{S}^n)$ and, e.g., the additive group $\mathbf{Z}$ of integers.

Definition 13. Consider an arbitrary simplex $S_j^m \in \mathbb{S}^m$ and the following sums:

$$\begin{aligned} \Delta S_j^m = \; & \varepsilon_j^1(m, \Delta) S_1^{m-1} + \varepsilon_j^2(m, \Delta) S_2^{m-1} + \dots \\ & + \varepsilon_j^{\alpha_{m-1}}(m, \Delta) S_{\alpha_{m-1}}^{m-1}, \end{aligned} \tag{3}$$

where $m = 1, 2, \dots, n$ and $j = 1, 2, \dots, \alpha_m$, as well as

$$\begin{aligned} \nabla S_j^m = \; & \varepsilon_1^j(m, \nabla) S_1^{m+1} + \varepsilon_2^j(m, \Delta) S_2^{m+1} + \dots \\ & + \varepsilon_{\alpha_{m+1}}^j(m, \nabla) S_{\alpha_{m+1}}^{m+1}, \end{aligned} \tag{4}$$

where $m = 0, 1, \dots, n-1$ and $j = 1, 2, \dots, \alpha_m$. These sums are called the *Δ-border (border) and ∇-border (co-border)* of the simplex S_j^m and denoted by ΔS_j^m and ∇S_j^m, respectively. □

For all $S_i^0 \in \mathbb{S}^0$, $i = 0, 1, \dots, \alpha_0$, and all $S_j^n \in \mathbb{S}^n$, $j = 1, 2, \dots, \alpha_n$, let us consider $\Delta S_i^0 = 0$ and $\nabla S_j^n = 0$, respectively. The formulas (3) and (4) can be more simplified. For example, let S_j^m be represented by the corresponding indices, i.e., $S_j^m = (j_0, j_1, \dots, j_k, \dots, j_m)$, and its faces be represented by $S_0^{m-1}, S_1^{m-1}, \dots, S_m^{m-1}$, where S_k^{m-1} is the face of S_j^m being opposite to the vertex j_k. Then, according to the definition of coherence of simplices S_j^m and $(-1)^k S_k^{m-1}$, i.e., $\varepsilon_j^k(m, \Delta) = (-1)^k$, formula (3) can be written as follows:

$$\begin{aligned} \Delta S_j^m = \; & (-1)^0 S_0^{m-1} + (-1)^1 S_1^{m-1} + \dots + (-1)^k S_k^{m-1} + \dots \\ & + (-1)^m S_m^{m-1}. \end{aligned} \tag{3'}$$

Further coefficients of Δ-incidence (which do not occur in (3′)) are, by definition, equal to zero.

In the same way (4) can be simplified. Indeed, let $S_{l_1}^{m+1}, S_{l_2}^{m+1}, \dots, S_{l_{j(m)}}^{m+1}$ be a set of simplices of st S_j^m; then we obtain

$$\nabla S_j^m = \varepsilon_{l_1}^j(m,\nabla)S_{l_1}^{m+1} + \varepsilon_{l_2}^j(m,\nabla)S_{l_2}^{m+1} + \dots + \varepsilon_{l_{j(m)}}^j(m,\nabla)S_{l_{j(m)}}^{m+1}. \tag{4'}$$

Again, by definition, the coefficients not occuring in (4′) are equal to zero.

The advantage of the fomulas (3) and (4) will be verified below.

Now let $f : \mathbb{S}^m \longrightarrow \mathbf{Z}$ be a ∇-mapping with the following property: if $S^m \in \mathbb{S}^m$ is a negatively oriented simplex (i.e., it can be written as $-S^m$), then $f(-S^m) = -f(S^m)$. Given $\mathbb{S}^m = \{S_1^m, S_2^m, \dots, S_\alpha^m\}$, we have $f(S_i^m) = g_i$, where $g_i \in \mathbf{Z}$, and for simplicity we will write $g_i S_i^m$ with $m = 0, 1, \dots, n$ and $i = 1, 2, \dots, \alpha_m$ (cf. [25] and [48]).

Definition 14. Given the family $\mathbb{S}^m = \{S_1^m, S_2^m, \dots, S_\alpha^m\}$, $m = 0, 1, \dots, n$. The sum

$$g_1 S_1^m + g_2 S_2^m + \dots + g_{\alpha_m} S_{\alpha_m}^m$$

is called the *m-dimensional chain* of the comlex of relations K^n and denoted by L^m. □

Let $L_1^m = g_1^1 S_1^m + g_2^1 S_2^m + \dots + g_{\alpha_m}^1 S_{\alpha_m}^m$, $L_2^m = g_1^2 S_1^m + g_2^2 S_2^m + \dots + g_{\alpha_m}^2 S_{\alpha_m}^m$ be two m-dimensional chains. For example, the chain $L^1(x_i, x_j)$ from Definition 3 can be written as $L^1(\Delta) = g_{t_1} S_{t_1}^1 + g_{t_2} S_{t_2}^1 + \dots + g_{t_s} S_{t_s}^1$, where $g_{t_1}, g_{t_2}, \dots, g_{t_s}$ are equal to $+1$ or -1, depending on the fact whether or not the directions of the chain and the simplices of dimension 1 coincide.

Definition 15. By

$$L_1^m + L_2^m = (g_1^1 + g_1^2)S_1^m + (g_2^1 + g_2^2)S_2^m + \dots + (g_{\alpha_m}^1 + g_{\alpha_m}^2)S_{\alpha_m}^m \tag{5}$$

we describe the *sum of the chains* L_1^m and L_2^m. □

Theorem 4. *With respect to the operation defined by (5), the set $\mathcal{L}^m$ of all m-dimensional chains of the complex of relations K^n forms a commutative group.*

By (5) the proof of this statement is obvious.

