The Ramsey Numbers r(mC_4, nC_5)

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ABSTRACT. If G and H are graphs, define the *Ramsey number* $r(G, H)$ to be the least number p such that if the edges of the complete graph K_p are colored red and blue(say), either the red graph contains a copy of G, or the blue graph contains a copy of H. In this paper, we determined the *Ramsey number* $r(mC_4, nC_5)$ for any $m \geq 1, n \geq 1$.

1. Introduction

Let G and H be simple graphs. Define the *Ramsey number* $r(G, H)$ to be the least number p such that if the edges of K_p are two-colored, say red and blue, either the red graph contains a copy of G, or the blue graph contains a copy of H(we as well call that K_n contains a red G or a blue H) .

In this paper, we will generally follow the notation of Harary [4]. A cycle with n (≥ 3) vertices is called an *n-cycle* and denoted by C_n , mC_n denotes the union of m vertex-disjoint copies of C_n . If A is a *vertex set*, we denote the induced subgraph of A by $< A >$. We call *vertex* u a red-neighbor of *vertex* v if it is joined to v by a red edge. Likewise we define a blue-neighbor. Let G be a graph, $\beta_0(G)$ denote the maximum cardinality among the independent sets of vertices in graph G.

Mizuno and Sato[6] decided that: $r(mC_4, nC_k) = nk + 2m - 1$, where

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$k \geq 6$, $n \geq m \geq 1$; $m(k+2) - 1 \leq r(mC_4, mC_k) \leq m(k+2)$, where $k = 4, 5$, $m \geq 2$. The authors determined $r(mC_4, nC_4)$ in [5]: $r(mC_4, nC_4) = 2m + 4n - 1$, where $n \geq m \geq 1$, $(m, n) \neq (1, 1)$. In this paper, we establish the values of $r(mC_4, nC_5)$ for any $m \geq 1, n \geq 1$.

Some of useful theorems are given below.

Theorem A [1]

$$r(G, H) \geq |V(G)| + |V(H)| - min(\beta_0(G), \beta_0(H)) - 1$$

Theorem B [1] *Let $i = min(\beta_0(H), \beta_0(K))$, and suppose that any two-colored complete graph containing a mutually disjoint red H and blue K contains a red H and a blue K having at least i vertices in common, then*

$$r((m+1)H, (n+1)K) \leq r(mH, nK) + |V(H)| + |V(K)| - min(\beta_0(H), \beta_0(K))$$

Theorem C [1]

$$r(G, mH) \leq r(G, (m-1)H) + |V(H)|, m \geq 2$$

Theorem D [6] *If a two-colored complete graph K_{m+4} $(m \geq 4)$ contains a red C_m $(m \geq 4)$ and a blue C_4 being disjoint, then it contains a red C_m and a blue C_4 having at least two vertices in common.*

2. Some lemmas

Lemma 1. *Suppose that a two-colored complete graph K_6 contains neither red C_4 nor blue C_5, then it has as subgraph a red $K_3 \cup K_3$ and a blue $K_{3,3} - e$, where $K_{3,3} - e$ denotes a $K_{3,3}$ deleted one edge.*

Proof. Because K_6 doesn't contains red C_4, by $r(C_4, C_4) = 6$[2], it contains a blue C_4. Denote the blue C_4 by C^*: $w_1 w_2 w_3 w_4 w_1$. Set the other two vertices to be w_5 and w_6. Then,

If $w_i w_j$ $(i = 5, 6, j = 1, 2, 3, 4)$ are all red edges, then there is a red C_4. Let $w_5 w_1$ be a blue edge.

If $w_5 w_2$ and $w_5 w_4$ are not both red edges, then there is a blue C_5_$w_5 w_2 w_3 w_4 w_1 w_5$ or $w_5 w_1 w_2 w_3 w_4 w_5$. Therefore $w_5 w_2$ and $w_5 w_4$ are red edges.

If both w_6w_2 and w_6w_4 are red edges, then there is a red C_4 : $w_5w_2\, w_6w_4\, w_5$. Let w_6w_2 be a blue edge.

If w_6w_1 and w_6w_3 are not both red edges, then there is a blue C_5_$w_6w_2\, w_3w_4\, w_1w_6$ or $w_6w_3w_4\, w_1w_2\, w_6$. Let w_6w_1 and w_6w_3 be red edges.

