A Note on Nonplanar Sequences of Iterated Jump Graphs

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ABSTRACT

For a graph G, the jump graph $J(G)$ is that graph whose vertices are the edges of G and where two vertices of $J(G)$ are adjacent if the corresponding edges are not adjacent. For $k \geq 2$, the kth iterated jump graph $J^k(G)$ is defined as $J(J^{k-1}(G))$, where $J^1(G) = J(G)$. An infinite sequence $\{G_i\}$ of graphs is planar if every graph G_i is planar; while the sequence $\{G_i\}$ is nonplanar otherwise. All connected graphs G for which $\{J^k(G)\}$ is planar have been determined. In this paper, we investigate those connected graphs G for which $\{J^k(G)\}$ is nonplanar. It is shown that if $\{J^k(G)\}$ is a nonplanar sequence, then $J^k(G)$ is nonplanar for all $k \geq 4$. Furthermore, there are only six connected graphs G for which $\{J^k(G)\}$ is nonplanar and $J^3(G)$ is planar.

Key Words: jump distance, jump graph, planar graph

AMS Subject Classification: 05C12

1 Introduction

The *jump graph* $J(G)$ of a graph G (see [3]) is that graph whose vertices are the edges of G, and where two vertices are adjacent in $J(G)$ if the corresponding edges of G are not adjacent. For $k \geq 2$, the kth *iterated jump graph* $J^k(G)$ is defined as $J(J^{k-1}(G))$, where $J^1(G) = J(G)$. For $i = 1, 2$, the graphs F_i, $J(F_i)$, $J^2(F_i)$, and $J^3(F_i)$ are shown in Figure 1.

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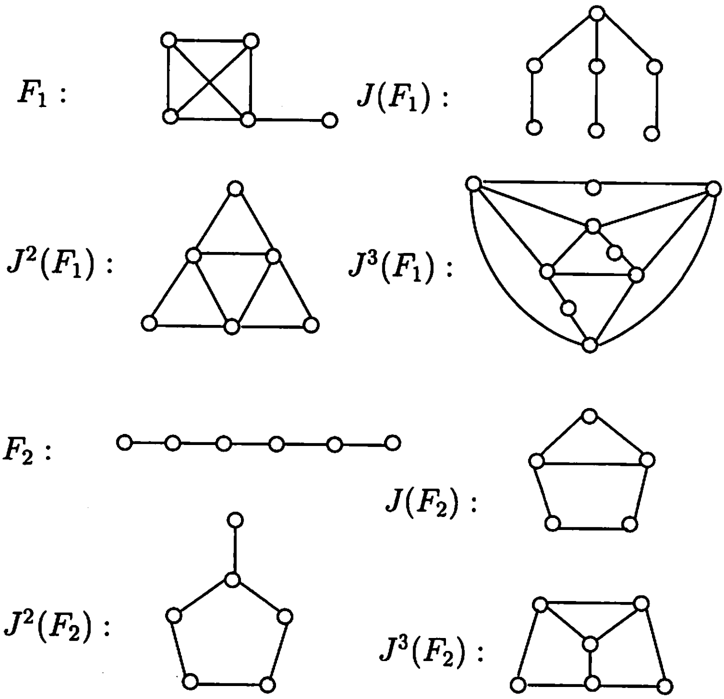


Figure 1: The graphs F_i, $J(F_i)$, $J^2(F_i)$

An infinite sequence $\{G_k\}$ of graphs is said to *converge* (see [1]) if there exists a graph G and a positive integer N such that G_k is isomorphic to G for all $k \geq N$. The graph G is then called the *limit graph* of the sequence $\{G_k\}$. An infinite sequence that does not converge is said to *diverge*. A finite sequence $\{G_k\}$ is said to *terminate*. An infinite sequence $\{G_k\}$ is called *planar* if G_k is planar for every positive integer k and *nonplanar* if G_k is nonplanar for some positive integer k. We refer to the book [4] for graph theory notation and terminology not described here.

The planarity of iterated jump graphs was studied in [2, 5, 6]. Let $cor(K_3)$ denote the corona of K_3, which is the graph obtained by adding a pendant edge to each vertex of K_3. The following two results were established in [5] and [1], respectively.