Definition 16. Let $L^m \in \mathcal{L}^m$ be an arbitrary chain, $m = 0, 1, \ldots, n$. Then the equality

$$\Delta L^m = g_1 \Delta S_1^m + g_2 \Delta S_2^m + \ldots + g_{\alpha_m} \Delta S_{\alpha_m}^m \tag{6}$$

is called the Δ-*border (border)* of the chain L^m. □

Consider $\Delta L^0 = 0$. It is natural to call any $L^m \in \mathcal{L}^m$ a Δ-*chain*, too. This will be done below. So the notations L^m and ΔL^m are equivalent.

Corollary 5. *The operation of creating a Δ-border of the Δ-chain $L^m \in \mathcal{L}^m$ is a homomorphism, denoted by*

$$\Delta(m) : \mathcal{L}^m \longrightarrow \mathcal{L}^{m-1}, \; m = 1, 2, \ldots, n.$$

It is natural to say that this homomorphism is a Δ-*homomorphism*. This is even necessary since we will use also so-called ∇-*homomorphisms*, which immediately can be obtained by applying the respective operations for creating the ∇-border with respect to ∇S_i^m, $i = 1, 2, \ldots, \alpha_m$, and doing the necessary permutations on the left side of (6) for $m = 0, 1, \ldots, n$.

Denote by $I_m \Delta(m)$ the *image* and by Ker $\Delta(m)$ the *kernel* of the homomorphism $\Delta(m)$.

Theorem 5. *For all $L^m \in \mathcal{L}^m$ the following equality holds:*

$$\Delta\Delta L^m = 0, \; m = 0, 1, \ldots, n.$$

Proof. To verify this, it suffices to show that $\Delta\Delta S_j^m = 0$ for any $S_j^m \in \mathbb{S}^m$. Let S_j^m be represented by the indices $(j_0, j_1, \ldots, j_m)$. According to the definition of the coefficient of Δ-incidence and (3) we have $\Delta\Delta S_j^m = \Delta(\sum_{k=0}^m (-1)^k S_{i_k}^{m-1}) = \Delta(\sum_{k=0}^m (j_0, j_1, \ldots, \hat{j}_k, \ldots, j_m))$, where the symbol $\hat{}$ again denotes the absence of the element j_k. Hence

$$\Delta\Delta S_j^m = \textstyle\sum_{k=0}^m (-1)^k (\sum_{s=0}^{k-1} (-1)^s \cdot (j_0, j_1, \ldots, \hat{j}, \ldots, \hat{j}_k, \ldots, j_m) +$$

$$\textstyle\sum_{s=k+1}^m (-1)^{s-1} (j_0, j_1, \ldots, \hat{j}_s, \ldots, \hat{j}_k, \ldots, j_m)).$$

Observe that the $(m-2)$-dimensional simplex $(j_0, j_1, \ldots, \hat{j}_s, \ldots, \hat{j}_k, \ldots, j_m)$ occurs twice in this sum; first with the Δ-coefficient $(-1)^k \cdot (-1)^s$, and second with the opposite Δ-coefficient $(-1)^k \cdot (-1)^{s-1}$. So it turns out that $\Delta\Delta S_j^m = 0$ with $j = 1, 2, \ldots, \alpha_m$ (see [25]). ■

Based on this proof we observe

Corollary 6. *For the complex of relations K^n the following equality holds:*

$$I^{m-1}(\Delta) \cdot I^m(\Delta) = 0, \ \ m = 1, 2, \ldots, n. \tag{7}$$

Proof. Indeed, we have the following two observations:

1) Using the form in which ΔS_j^m was defined (see (3)) and $\Delta(\Delta S_j^m)$, $m = 1, 2, \ldots, n$, and grouping suitably the $(m-2)$-dimensional simplices $S_1^{m-2}, S_2^{m-2}, \ldots, S_{\alpha_m}^{m-2}$, it turns out that the coefficient of S_j^{m-2} is the intersection element of the i-th line and j-th column of the matrix $I^{m-1}(\Delta) \cdot I^m(\Delta)$.

2) Denoting by $\varepsilon^i(m-1, \Delta)$ the i-th line of the matrix $I^{m-1}(\Delta)$ and by $\varepsilon_j(m, \Delta)$ the j-th column of $I^m(\Delta)$, we obtain the coefficient of S_j^{m-2} in the matrix $I^{m-1}(\Delta \cdot I^m(\Delta)$. Then, according to the equality $\Delta\Delta S_j^m = 0$, the scalar product $\langle \varepsilon^i(m-1, \Delta), \varepsilon_j(m, \Delta) \rangle$ equals zero, which proves the equality (7). ■

Definition 17. The chain $L^m \in \mathcal{L}^m$ with the property $\Delta L^m = 0$ is said to be the Δ-*cycle of dimension* m of the complex K^n, and it is denoted by $Z^m(\Delta) = L^m$, $m = 0, 1, \ldots, n$. □

For example, if in the chain $L^1(x_i, x_j)$, represented by $L^1(\Delta) = g_{t_1} S_{t_1}^1 + g_{t_2} S_{t_2}^1 + \ldots + g_{t_s} S_{t_s}^1$, the equality $x_i = x_j$ holds, then we have $\Delta L^1(\Delta) = 0$, i.e., $\Delta L^1(\Delta) = Z^1(\Delta)$ is a Δ-cycle of dimension one.

Theorem 6. *With respect to the addition of Δ-chains, the set of all Δ-cycles of dimension m forms a commutative subgroup of the group $\mathcal{L}^m$.*

The proof of this theorem is trivial.

Denote the subgroup of Δ-cycles of $\mathcal{L}^m$ by $\mathcal{Z}^m(\Delta)$, $m = 0, 1, \ldots, n$.