If both w_5w_3 and w_6w_4 are red edges, then there is a red C_4 : $w_5w_3w_6w_4\, w_5$. By the symmetry of w_5w_3 and w_6w_4 in K_6. Let $w_5\, w_3$ be a blue edge.

If $w_2\, w_4$ is blue edges, then there is a blue C_5: $w_1w_5w_3w_2\, w_4\, w_1$. Let $w_2\, w_4$ be a red edge.

If both w_6w_4 and w_6w_5 are red, then there is a red C_4 : $w_2\, w_4\, w_6w_5w_2$. Thus either w_6w_4 or w_6w_5 is a blue edge.

If w_6w_4 is blue, then w_1w_3 is red. Otherwise, there is a blue C_5: $w_1w_2\, w_6w_4\, w_3w_1$; if w_6w_5 is blue, then w_1w_3 is red as well. Otherwise, there is a blue C_5: $w_1w_2\, w_6w_5w_3w_1$.

By now, we have two red K_3: $<\{w_1, w_3, w_6\}>$ and $<\{w_2, w_4, w_5\}>$, and there are eight blue edges between the two red K_3, therefore Lemma 1 is proved.

Lemma 2. *Suppose that a two-colored complete graph K_{13} has as subgraph a red C_4 and a blue C_5 having three common vertices, then it has a red $2C_4$ or a blue $2C_5$ as subgraph.*

Proof. Let W be the induced subgraph of the rest seven vertices in K_{13} other than those in red C_4 and blue C_5. By $r(C_4, C_5) = 7$[2], W contains a red C_4 or a blue C_5, then K_{13} contains a red $2C_4$ or blue $2C_5$.

Lemma 3. *Suppose that a two-colored complete graph K_{13} has as subgraph a red C_4 and a blue C_5 having just two common vertices, then it has a red $2C_4$ or a blue $2C_5$ as subgraph .*

Proof. Let G and H be the red C_4 and the blue C_5 having just two common vertices in K_{13}. respectively. Set H: $v_1v_2\, v_3v_4\, v_5v_1$ to be the blue C_5. Set

$u_i (i = 1, 2, 3, 4, 5, 6)$ to be the rest six vertices in K_{13} other than those in G and H, and denote the induced subgraph of them by P.

If P contains a red C_4 or a blue C_5, then, K_{13} contains a $2C_4$ or a blue $2C_5$, and Lemma 3 follows. So we assume that P contains neither red C_4 nor blue C_5. By Lemma 1, P has as subgraph a red $K_3 \bigcup K_3$ and a blue $K_{3,3}$ - e. Let $\langle U_1 \rangle$ and $\langle U_2 \rangle$ be the two red K_3 in P, where $U_1 = \{u_1, u_2, u_3\}$, $U_2 = \{u_4, u_5, u_6\}$, and $u_j u_k$ $(1 \leq j \leq 3,\ 4 \leq k \leq 6)$ are red except $u_3 u_6$. We divide the proof into the following two cases: (1) The two common vertices of G and H are adjacent in H; (2) The two vertices are not adjacent in H.

Case 1. The two common vertices of G and H are adjacent in H.

Set v_1 and v_2 to be the common vertices. The following conclusions clearly hold:

Conclusion *a* v_3 has at most one red-neighbor in U_1, thus it has at least two blue-neighbors in U_1. By the symmetry of U_1 and U_2, v_3 also has at least two blue-neighbors in U_2. The same results hold for v_4 and v_5, respectively.

Then, If $v_1 u_1$ is blue, by Conclusion *a*, v_3 has at least two blue-neighbors in U_2, we may let $v_3 u_4$ be blue, then there exists a blue C_5: $v_1 v_2 v_3 u_4 u_1 v_1$. By Conclusion *a*, v_4 and v_5 have at least two blue-neighbor in U_1 and U_2, respectively, we consider the following three cases:

(1.1) u_2 and u_3 are blue-neighbors of v_4 and v_5, respectively.

Clearly, $v_4 v_5 u_3 u_5 u_2 v_4$ is a blue C_5, so, adding it to $v_1 v_2 v_3 u_4 u_1 v_1$ yields a blue $2C_5$.

(1.2) u_5 and u_6 are blue-neighbors of v_4 and v_5, respectively.