Theorem 1.1 *Let G be a connected graph such that $J^k(G)$ is defined for every positive integer k. Then the sequence $\{J^k(G)\}$ is planar if and only if $G = C_5$ or $G = cor(K_3)$.*

Theorem 1.2 *Let G be a connected graph. The sequence $\{J^k(G)\}$ terminates if and only if G is a subgraph of one of the graphs G_i $(1 \leq i \leq 6)$*

of Figure 2 or is a subgraph of $H_n = K_{1,n} + e$ *for some* $n \geq 5$.

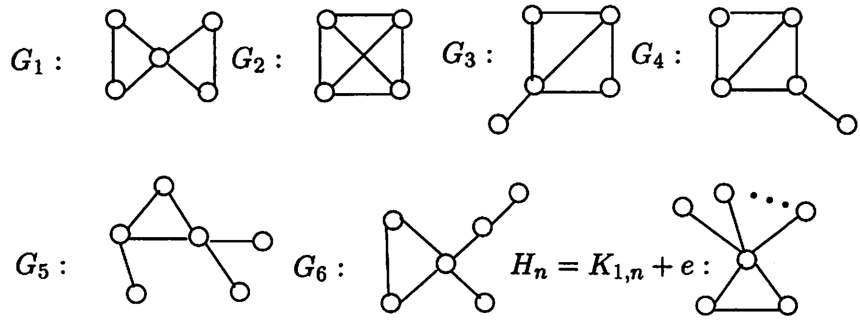


Figure 2: Graphs G for which $\{J^k(G)\}$ terminates

The goal of this paper is to study connected graphs G such that the sequence $\{J^k(G)\}$ is nonplanar. Such sequences are necessarily infinite then. We show that if G is a connected graph for which $\{J^k(G)\}$ is nonplanar, then nonplanar graphs in the sequence $\{J^k(G)\}$ are usually arrived at quickly. In fact, we show that if $\{J^k(G)\}$ is a nonplanar sequence, then $J^k(G)$ is nonplanar for all $k \geq 4$. Furthermore, there are only six connected graphs G for which $\{J^k(G)\}$ is nonplanar and $J^3(G)$ is planar. In order to do this, we review some known results and establish several useful lemmas in Section 2. We present the main theorem of this paper in Section 3.

2 Preliminary Results

In this section, we first establish several lemmas that are necessary for the proof of our main result. Throughout this paper we will use Kuratowski's famed characterization [7] of planar graphs.

Kuratowski's Theorem *A graph is planar if and only if it contains no subgraph isomorphic to* K_5 *or* $K_{3,3}$ *or a subdivision of one of these graphs.*

Lemma 2.1 *If* G *is a graph containing two disjoint subgraphs* F *and* H, *each of size* 3, *then* $J(G)$ *is nonplanar.*

Proof. If G is a graph containing two disjoint subgraphs F and H, each of size 3, then $J(G)$ contains $K_{3,3}$ as a subgraph with partite sets $E(F)$ and $E(H)$ and so $J(G)$ is nonplanar. ∎

Lemma 2.2 *If* G *is a graph containing two disjoint subgraphs* F *and* H, *one of size* 2 *and the other of size* 3, *then* $J^4(G)$ *is nonplanar.*

Proof. Since G contains the subgraphs F and H, it follows that $J(G)$ contains $K_{2,3}$ as a subgraph. The jump graph $J(K_{2,3})$ contains a 6-cycle as a subgraph, implying that $J^3(G)$ contains $2K_3$ as a subgraph, and so $J^4(G)$ contains $K_{3,3}$ as a subgraph. ∎

By Lemma 2.1 and the proof of Lemma 2.2, we have the following corollary.