Definition 18. Given two Δ-chains $L^m \in \mathcal{L}^m$ and $L^{m+1} \in \mathcal{L}^{m+1}$ with the property $L^m = \Delta L^{m+1}$. The Δ-chain with this property is called Δ-*cycle* *of dimension m being* Δ*-homologic with* 0, and the property itself is denoted by $L^m = Z(\Delta) \sim 0$. Further on, two Δ-cycles $Z_1^m(\Delta), Z_2^m(\Delta) \in \mathcal{Z}^m(\Delta)$ are said to be Δ*-homologic* if $Z_1^m(\Delta) - Z_2^m(\Delta) \sim 0$, $m = 0, 1, \ldots, n-1$ (see [1], [2], [14], [25], [28], [50], and [54]). □

Theorem 15. *With respect to the additive operation of* $\mathcal{L}^m$*, the set of all* Δ*-cycles of dimension m and being* Δ*-homologic with* 0 *forms a subgroup of the group* $\mathcal{Z}^m(\Delta)$*, which is denoted by* $\mathcal{Z}_0^m(\Delta)$*.*

Again the proof is obvious.

It is interesting to mention that the existence of the group $\mathcal{Z}_0^m(\Delta)$ results from the formula $\Delta\Delta L^m = 0$, $m = 0, 1, \ldots, n$. Also it is obvious that $\mathcal{Z}_0^m(\Delta) \sim 0$, since for the complex of relations K^n there are no Δ-chains of dimension $n+1$.

Now we are ready for the important notion of the quotient group of K^n.

Definition 19. The *quotient group* $\mathcal{Z}^m(\Delta)/\mathcal{Z}_0^m(\Delta)$ of the complex of relations K^n is called the *group of* Δ*-homologies* (simply *homologies*) *of dimension m* over the group **Z** and denoted by $\Delta^m(K^n)$, $m = 0, 1, \ldots, n$. The ranks of these groups are called *Betti numbers.* □

It should be remarked that in [1], [2], [48], [50], and [54] such groups are denoted by $\mathcal{H}_m(K^n)$, $m = 0, 1, \ldots, n$.

Now, applying **Z**, let us form the so-called groups of cohomologies (cf. [14] and [25]) of the complex of relations K^n.

Simplifying the notation, we will introduce (as above) the notion of a ∇-chain (co-chain), which is analogous to that of a chain of respective dimension.

Definition 20. Let $L^m \in \mathcal{L}^m$ be an arbitrary ∇-chain, $m = 0, 1, \ldots, n$. The ∇*-border (co-border)* of the ∇-chain L^m is given by the equality

$$\nabla L^m = g_1 \nabla S_1^m + g_2 \nabla S_2^m + \ldots + g_{\alpha_m} \nabla S_{\alpha_m}^m . \quad \square \tag{8}$$

Consider $L^n = 0$.

Corollary 7. *The operation of forming the* ∇*-border of a* ∇*-chain* $L^m \in \mathcal{L}^m$ *represents a* ∇*-homomorphism, denoted by*

$$\nabla(m) : \mathcal{L}^m \longrightarrow \mathcal{L}^{m+1}, \; m = 0, 1, \ldots, n-1 .$$

The proof is trivial.

Denote by $I_m\nabla(m)$ the *image* and by Ker $\nabla(m)$ the *kernel* of the ∇-homomorphism $\nabla(m)$.

Definition 21. The ∇-chain $L^m \in \mathcal{L}^m$ with the property $\nabla L^m = 0$ is called ∇-*cycle of dimension* m of the complex of relations K^n and denoted by $\mathcal{Z}^m(\nabla) = L^m$, $m = 0, 1, \ldots, n$. □

Theorem 7. *With respect to the addition of* ∇*-cycles from* $\mathcal{L}^m$*, the set of all* ∇*-cycles of dimension* m *forms a subgroup of* $\mathcal{L}^m$*, which is denoted by* $\mathcal{Z}^m(\nabla)$, $m = 0, 1, \ldots, n$.

Again the proof is trivial.

Definition 22. Given two ∇-chains $L^m \in \mathcal{L}^m$ and $L^{m-1} \in \mathcal{L}^{m-1}$ with the property $\nabla L^{m-1} = L^m$. The ∇-chain L^m with this property is called ∇-*cycle of dimension* m *being* ∇*-homologic with* 0, and the property itself is denoted by $L^m = Z^m(\nabla) \sim 0$. Two ∇-cycles $Z_1^m(\nabla)$, $Z_2^m(\nabla) \in \mathcal{Z}^m(\nabla)$ are said to be ∇-*homologic* if $Z_1^m(\nabla) - Z_2^m(\nabla) \sim 0$, $m = 0, 1, \ldots, n-1$. □

Theorem 8. *With respect to the additive operation from* $\mathcal{L}^m$*, the set of all* ∇*-cycles of dimension* m *being* ∇*-homologic with* 0 *forms a subgroup of* $\mathcal{Z}^m(\nabla)$, $m = 0, 1, \ldots, n-1$. *This subgroup is denoted by* $\mathcal{Z}_0^m(\nabla)$*, and for* $m = n$ *we have* $\mathcal{Z}_0^n(\nabla) = \mathcal{Z}_0^m(\nabla)$.

The proof of this statement is also obvious.

Definition 23. The quotient group $\mathcal{Z}^m(\nabla)/\mathcal{Z}_0^m(\nabla)$ of the complex of relations K^n is called the *group of* ∇*-homologies* (simply *cohomologies*) *of dimension* m of this complex over $\mathbf{Z}$ and denoted by $\nabla_m(K^n)$, $m = 0, 1, \ldots, n$. □

Again we mention that in [1], [2], [25], [48], [50], and [54] these groups are denoted by $\mathcal{H}_m(K^n)$, $m = 0, 1, \ldots, n$.

The ranks of these groups of ∇-homologies (cohomologies) are certain integers. We call these ranks the *Betti* ∇*-coefficients* of the corresponding dimension.