Similar to Case (1.1), there exist a blue $2C_5$.

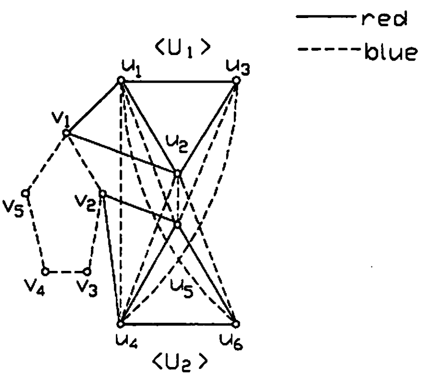
(1.3) Two of the vertices in $\{u_2, u_3, u_5, u_6\}$ are common red-neighbors of v_4 and v_5.

Then, the induced subgraph of v_4, v_5 and their two common red-neighbors contains a red C_4, so adding it to G yields a red $2C_4$.

Now, let $v_1 u_1$ be a red edge.

By the symmetry of u_1 and u_2 in $\langle U_1 \rangle$, $v_1 u_2$ is red as well, so $\langle \{v_1\} \bigcup U_1 \rangle$ contains a red C_4, denote it by C_4^*.

By the symmetry of v_1 and v_2 in H, and by the symmetry of U_1 and U_2, $<\{v_2\} \bigcup U_2>$ contains a red C_4, so adding it to $C_4{}^*$ yields a red $2C_4$ (See Figure 1).



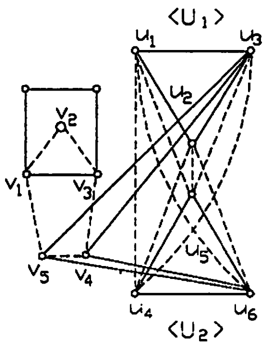


Figure 1 Figure 2

Case 2. The two common vertices of G and H are not adjacent in H.

Set v_1 and v_3 to be the common vertices. Following conclusion holds clearly.

Conclusion *b* v_2, v_4 and v_5 have at most one red-neighbors in U_1, respectively, thus they have at least two blue-neighbors in U_1; By the symmetry of U_1 and U_2, the same result holds for U_2.

Then, If $v_4 u_3$ is blue, by Conclusion *b*, either $v_2 u_4$ or $v_2 u_5$ is a blue edge, by the symmetry of $v_2 u_4$ and $v_2 u_5$, we may set $v_2 u_4$ to be a blue edge, then $v_2 v_3 v_4 u_3 u_4 v_2$ is a blue C_5. Also, by Conclusion *b*_set $v_5 u_1$ and $v_5 u_5$ to be blue edges, then $v_5 u_1 u_6 u_2 u_5 v_5$ is a blue C_5_so adding it to $v_2 v_3 v_4 u_3 u_4 v_2$ yields a blue $2C_5$. Let $v_4 u_3$ be a red edge.

By the symmetry of u_3 and u_6 in P, Let $v_4 u_6$ be red edge as well.

By the symmetry of v_4 and v_5 in H, Let $v_5 u_3$ and $v_5 u_6$ be red edges. thus $v_4 u_3 v_5 u_6 v_4$ is a red C_4_so adding it to G yields a red $2C_4$ (See Figure 2).

Hence Lemma 3 is proved.

Theorem 1. $r(C_4, 2C_5) = 11$.

Proof. Firstly, we show that $r(C_4, 2C_5) \geq 11$. By Theorem A.

$$r(C_4, 2C_5) \geq |V(C_4)| + |V(2C_5)| - min(\beta_0(C_4), \beta_0(2C_5)) - 1 = 11.$$

Next we show $r(C_4, 2C_5) \le 11$. If two-colored K_{11} contains a red C_4, then the result follows. Otherwise, by $r(C_4 \_K_4) = 10$[3], K_{11} contains a blue K_4, denote it by K_4^*. Also, by $r(C_4, C_5) = 7$, $K_{11} - K_4^*$ contains a blue C_5, denote it by C_5^*. Set $V(K_{11} - (K_4^* \bigcup C_5^*)) = \{w_1, w_2\}$. Clearly, $<\{w_1, w_2\} \bigcup K_4^*>$ contains either a red C_4 or blue C_5. If it contains a red C_4, we get the proof; if it contains a blue C_5, adding it to C_5^* yields a blue $2C_5$, we get the proof as well.