Corollary 2.3 *Let G be a connected graph.*

(*a*) *If G contains a path of order 8 or more, then $J(G)$ is nonplanar.*

(*b*) *If G contains a cycle of order 6 or more, then $J^2(G)$ is nonplanar.*

(*c*) *If G contains a path of order 7, then $J^2(G)$ is nonplanar.*

(*d*) *If G contains the graph $K_{2,4}$ as a subgraph, then $J^2(G)$ is nonplanar.*

Proof. For (a), if G contains a path of order 8 or more, then G contains two disjoint subgraphs, each of size 3, and so $J(G)$ is nonplanar by Lemma 2.1. The proof of Lemma 2.2 yields (b). Since $2K_3$ is a subgraph of $J(P_7)$, it follows from Lemma 2.1 that (c) holds. If G contains $K_{2,4}$ as a subgraph, then $J(G)$ contains two disjoint subgraphs, each of size 3 and so $J^2(G)$ is nonplanar by Lemma 2.1. ∎

Lemma 2.4 *Let G be a graph containing two disjoint subgraphs F' and F'' of sizes 3 and 2, respectively, and additionally having three independent edges, which may or may not be incident to the vertices of F' and F''. Then $J(G)$ is nonplanar.*

Proof. Let the three edges of F' be e_1, e_2, and e_3, and let the two edges of F'' be e_4 and e_5. Let e_6, e_7, and e_8 be the three independent edges. If even one of e_6, e_7, e_8 is adjacent to no edge of F', then, by Lemma 2.1, $J(G)$ is nonplanar. Hence we may assume that each of e_6, e_7, and e_8 is adjacent to at least one edge of F'. We consider two cases.

Case 1. *One of e_6, e_7, e_8, say e_6, is adjacent to exactly one edge of F', say e_1.* Hence, in $J(G)$, e_6 is adjacent to e_2 and e_3. Since e_6, e_7, and e_8 are independent, at least one of e_7 and e_8, say e_7, is adjacent to e_1 and e_6 in $J(G)$. Consequently, $J(G)$ contains the subgraph (shown in Figure 3), which is isomorphic to a subdivision of $K_{3,3}$. Hence $J(G)$ is nonplanar.

Case 2. *Each of e_6, e_7, and e_8 is adjacent to exactly two edges of F'.* Hence, we may assume that e_6 is adjacent to e_2 and e_3, e_7 is adjacent to e_1 and e_3, and e_8 is adjacent to e_1 and e_2. Consequently, $J(G)$ contains the subgraph (shown in Figure 4), which is isomorphic to a subdivision of $K_{3,3}$ and so $J(G)$ is nonplanar. ∎

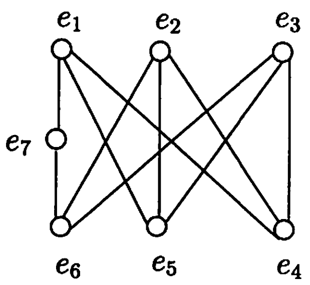


Figure 3: The subgraph in Case 1

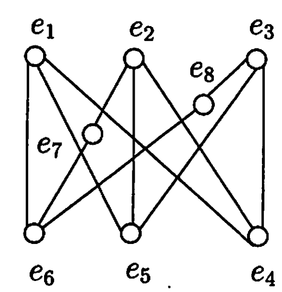


Figure 4: The subgraph in Case 2

Lemma 2.5 *If G is a graph obtained by adding a pendant edge to $K_{2,3}$, then $J^2(G)$ is nonplanar.*

Proof. The graph G is one of the graphs X_1 and X_2 shown in Figure 5. The jump graphs $J(X_1)$ and $J(X_2)$ are also shown in Figure 5. If $G = X_1$, then $J(G)$ contains two disjoint subgraphs of size 3. By Lemma 2.1, $J^2(G)$ is nonplanar. If $G = X_2$, then $J^2(G)$ contains a subdivision of $K_{3,3}$. Thus $J^2(G)$ is nonplanar as well. ∎

The following two lemmas were established in [6] and [5], respectively.