Remember that the existence of groups was proved by the formula $\Delta\Delta L^m = 0$, $m = 0, 1, \ldots, n$. Thus, the same will be done for $\mathcal{Z}_0^m(\nabla)(K^n)$, since otherwise the corresponding construction is questionable.

Theorem 9. *For any ∇-chain $L^m \in \mathcal{L}^m$ of the complex of relations K^n the following relation holds:*

$$\nabla\nabla L^m = 0, \;\; m = 0, 1, \ldots, \alpha_m \,.$$

Proof. According to the definition of ∇L^m (cf. (8)) it is sufficient to verify that

$$\nabla\nabla S_i^m = 0, \; i = 1, 2, \ldots, \alpha_m \,.$$

Assume $I^{m+1}(\Delta) \cdot I^{m+2}(\Delta) = 0$. The matrix on the left side has the scalar product $\langle \varepsilon^i(m{+}1, \Delta), \varepsilon_j(m{+}2, \Delta)\rangle$ as generator element, where $\varepsilon^i(m{+}1, \Delta)$ and $\varepsilon_j(m+2, \Delta)$ describe the i-th line of $I^{m+1}(\Delta)$ and the j-th column of $I^{m+2}(\Delta)$, respectively. Transposing the matrix $I^{m+1}(\Delta) \cdot I^{m+2}(\Delta)$, we get $(I^{m+1}(\Delta) \cdot I^{m+2}(\Delta))^* = I_*^{m+2}(\Delta) \cdot I_*^{m+1}(\Delta)$ (by putting down the index of transposition), and so we have $I_*^{m+2}(\Delta) \cdot I_*^{m+1}(\Delta) = 0$. The generator element of the matrix from the left side is $\langle e_j(m+2, \Delta), \varepsilon^i(m+1, \Delta)\rangle = 0$, where $\varepsilon_j(m+2, \Delta)$ is j-the line of the matrix $I_*^{m+2}(\Delta)$ and $\varepsilon^i(m+1, \Delta)$ is the i-the column of $I_*^{m+1}(\Delta)$. By Corollary 3 we obtain

$$\varepsilon_j^i(m+2, \Delta) = \varepsilon_i^j(m, \Delta) \,,$$

where $m = 1, 2, \ldots, n-1$, and

$$\varepsilon_j^l(m+1, \nabla) = \varepsilon_l^j(m+2, \Delta) \,,$$

where $m = 0, 1, \ldots, n-2$, $i = 1, 2, \ldots, \alpha_m$, $j = 1, 2, \ldots, \alpha_{m+1}$, and $l = 1, 2, \ldots, \alpha_{m+2}$.

From the transposition of matrices (and by a suitable change of indices) we get

$$\langle \varepsilon^l(m+1, \nabla), \varepsilon_i(m+1, \nabla)\rangle = 0, \, i = 1, 2, \ldots, \alpha_m, \, l = 1, 2, \ldots, \alpha_{m+2} \,,$$

$$I_*^{m+1}(\Delta) = \varepsilon_j^i(m+2, \Delta) = \varepsilon_i^j(m, \nabla) = I^m(\nabla), \, m = 0, 1, \ldots, n-1 \,,$$

$$I_*^{m+2}(\Delta) = \varepsilon_j^l(m+1, \nabla) = \varepsilon_l^j(m+2, \Delta) = I^{m+1}(\nabla) \,,$$

with $m = -1, 0, 1, \ldots, n-2$, $i = 1, 2, \ldots, \alpha_m$, $j = 1, 2, \ldots, \alpha_{m+1}$, and $l = 1, 2, \ldots, \alpha_{m+2}$.

This yields the equality

$$I^{m+1}(\nabla)I^m(\nabla) = 0\,, \quad m = 0, 1, \ldots, n-1\,. \tag{9}$$

The generator element of the matrix of this equality is $\langle \varepsilon^l(m+1, \Delta), \varepsilon_i(m+1, \nabla)\rangle = 0$, with $i = 1, 2, \ldots, \alpha_m$ and $l = 1, 2, \ldots, \alpha_{m+2}$.

Now we return to the sum $\nabla\nabla S_i$, $i = 1, 2, \ldots, \alpha_m$, and group the same elements with respect to S_l^{m+2}. We obtain the coefficient $\langle \varepsilon^l(m+1, \nabla), \varepsilon_i(m+1, \nabla)\rangle = 0$ for $i = 1, 2, \ldots, \alpha_m$ and $l = 1, 2, \ldots, \alpha_{m+2}$. Thus the theorem is proved. ■

Remark 3. For the group of homologies and cohomologies of the complex of relation K^n the determination of the directions of its simplices is a technical problem, and it does not depend on the construction of groups (cf. [14], [25], and [47]).

Definition 24. Let $K^n, n \geq 1$, be a complex of relations. We call the complex K^n an *acyclic complex* if $\Delta^1(K^n) = \Delta^2(K^n) = \ldots = \Delta^n(K^n) \simeq 0$. □

Before we can formulate the next theorem, we have to introduce a classical notion (cf. [14] and [48]), and according to this we will also prove a lemma.