Theorem 2. $r(2C_4, 2C_5) = 13$.

Proof. By Theorem A, $r(2C_4, 2C_5) \ge 13$.

Now we show that $r(2C_4, 2C_5) \le 13$. By $r(C_4, 2C_5) = 11$, K_{13} contains either a red C_4 or a blue $2C_5$. If K_{13} contains a blue $2C_5$, we get the result. Otherwise, K_{13} contains a red C_4, denote it by C. By $r(C_4, C_5) = 7$, $K_{13} - C$ contains either a red C_4 or a blue C_5. If $K_{13} - C$ contains a red C_4, then adding it to C yields a red $2C_4$: if $K_{13} - C$ contains a blue C_5, by Theorem D, K_{13} contains a red C_4 and a blue C_5 having two vertices in common. Then, by Lemma 2 and Lemma 3, we can prove the result.

Theorem 3. *For* $m \ge 1$, $r(mC_4, C_5) = 4m + 3$.

Proof. By Theorem C, $r(mC_4, C_5) \le r((m-1)C_4, C_5) + |V(C_4)|$

$$\le r(C_4, C_5) + \_m\text{-}1\_|V(C_4)|$$
$$= 4m + 3.$$

Next we show that $r(mC_4, C_5) \ge 4m + 3$. Color complete graph K_{4m+2} as below: (1) Color a K_3 to be red, and denote it by K_3^*. (2) Color $K_{4m+2} - K_3^*$ to be red. (3) Color the edges joining K_3^* to $K_{4m+2} - K_3^*$ to be red. Then, the K_{4m+2} contains neither red mC_4 nor blue C_5. So, we can prove that $r(mC_4, C_5) \ge 4m + 3$.

Theorem 4. *For* $n \ge 2$, $r(C_4, nC_5) = 5n + 1$.

Proof. The lower bound follows from Theorem A.

Next we show $r(C_4, nC_5) \ge 5n + 1$.

By Theorem C, $r(C_4, nC_5) \le r(C_4, (n-1)C_5) + |V(C_5)|$

$$\le r(C_4, 2C_5) + (n-2)|V(C_5)| = 5n + 1.$$

Hence the theorem is proved.

Theorem 5. *For* $m \geq 2$_$r(mC_4, mC_5) = 7m - 1$.

Proof. The lower bound follows from Theorem A.

Next we show $r(mC_4, mC_5) \leq 7m - 1$.

By Theorem B, it follows that

$$r(mC_4, mC_5) \leq r((m-1)C_4, (m-1)C_5) + |V(C_4)| + |V(C_5)| - min(\beta_0(C_4), \beta_0(C_5)) = r((m-1)C_4, (m-1)C_5) + 7,$$

By recursion, we get

$$r(mC_4, mC_5) \leq r(2C_4, 2C_5) + 7(m-2) = 7m - 1.$$

Theorem 6 For $n \geq m \geq 2$_$r(mC_4, nC_5) = 2m + 5n - 1$.

Proof. The lower bound follows from Theorem A.

Now we show that $r(mC_4, nC_5) \leq 2m + 5n - 1$.

By theorem C, $r(mC_4, nC_5) \leq r(mC_4, (n-1)C_5) + |V(C_5)|$.

By recursion, $r(mC_4, nC_5) \leq r(mC_4, mC_5) + (n-m)\,|V(C_5)| = 2m + 5n - 1$.

Theorem 7 For $m \geq n \geq 2$_$r(mC_4, nC_5) = 4m + 3n - 1$.

Proof. The result can be proved with the same method used in Theorem 6.

We obtain the following theorem on the basis of Theorem 4, 5, 6, 7.

Theorem 8

$$r(mC_4, nC_5) = \begin{cases} 4m+3, & if \;\; m \geq 1, n = 1, \\ 2m+5n-1, & if \;\; n > m \geq 1, \\ 4m+3n-1, & if \;\; m \geq n \geq 2. \end{cases}$$

Hence we determin the Ramsey number $r(mC_4, nC_5)$ for any number m and n, and Along with the result about $r(mC_4, nC_4)$ in [5], we improve the result of Minuzo and Sato for $r(mC_4, nC_k)$ by reducing k from 6 to 4.

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