Lemma 2.6 *Let G be a connected graph. If G is nonplanar, then $J(G)$ is nonplanar.*

Lemma 2.7 *Let G be a connected graph having a pendant edge $e = uv$, where $\deg_G v = 1$, and let H be the graph obtained by identifying v with a vertex of G that is not adjacent to u. If $J^k(H)$ is nonplanar, then $J^k(G)$ is nonplanar, where $k \geq 1$.*

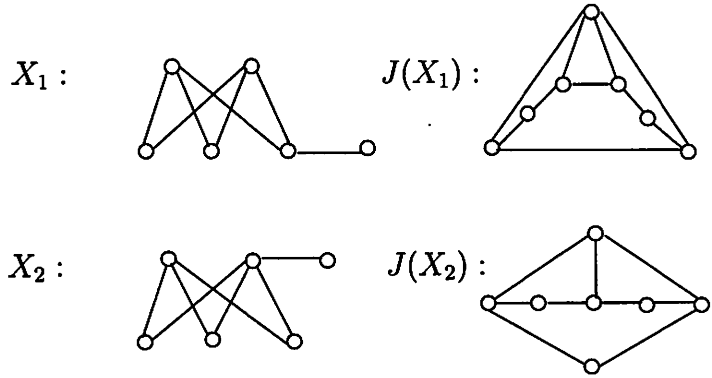


Figure 5: Two graphs in the proof of Lemma 2.5

3 Main Result

We are now prepared to present the main result of this paper. In the proof of this result, the length of a longest cycle in a connected graph that is not a tree is called the *circumference* of G and is denoted by $c(G)$.

Theorem 3.1 *Let G be a connected graph for which $\{J^k(G)\}$ is infinite. If $G \neq C_5$ and $G \neq cor(K_3)$, then $J^k(G)$ is nonplanar for all $k \geq 4$. Furthermore, $J^3(G)$ is planar if and only if G is one of the six graphs F_i $(1 \leq i \leq 6)$ in Figure 6.*

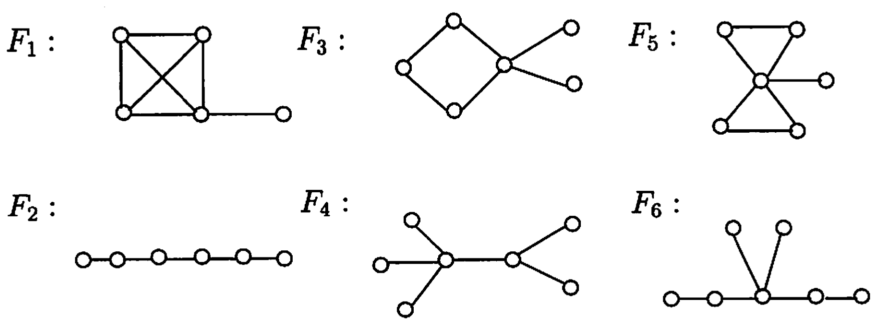
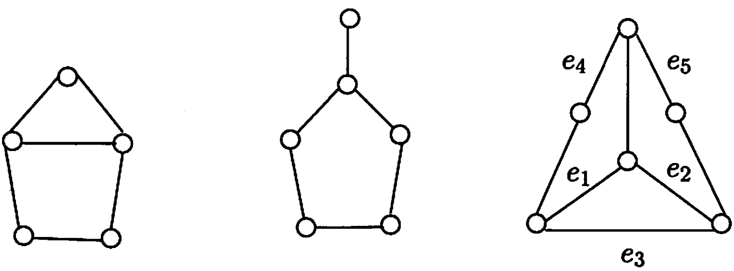


Figure 6: Graphs F_i $(1 \leq i \leq 6)$ in Theorem 3.1

Proof. By Lemma 2.6, it suffices to show that if $G \neq C_5$ and $G \neq cor(K_3)$, then $J^k(G)$ is nonplanar for some $k \leq 4$. We proceed by cases, according to whether G is a tree or according to the value of $c(G)$ if G is not a tree. First, suppose that G is not a tree.

Case 1. $c(G) \geq 6$. By Corollary 2.3, $J^2(G)$ is nonplanar in this case.

Case 2. $c(G) = 5$. Since $G \neq C_5$, it follows that G contains at least one of the graphs H_1 and $H_2 = J(H_1)$ shown in Figure 7 as a subgraph. The graph $J^2(H_1) = J(H_2)$ is also shown in Figure 8. Let F' be the subgraph of $J^2(H_1)$ induced by $\{e_1, e_2, e_2\}$ and F'' the subgraph induced by $\{e_4, e_5\}$. By Lemma 2.4, $J^3(H_1) = J^2(H_2)$ is nonplanar and so $J^3(G)$ is nonplanar in this case.