Definition 25. Let $K^n = (\mathbb{S}^0, \mathbb{S}^1, \ldots, \mathbb{S}^n)$ be a complex of relations, $\mathbb{S}^0 = (S_1^0, S_2^0, \ldots, S_{\alpha_m}^0)$ the family of 0-dimensional simplices of this complex, and $L^0 = g_1 S_1^0 + g_2 S_2^0 + \ldots + g_{\alpha_0} S_{\alpha_0}^0$ an arbitrary Δ-chain from $\mathcal{L}^0$. The index of the Δ-chain L^0 is said to be the *operator*

$$I : L^0 \longrightarrow \mathbf{Z} \tag{10}$$

with the property $I(L^0) = g_1 + g_2 + \ldots + g_{\alpha_0}$. □

It is obvious that if $L_1^0, L_2^0 \in \mathcal{L}^0$ are two arbitrary Δ-chains, then the following equality holds:

$$I(L_1^0 + L_2^0) = I(L_1^0) + I(L_2^0)\,. \tag{11}$$

Lemma 1. *Given the connected complex of relation $K^n = (\mathbb{S}^0, \mathbb{S}^1, \ldots, \mathbb{S}^n)$ and the group $\mathcal{L}^0$ of 0-dimensional Δ-chains for K^n. The Δ-chain $L^0 \in \mathcal{L}^0$ is homologic with 0 if and only if $I(L^0) = 0$.*

Proof. Let $S^1 \in \mathbb{S}^1$ be a simplex that is positively oriented and can be written as $S^1 = (S_i^0, S_j^0)$, where $S_i^0, S_j^0 \in \mathbb{S}^0$, $i \neq j$. In this case $\Delta(g \cdot S^1) = gS_j^0 - gS_i^0$, i.e., we have $I(\Delta(g \cdot S^1)) = 0$. According to (10), for any $z^0(\Delta) \in \mathcal{Z}(\Delta)$ the relation $I(\Delta z^0(\Delta)) = 0$ holds. Now let $I(L^1) = 0$, where $L^1 \in \mathcal{L}^0$ is an arbitrary Δ-chain. One has to show that $L^1 \sim 0$. The connectedness of K^n leads to the fact that for any $S_i^0, S_j^0 \in \mathbb{S}^0$ from K^n there exists a succession of 1-dimensional simplices $S_{i_1}^1, S_{i_2}^1, \ldots, S_{i_t}^1$ such that $S_{i_k}^1$ and $S_{i_{k+1}}^1$, $k = 1, 2, \ldots, t-1$, are adjacent, the forerunner of $S_{i_1}^1$ coincides with S_i^0, and the extremity of $S_{i_t}^1$ coincides with S_j^0. Moreover, the elements of this sequence may be oriented in such a way that it is possible to present all of them positively, i.e., as a Δ-*chain* $L^1 = gS_{i_1}^1 + gS_{i_2}^1 + \ldots + gS_{i_t}^1$, where $g \in \mathbf{Z}$ is positive.

Instantly it can be checked that $\Delta L^1 = gS_j^0 - gS_i^0$, and so $gS_j^0 \sim gS_i^0$. But this yields the statement that, whatever $L^1 \in \mathcal{L}^1$ is, this Δ-chain is homologic with gS_i^0. Based on this we obtain $I(L^1) = g$ and also $L^1 \sim I(L^1)S_i^0$, and (11) yields $I(L^1) = 0$, i.e., $L^1 \sim 0$. ■

Now we have

Theorem 10. *If the complex of relations K^n is connected, then $\Delta^0(K^n)$ is isomorphic to the group of integers $\mathbf{Z}$.*

Proof. According to (10), the operator $I : \mathcal{L}^0 \longrightarrow \mathbf{Z}$ represents an isomorphism of the group $\mathcal{L}^0 = \mathcal{Z}^0(\Delta)$ in $\mathbf{Z}$. But since for any $g \in \mathbf{Z}$ there is a Δ-cycle gS_i^0 in $\mathcal{Z}^0(\Delta)$ whose index equals g, we have $I(\mathcal{Z}^0(\Delta)) = \mathbf{Z}$. Thus, according to Lemma 1, the kernel of the homomorphism I is $\mathcal{Z}_0^0(\Delta)$, and therefore $\Delta^0(K^n) = \mathcal{Z}^0(\Delta)/\mathcal{Z}_0^0(\Delta)$ is isomorphic to $\mathbf{Z}$. ■

Corollary 8. *If for the complex of relations K^n the equality (1) holds, then*

$$\Delta^m(K^n) \cong \underbrace{\mathbf{Z} \oplus \mathbf{Z} \oplus \cdots \oplus \mathbf{Z}}_{q}. \tag{12}$$

Definition 25. A connected and acyclic complex of relations K^n is said to be a *directed n-tree of relations.* □

The importance of this notion will be shown in forthcoming papers of the authors. For $n = 1$, the corresponding constructon represents a directed connected graph without cycles, i.e., if K^1 is connected, then it represents a directed tree (see [8], [14], and [15]).

Definition 26. If $K^n = \{\mathbb{S}^0, \mathbb{S}^1, \ldots, \mathbb{S}^n\}$ is a complex of relations being locally complete (cf. Definition 6) which is transformed into an abstract simplicial complex (see Definition 7 and after it), then K^n will be called a *symmetrical complex.* □

Clearly, in this case K^n is not directed. But if necessary, each element of $\mathbb{S}^m$, $m = 0, 1, \ldots, n$, can be directed as it is described in § 2 above.

In § 1 we introduced the Euler characteristic for the complex of relations K^n. Its importance can also be seen by the following observation: If, for example, $\rho_m(\Delta^m(K^n))$, $i = 0, 1, \ldots, n$, represents the rank of the group $\Delta^m(K^n)$, then we have

Theorem 11. *For the complex of relations* $K^n = (\mathbb{S}^0, \mathbb{S}^1, \ldots, \mathbb{S}^n)$ *the equality*

$$\mathcal{X}(K^n) = \sum_{m=0}^{n} \rho_m(\Delta^m(K^n))$$

holds.

Proof. From the algbra of groups (cf. [48]) it is know that, if A is group of finite rank and $B \subset A$ denotes a commutative subgroup of A, then for the quotient group $C = A/B$ the equality

$$\rho(A) = \rho(B) + \rho(C)$$

holds. So we have $\rho_m(\mathcal{L}^m) = \alpha_m = \rho_m(\mathcal{Z}^m(\Delta)) + \rho_m(\mathcal{L}^m/\mathcal{Z}^m(\Delta))$ for $m = 0, 1, \ldots, n$.

For $m > 0$ the quotient group $\mathcal{L}^m/\mathcal{Z}^m(\Delta) \cong \Delta^{m-1}(K^n)$. The kernel Ker $\Delta(m)$ of the homomorphism $\Delta(m) : \mathcal{L}^m \longrightarrow \mathcal{L}^{m-1}$ is exactly the group $\mathcal{Z}^m(\Delta)$, and $I^m\Delta(m)$ represents the subgroup $\mathcal{Z}_0^{m-1}(\Delta) \subset \mathcal{L}^{m-1}$ (cf. § 1). So we have $\mathcal{L}^m/\mathcal{Z}^m(\Delta) = \mathcal{Z}_0^{m-1}(\Delta)$. Further on, $\alpha_m = \rho_m(\mathcal{Z}^m(\Delta) + \rho_m(\mathcal{Z}_0^m(\Delta))$ for $m = 0, 1, \ldots, n$. By definition, for $m = 0$ we have $\mathcal{Z}^0(\Delta) = \mathcal{L}^0$, and hence $\rho_0(\mathcal{L}^0) = \rho_0(\mathcal{Z}^0)$. By $I^m\Delta(0) = \mathcal{Z}_0^1(\Delta) = 0$ we have $\rho_1(\mathcal{Z}_0^1(\Delta)) = 0$, which yields $\alpha_m = \rho_m(\mathcal{Z}^m(\Delta)) + \rho_{m-1}(\mathcal{Z}_0^{m-1}(\Delta))$, $m = 0, 1, \ldots, n$.

By suitable substitutions we get the equalities $\rho_m(\mathcal{Z}^m(\Delta)) = \rho_m(\mathcal{Z}_0^m(\Delta)) + \rho_m(\mathcal{Z}^m(\Delta)/\mathcal{Z}_0^m(\Delta)) = \rho_m(\mathcal{Z}_0^m(\Delta)) + \rho_m(\Delta^m(K^n)) = \rho_m(\mathcal{Z}_0^m(\Delta)) + r^m(\Delta)$ for $m = 0, 1, \ldots, n$. As $\mathcal{Z}_0^m(\Delta) = 0$ (see § 1), we have $\alpha_m = \rho_m(\mathcal{Z}_0^m(\Delta)) +$

$r^m(\Delta) + \rho_1(\mathcal{Z}_0^m(\Delta))$, and substitutions yield $\mathcal{X}(K^n) = \sum_{m=0}^{n}(-1)^m\alpha_m = r^0(\Delta) + \rho_1(\mathcal{Z}_0^1(\Delta)) + \rho_0(\mathcal{Z}_0^0(\Delta)) - (r^1(\Delta) + \rho_0(\mathcal{Z}_0^0(\Delta)) + \rho_1(\mathcal{Z}_0^1(\Delta))) + \ldots + (-1)^m(r^m(\Delta) + \rho_{m-1}(\mathcal{Z}_0^{m-1}(\Delta)) + \rho_m(\mathcal{Z}_0^m(\Delta)) + \ldots + (-1)^n(r^n(\Delta) + \rho_{n-1}(\mathcal{Z}_0^{n-1}(\Delta)) + \rho_n(\mathcal{Z}_0^n(\Delta))) = r^0(\Delta) - r^1(\Delta) + r^2(\Delta) + \ldots + (-1)^m r^m(\Delta) + \ldots + (-1)^n r^n(\Delta) = \sum_{m=0}^{n}(-1)^m r^m(\Delta)$. ■

§ 4 Concluding remarks

1. The complex of relations $K^n = \{\mathbb{S}^0, \mathbb{S}^1, \ldots, \mathbb{S}^n\}$ is, as mentioned in § 1, not an abstract simplicial graph. The facts exposed in § 3 allow us to explain what K^n really does represent.

First we pick up the following notions from [25]:

A family C^n of objects ξ, called *abstract cells*, to which certain numbers p with $0 \leq p \leq n$ correspond (in each case called the *dimension* of such an object and denoted, if necessary, by $\dim \xi_p = p$), is said to be a *finite-dimensional abstract cellular complex.* If in C^n there exists at least one cell ξ_n of dimension n, then the whole family C^n has the dimension n, i.e., $\dim C^n = n$. For the family C^n one might consider the following order type relation $\prec$: if $\xi \prec \xi'$, then ξ is called a *face* of the cell ξ'. If there are two cells ξ_p and ξ_{p+1} satisfying the relation $\xi_p \prec \xi_{p+1}$, then for this pair of cells the coefficient of incidence $[\xi_{p+1} : \xi_p]$ (or the coefficient of coincidence $[\xi_p : \xi_{p+1}]$) may be defined, which is an integer. The given objects, the order type relation $\prec$ and the coefficient of incidence (of coincidence) satisfy the following axioms:

AC1: The relation $\prec$ is strict and partial.

AC2: The relations $\xi_p \prec \xi_q$ and $\xi_p \neq \xi_q$ lead to the relation $p < q$.

AC3: For any pair ξ_p, ξ_q, where $\xi_p \prec \xi_q$ and $p < q - 1$, there exists a finite set of cells ξ' such that $\xi_p \prec \xi' \prec \xi_q$ holds for each member of this finite set.

AC4: The inequality $[\xi_{p+1} : \xi_p] \neq 0$ ($[\xi_p : \xi_{p+1}] \neq 0$) yields $\xi_p \prec \xi_{p+1}$.

AC5: For any pair ξ_{p-1}, ξ_{p+1} the relation

$$\sum_{\xi_p}[\xi_{p+1} : \xi_p] \cdot [\xi_p : \xi_{p-1}] = 0 \tag{13}$$

$$\left(\sum_{\xi_p}[\xi_{p-1}:\xi_p]\cdot[\xi_p:\xi_{p+1}]=0\right) \tag{14}$$

holds.