H_1 $\quad$ $J(H_1) = H_2$ $\quad$ $J^2(H_1) = J(H_2)$

Figure 7: Graphs H_1, $J(H_1) = H_2$, and $J^2(H_1)$ in Case 2

Case 3. $c(G) = 4$. By Theorem 1.2, G must contain a subgraph that is isomorphic to one of the graphs $Y_1, Y_2, \cdots, Y_5$ shown in Figure 8, where Y_1 is the graph F_1 in Figure 1 (and Figure 6).

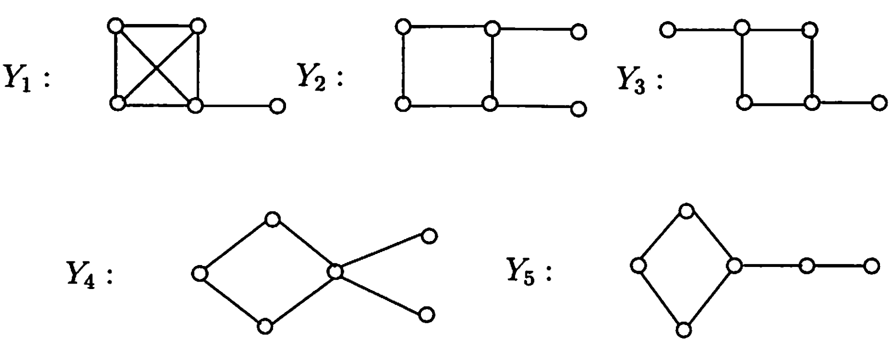


Figure 8: Graphs Y_i for $1 \leq i \leq 5$ in Case 3

The planar graphs $Y_1 = F_1, J(Y_1), J^2(Y_1)$, and $J^3(Y_1)$ are shown in Figure 1. Since $J^3(Y_1)$ contains two disjoint subgraphs, each of size 3, it follows from Lemma 2.1 that $J^4(Y_1)$ is nonplanar.

The planar graph $J(Y_2)$ is shown in Figure 9. Observe that $J^2(F_2)$ (shown in Figure 1) is a subgraph of $J(Y_2)$. Applying Lemma 2.4 to $J^2(F_2)$, we see that $J^4(F_2)$ is nonplanar, and so $J^3(Y_2)$ is nonplanar.

$J(Y_2)$:

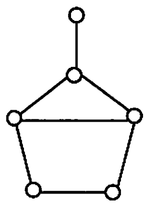


Figure 9: The graph $J(Y_2)$ in Case 3

The graphs $J(Y_3)$ and $J^2(Y_3)$ are shown in Figure 10. Since $J^2(Y_3)$ contains $2K_3$, it follows from Lemma 2.1 that $J^3(Y_3)$ is nonplanar.

$J(Y_3)$: $J^2(Y_3)$:

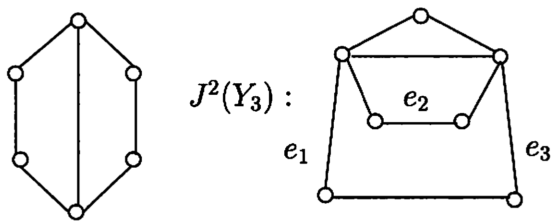


Figure 10: Graphs $J(Y_3)$ and $J^2(Y_3)$ in Case 3

Since $J(Y_4) = Y_3$, it follows that $J^4(Y_4) = J^3(Y_3)$, which is nonplanar. However, $J^3(Y_4) = J^2(Y_3)$ is planar, as shown in Figure 10. The graph Y_4 is also the graph F_3 shown in Figure 6.

The graph $J(Y_5)$ is shown in Figure 11. Since $J^2(Y_5)$ contains the path $e_1, e_5, e_2, e_7, e_3, e_6, e_8$ of order 7, it follows that $J^4(Y_5)$ is nonplanar by Corollary 2.3.