For the complex of relations define two sets of abstract simplices $\mathbb{S}^{-1} = \{S^0 = \emptyset\}$, $\mathbb{S}^{n+1} = \{S^{n+1} = \emptyset\}$ (with dim $\emptyset = -1$, cf. [28]) which satisfy

$$[S_i^0 : S_j^{-1}] = 0, \qquad \text{where} \quad S_j^{-1} = S^{-1} \in \mathbb{S}^{-1},$$

$$[S_j^{n+1}] : S_i^n = 0, \qquad \text{where} \quad S_j^{n+1} = S^{n+1} \in \mathbb{S}^{n+1} = \emptyset\,.$$

Then, by the equalities (8) and (9), we obtain

Theorem 12. *The complex of relations $K^n = \{\mathbb{S}^0, \mathbb{S}^1, \dots, \mathbb{S}^n\}$ represents a model of a finite-dimensional abstract cellular complex.*

The construction of $K^n = \{\mathbb{S}^0, \mathbb{S}^1, \dots, \mathbb{S}^n\}$ and of the homologies as done above yields a possibility to develop (in a purely combinatorial way) the topology of multi-ary relations, to generalize the corresponding classical theorems and to formulate new results in combinatorial topology.

2. As we mentioned in § 1, the complex of relations $K^n = \{\mathbb{S}^0, \mathbb{S}^1, \dots, \mathbb{S}^n\}$ represents a directed hypergraph. In such a form, this notion is not used in the literature. By using some slight modifications, K^n can be transformed into a usual hypergraph in the sense of Berge (see [6]). But the investigations here go beyond this, they are more general.

First we give an observation. In [19], the normal homologies of hypergraphs are discussed, but the constrain is made on the topological space by using the notion of a *nerv*, which is a usual hypergraph of some covering of this space, see [1] and [48].

Let us give an elementary example.

Introduce the notion of isomorphism of two complexes of relations. Applying this notion in the transformations of K^n into itself and using the Hopf formula (cf. [25] and [48]), we can show when a directed or a usual hypergraph has the fixed elements.

For example, if K^n consists only of the m-dimensional faces of a simplex S^n, $0 \le m \le n-1$, where n is an even number, then any isomorphism of

K^n into itself, keeping the orientation of S^n and permuting its vertices in the cyclic way, has the fixed relations.

Thus, from our point of view it seems to be important to complete the theory of hypergraphs by the new topological elements, also because this theory is closely related to many different problems from (applied) mathematics (cf. [3], [9], [11], [12], [13], [15], [16], [21], [22], [23], [26], [27], [30], [36], [37], [38], [39], [42], [43], [46], [49], [52], [55], [56], [57], and [59]).

References

[1] ALEXANDROFF, P.: *Combinatorial Topology.* Moscow, 1947 (in Russian)

[2] ALEXANDROFF, P., HOPF, H.: *Topologie,* I. Springer, Berlin 1935.

[3] BARANYAI, Z.: On the factorization of the complete uniform hypergraph. In: *Infinite and Finite Sets* (Eds.: A. Hajnal, R. Rado, V. T. Sos), North-Holland, Amsterdam 1975, 91-108.

[4] BAUSE, H. J.: *Combinatorial Foundation of Homology and Homotopy.* Springer, 1998.

[5] BERGE, C.: *Graphs and Hypergraphs.* North-Holland, 1973.

[6] BERGE, C.: *Hypergraphs: Combinatorics of Finite Sets.* North-Holland, 1981.

[7] BERGE, C.: *Theorie des graphes et des applications.* DUNOD, 277, Paris 1958.

[8] BJØRNER, A.: Topological methods. In: *Handbook of Combinatorics,* Chapter 34, 1995.

[9] BJØRNER, A., ZIEGLER, G. M.: Broken circuit complexes: factorizations and generalizations. *J. Combin. Theory, Ser. B,* 51 (1991), 96-126.

[10] BJØRNER, A., ZIEGLER, G. M.: Introduction to greedoids. In: *Matroid Applications* (Ed. N. White), pp. 284-357. Cambridge University Press, Cambridge 1992.

[11] BJØRNER, A., LAS VERGNAS, M., STURMFELS, B., WHITE, N., ZIEGLER, G. M.: *Oriented Matroids.* Cambridge University Press, Cambridge 1995.

[12] BOLLOBAS, B.: On generalized graphs. *Acta Math. Acad. Sci. Hung.* 16 (1965), 445-452.

[13] BOLLOBAS, B.: *Modern Graph Theory.* NM, USA, 1998.

[14] BOLTYANSKI, V.: *Homotopic Theory of Continuous Images and Vector Fields.* Trudy Matem. Inst. Steklov, AN SSSR, XLVII, Moscow 1995 (in Russian).

[15] BOLTYANSKI, V., MARTINI, H., SOLTAN, P.: *Excursions into Combinatorial Geometry.* Springer, 1996.

[16] BOLTYANSKI, V., MARTINI, H., SOLTAN, V.: *Geometric Methods and Optimization Problems.* Kluwer Acad. Publishers, Dordrecht, 1999.

[17] CARTAN, H., EILENBERG, S.: *Homological Algebra.* Princeton, New Jersey, 1956.

[18] DEWDNEY, A. K.: Degree sequences in complexes and hypergraphs. *Proc. Amer. Math. Soc.* 53 (1975), 535-540.

[19] DODSON, C. T. I., LOK, R.: Hypergrahs, homotopy and neighbourhood homology. *Ars Combin.* 16 (1983), 107-130.

[20] DOWKER, C. H.: Homology groups of relations. *Annals Math.* 56 (1952), 52-57.

[21] ERDŐS, P.: On extremal problems on graphs and generlized graphs. *Israel J. Math.* 2 (1964), 183-190.

[22] FRITSCH, R., FRITSCH, G., PESCHKE, E.: *The Four-Color Theorem.* Springer, 1998.

[23] GANTMAHER, F. R.: *Matrix Theory.* Moscow 1967 (in Russian).

[24] GHOUILA-HOURI, A.: Caracterisation des matrices totalement unimodulaires. *C. R. Acad. Sci. (Paris)* 254 (7), (1962), 1192-1194.

[25] HILTON, P. I., WYLIE, S.: *Homology Theory (An Introduction to Algebraic Topology).* Cambridge, 1960.

[26] HOFFMAN, A. J., KRUSKAL, J. B.: Integral boundary points of convex polyhedra. *Ann. Math. Studies* 28 (1956), 230-246.

[27] HOLL, M.: *Combinatorics.* Moscow, 1970 (in Russian).

[28] HUREWICZ, W., WALLMAN, H.: *Dimension Theory.* Moddison, 1941.

[29] JAMES, R. MUNKRES: Topological results in combinatorics. *Michigan Math. J.* 31 (1984), 113-128.

[30] KENMOSHI, Y., ATSUSHI, I., ICHIKAWA, A.: Discrete combinatorial geometry. *Pattern Recognition* 30 (1997), 1719-1728.

[31] KNASTER, B., KURATOWSKI, K., MAZURKIEVICZ, S.: Ein Beweis des Fixpunktsatzes für n-dimensionale Simplexe. *Fundam. Math.* 14 (1929), 132-137.

[32] KUROSH, A. G.: *Course of Higher Algebra.* Moscow, 1968 (in Russian).

[33] KUROSH, A. G.: *Lecures on General Algebra.* Moscow, 1962 (in Russian).

[34] LANG, S.: *Algebra.* Reading, Mass., 1965.

[35] LOVASZ, L.: On the chromatic number of a finite set-system. *Acta Math. Acad. Sci. Hung.* 19 (1968), 59-67.

[36] MARTINI, H., HLIBICIUC, V., PRISAKARU, K., SOLTAN, P.: On d-convex partitions of polygonal regions. Proceedings of the Annual Converence SRMS, "Babesh-Bolyai" University, Cluj-Napoca, Romania, to appear.

[37] MARTINI, H., SOLTAN, P.: On convex partitions of polygonal regions. *Discrete Math.* 195 (1999), 167-180.

[38] MARTINI, H., SOLTAN, V.: Minimum convex partition of polygonal domains by guillotine cuts. *Discrete and Computational Geometry* 19 (1998), 291-304.

[39] MARTINI, H., SOLTAN, V.: Minimum number of pieces in a convex partition of a polygonal domain. *International Journal of Computational Geometry and Applications* 9 (1999), 599-614.

[40] MELNICOV, O., TYSHKEVICH, R., YEMELICHEV. V., SARVANOV, V.: *Lectures on Graph Theory.* B.I.-Verlag, Mannheim, 1994.

[41] MEYER, J. C.: Quelques probleme concernant les cliques des hypergraphes, h-complets et q-parti h-complets. Hypergraph Seminar (Berge and Ray-Chaudhuri, eds.), *Lecture Notes in Math. 411*, Springer-Verlag 1974, pp. 127-139 and 285-286.

[42] MILNOR, E. G.: A combinatorial theorem on systems of sets. *J. London Math. Soc.* 43 (1966), 204-206.

[43] MILNOR, E. G.: Groups which act on S^n without fixed points. *Amer. J. Math.* 79 (1967), 623-660.

[44] NAIK, R. N., RAO, S. B., SHRIKHANDE, S. S., SINGHI, N. W.: Intersection graphs on k-uniform linear hypergraphs. *Europ. J. Combin.* 3 (1982), 159-172.

[45] OLIVER, R.: Fixed-point sets of group actions on finite acyclic complexes. *Comment. Math. Helvet.* 50 (1977), 155-177.

[46] PADBERG, M. W.: Total unimodularity and the Euler subgraph problem. *Operations Research Letters* 7 (1988), 173-179.

[47] POINCARE, H.: Complement a l'analysis situs. *Rend. Circ. Mat. Palermo* 13 (1899), 285.

[48] PONTRYAGIN, A. S.: *Foundations of Combinatorial Topology.* Moscow, 1976 (in Russian).

[49] RINGEL, G.: *Map Color Theorem.* Springer-Verlag, Berlin-Heidèlberg-New York, 1974.

[50] ROYMAN, J. J.: *An Introduction to Algebraic Topology.* Springer-Verlag, 1998.

[51] SANDERS, MACLANE: *Homology.* Springer-Verlag, 1963.

[52] SOLTAN, P., ZAMBITKI, D. K., PRISACARU, K. F.: *Extremal Problems on Graphs and Algorithms for Solutions.* Kishinev, 1974 (in Russian).

[53] SOLTAN, V.: Partition of a planar set into a finite number of d-convex parts. *Cybernetics* 20 (1984), 855-860.

[54] TELEMAN, S.: *Elemente de topologie si varietati diferentiale.* Bucuresti, 1964.

[55] TOMESCU, I.: Sur le probleme du coloriage des graphes generalises. *C. R. Acad. Sci. (Paris)* 267 (1968), 250-252.

[56] TUTTE, W. T.: Lectures on Matroids. *J. Research Nat. Bureau of Standards* B 69: 1-2 (1965), 1-48.

[57] TUZA, Z.: Cirtical hypergraphs and intersecting set-pair systems. *J. Combin. Theory,* Ser. B, 39 (1985), 134-145.

[58] VEBLEN, O., ALEXANDER, I. W.: Manifolds of n dimensions. *Ann. Math.* 14 (1913), 163.

[59] VOLOSHIN, V. I.: On the upper chromatic number of a hypergraph. *Austr. J. Combin.* 11 (1995), 25-45.

[60] ZYKOW, A. A.: Hyperpgraphs (in Russian). In: *Successes in Mathematics*, Vol. XXIX, Edition 6 (180), Moscow 1974.