$J(Y_5)$:

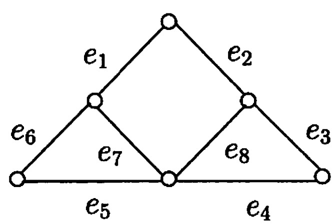


Figure 11: The graph $J(Y_5)$ in Case 3

Case 4. $c(G) = 3$. There are two cases.

Subcase 4.1. *G contains more than one triangle.* If G contains two disjoint triangles, then, by Lemma 2.1, $J(G)$ is nonplanar. Otherwise, since G is not the graph G_1 of Figure 2, it follows that G contains at least one of the graphs G' and $G'' = F_3$ of Figure 12 as a subgraph. The graph $J(G')$ is shown in Figure 12. Since $J^2(G')$ contains the path $e_1, e_2, \cdots, e_9$ of order 9, it follows that $J^3(G')$ is nonplanar by Corollary 2.3. On the other hand, $J^3(G'')$ is planar as shown in Figure 12. Since $J^3(G'')$ contains two disjoint subgraphs, each of size 3, $J^4(G'')$ is nonplanar by Lemma 2.1.

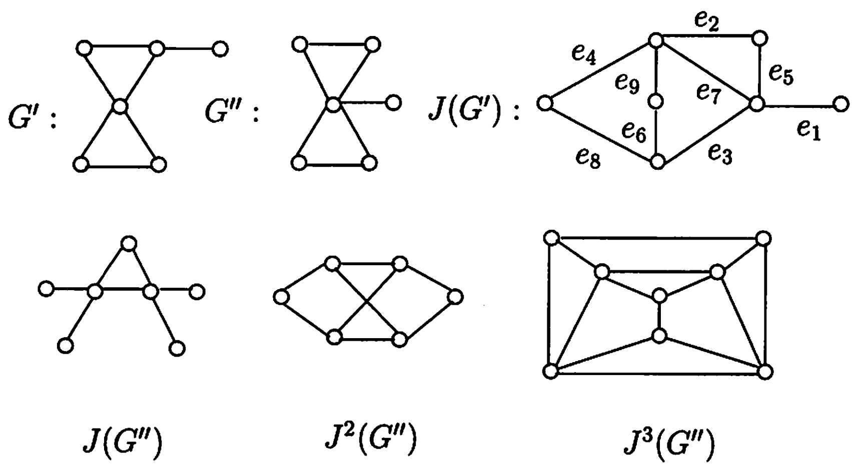


Figure 12: The graphs G' and G'' in Subcase 4.1

Subcase 4.2. *G contains exactly one triangle.* Since $G \neq cor(K_3)$ and G is not a subgraph of any of G_5, G_6, H_n of Figure 2, it follows that G contains at least one of the graphs A_i $(1 \leq i \leq 8)$ shown in Figure 13 as a subgraph.

We show that each graph $J^3(A_i)$ $(1 \leq i \leq 8)$ is nonplanar. Since $J(A_1)$ is the graph obtained by adding a pendant edge to $K_{2,3}$ at a vertex of degree 3, it the follows from Lemma 2.5 that $J^3(A_1)$ is nonplanar. Moreover, for each i with $2 \leq i \leq 6$, the graph $J(A_1)$ is a subgraph of $J(A_i)$ and so $J^3(A_i)$ is nonplanar as well. The graphs $J(A_7)$, $J^2(A_7)$, and $J(A_8)$ are shown in Figure 14. Thus $J^3(A_7)$ is nonplanar by Lemma 2.4. Since $J(A_8)$ contains two disjoint subgraphs of size 3, it follows from Lemma 2.1 that $J^2(A_8)$ is nonplanar. Therefore, $J^3(G)$ is nonplanar in this subcase.

Case 5. *G is a tree.* Since every star is a subgraph of H_n of Figure 3 for some $n \geq 4$ and $\{J^k(G)\}$ does not terminate, the tree G is not a star and therefore its diameter is at least 3. Since the jump graph of every path of length 5 or more contains the graph H_1 of Figure 7 as a subgraph, it follows from Case 2 that $J^4(G)$ is nonplanar if $\operatorname{diam} G \geq 5$. Moreover, $J(P_n)$ $(n \geq 8)$ and $J^3(P_7)$ are nonplanar, while $J^3(P_6)$ is planar. Moreover,

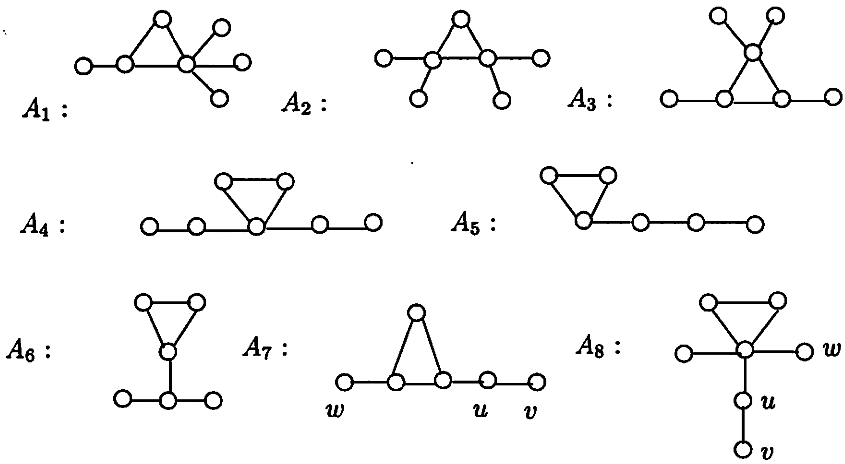


Figure 13: Graphs A_i $(1 \leq i \leq 8)$ in Subcase 4.2

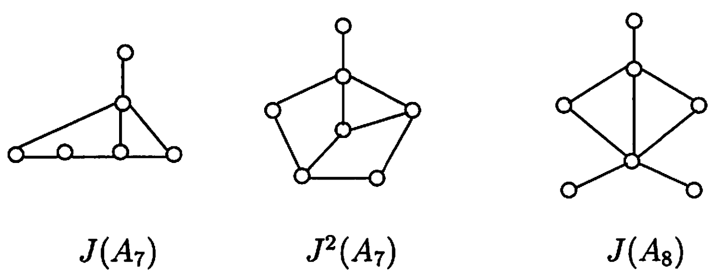


Figure 14: The graphs $J(A_7)$, $J^2(A_7)$, and $J(A_8)$ in Subcase 4.2

if $\operatorname{diam} G = 5$ and $G \neq P_6$, then $J(G)$ contains a 6-cycle and so $J^3(G)$ is nonplanar by Corollary 2.3. The graph P_6 is also the graph F_2 in Figure 6.

What remains then are trees G with $3 \leq \operatorname{diam} G \leq 4$. Assume first that $\operatorname{diam} G = 3$. Then G is a double star. Let u and v be the vertices of G that are not end-vertices. Since G is not a subgraph of G_5 or H_n of Figure 2, we may assume that $\deg v \geq 3$ and $\deg u \geq 4$. By Lemma 2.2, $J^4(G)$ is nonplanar. If $\deg v = 3$ and $\deg u > 4$, then $J(G)$ contains $K_{2,4}$ and so $J^3(G)$ is nonplanar. If $\deg v > 3$ and $\deg u \geq 4$, then $J(G)$ is nonplanar by Lemma 2.1. If $\deg v = 3$ and $\deg u = 4$, then $J(G) = K_{2,3}$, $J^2(G) = J(K_{2,3}) = C_6$, and $J^3(G) = J(C_6) = K_3 \times K_2$, which is planar. Hence the graph G is the graph F_4 of Figure 6.

Next, assume that $\operatorname{diam} G = 4$. Let $P : v_1, v_2, v_3, v_4, v_5$ be a path of length 4 in G. We consider three subcases.

Subcase 5.1. $\deg v_3 = 2$. Assume first that exactly one of v_2 and v_4 has degree 3, while the other has degree 2. Then $J(G)$ is a subgraph of the graph G_3 of Figure 2 and so $\{J^k(G)\}$ terminates. If (1) $\deg v_2 \geq 3$ and $\deg v_4 \geq 3$ or (2) $\deg v_2 \geq 4$ or $\deg v_4 \geq 4$, then $J^4(G)$ is nonplanar by Lemma 2.2. In fact, $J^3(G)$ is nonplanar, as we next show. If $\deg v_2 = 3$ and $\deg v_4 = 3$, then $J(G)$ contains a 6-cycle and so $J^3(G)$ is nonplanar by Corollary 2.3. If $\deg v_2 \geq 4$ or $\deg v_4 \geq 4$, then G contains a subgraph obtained from $K_{2,3}$ by adding a pendant edge at a vertex of degree 3. Then $J^3(G)$ is nonplanar by Lemma 2.5.

Subcase 5.2. $\deg v_3 = 3$. Assume first that $\deg v_2 = \deg v_4 = 2$. Then G is a subgraph of the graph G_6 of Figure 3 and so $\{J^k(G)\}$ terminates. Otherwise, $\deg v_2 \geq 3$ or $\deg v_4 \geq 3$. In this subcase, $J(G)$ contains $K_{2,3}$ with a pendant edge. Thus by Lemma 2.5, $J^3(G)$ is nonplanar.

Subcase 5.3. $\deg v_3 \geq 4$. Then G contains the subgraph F shown in Figure 15. Identifying v and w gives us a graph containing the graph G_4 of Figure 8 as a subgraph. By Lemma 2.7 and Case 3, $J^4(G)$ is nonplanar. On the other hand, $J^3(F)$ is planar as shown Figure 15. If $\deg v_3 \geq 5$, then $J(G)$ contain a subgraph obtained from $K_{2,3}$ by adding a pendant edge. Thus $J^3(G)$ is nonplanar by Lemma 2.5. The graph F shown in Figure 15 is the graph F_6 of Figure 6. This completes the proof. ∎

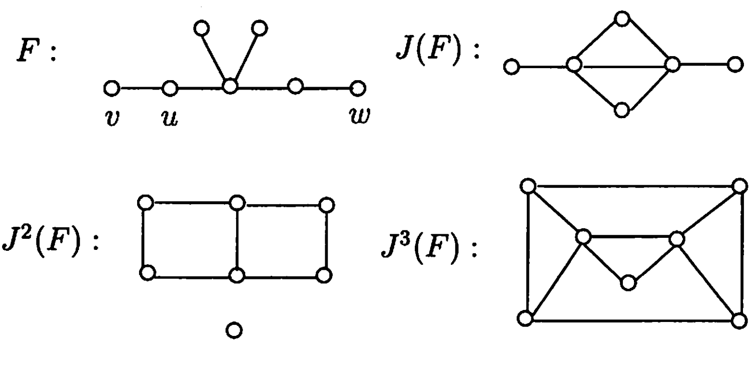


Figure 15: The subgraph F in Subcase 5.3 and its jump graphs

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References

[1] G. Chartrand, H. Gavlas, and M. Schultz, Convergent sequences of iterated H-graphs. *Discrete Math.* **147** (1995) 73-86.

[2] G. Chartrand, H. Gavlas, D. W. VanderJagt and P. Zhang, Which sequences of iterated jump graphs are planar? *Congr. Numer.* **139** (1999) 33-39.

[3] G. Chartrand, H. Hevia, E. B. Jarrett, and M. Schultz, Subgraph distance in graphs defined by edge transfers. *Discrete Math.* **170** (1997) 63-79.

[4] G. Chartrand and L. Lesniak, *Graphs & Digraphs*, third edition. Chapman & Hall, New York (1996).

[5] G. Chartrand, D. W. VanderJagt, and P. Zhang, On the planarity of iterated jump graphs. *Discrete Math.* **226** (2001) 93-106.

[6] H. Hevia, D. W. VanderJagt, and P. Zhang, On the planarity of jump graphs. *Discrete Math.* **220** (2000) 119-129.

[7] K. Kuratowski, Sur le probléme des courbes gauches en topologie. *Fund. Math.* **15** (1930) 270-283.