Degree Sum Conditions for the Hamiltonicity and Traceability of L_1 – Graphs

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Abstract

A graph G is called an L_1 – graph if, for each triple of vertices u, v, and w with $d(u,v) = 2$ and $w \in N(u) \cap N(v)$, $d(u) + d(v) \geq |N(u) \cup N(v) \cup N(w)| - 1$. Let G be a 2 – connected L_1 – graph of order n. If $\sigma_3(G) \geq n - 2$, then G is hamiltonian or $G \in \mathcal{K}$, where $\sigma_3(G) = \min\{d(u) + d(v) + d(w) : \{u, v, w\}$ is an independent set in $G\}$, $\mathcal{K} = \{G : K_{p,p+1} \subseteq G \subseteq K_p + (p+1)K_1$ for some p $\geq 2\}$. A similar result on the traceability of connected L_1 – graphs is also obtained.

1. Introduction

We consider only finite undirected graphs without loops and multiple edges. Notation and terminology not defined here follow that in [6]. If $S \subseteq V(G)$, then $N(S)$ denotes the neighbors of S, that is, the set of all vertices in G adjacent to at least one vertex in S. For a subgraph H of G and $S \subseteq V(G) - V(H)$, let $N_H(S) = N(S) \cap V(H)$ and $|N_H(S)| = d_H(S)$. If $S = \{s\}$, then $N_H(S)$ and $|N_H(S)|$ are written as $N_H(s)$ and $d_H(s)$ respectively. For disjoint subsets A, B of the vertex set $V(G)$ of a graph G, let $e(A,B)$ be the number of the edges in G that join a vertex in A and a vertex in B. A graph G is 1 – tough if $\omega(G - S) \leq |S|$ for every subset S of $V(G)$ with $\omega(G - S) > 1$, where $\omega(G - S)$ denotes the

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number of components in the graph $G - S$. For a graph G, $\sigma_3(G)$ is defined as $\min\{d(u) + d(v) + d(w) : \{u, v, w\}$ is an independent set in $G\}$ and $\mathcal{K}$ is defined as $\mathcal{K} = \{G : K_{p,p+1} \subseteq G \subseteq K_p + (p+1)K_1$ for some $p \geq 2\}$. If C is a cycle of G, let $\overrightarrow{C}$ denote the cycle C with a given orientation. For u, $v \in C$, let $\overrightarrow{C}[u, v]$ denote the consecutive vertices on C from u to v in the direction specified by $\overrightarrow{C}$. We use x^- and x^+ to denote the predecessor and successor of a vertex x on C along the orientation of C. If $A \subseteq V(C)$, then A^- and A^+ are defined as $\{v^- : v \in A\}$ and $\{v^+ : v \in A\}$ respectively. The analogous notation is used when the cycle C is replaced by a path P. A graph G is called claw – free if G has no induced subgraph isomorphic to $K_{1,3}$. For an integer i, a graph G is called an L_i – graph if $d(u) + d(v) \geq |N(u) \cup N(v) \cup N(w)| - i$, or equivalently $|N(u) \cap N(v)| \geq |N(w) - (N(u) \cup N(v))| - i$ for each triple of vertices u, v, and w with $d(u, v) = 2$ and $w \in N(u) \cap N(v)$. It can easily be verified that every claw – free graph is an L_1 – graph (see [2]).

The long time interest in claw – free graphs motivates our study of L_1 – graphs. In recent years several authors already obtained results on the hamiltonian properties of L_i – graphs. Asratian and Khachatrian [4] proved that all connected L_0 – graphs of order at least three are hamiltonian and Saito [9] shown that if a graph G is a 2 – connected L_1 – graph of diameter two then either G is hamiltonian or $G \in \mathcal{K}$. More results related to the hamiltonian properties of L_i – graphs can be found in [1], [2], [3] and [5].

Recently, Li and Schelp [7] extended Matthews and Sumner's theorems [8] on the hamiltonicity and traceability of claw – free graphs to L_1 – graphs.

Theorem 1 *[7] Let G be a 2 – connected L_1 – graph of order n. If $\delta(G) \geq (n-2)/3$, then G is hamiltonian or $G \in \mathcal{K}$.*

Theorem 2 *[7] Let G be a connected L_1 – graph of order n. If $\delta(G) \geq (n-2)/3$, then G is traceable.*

In this note, we observe that the above two theorems can be further strengthened respectively as follows.

Theorem 3 *Let G be a 2 – connected L_1 – graph of order n. If $\sigma_3(G) \geq n-2$, then G is hamiltonian or $G \in \mathcal{K}$.*

Theorem 4 *Let G be a connected L_1 – graph of order n. If $\sigma_3(G) \geq n-2$, then G is traceable.*

2. Proofs

The following result is used as a lemma in the proof of Theorem 3.

Lemma 1 *[2] If G is a 2 - connected L_1 - graph, then either G is 1 - tough or $G \in \mathcal{K}$.*

Proof of Theorem 3. Let G be a graph satisfying the conditions in Theorem 3. Suppose that G is not hamiltonian and $G \notin \mathcal{K}$. Choose a longest cycle C in G and specify an orientation of C. Then Lemma 1 implies that G is 1 - tough. Assume that H is a connected component of the graph $G[V(G) - V(C)]$ and that $N(V(H)) \cap V(C) := \{a_1, a_2, ..., a_l\}$ with $h_i a_i \in E$, where $h_i \in V(H)$ for $1 \leq i \leq l$. We also assume that $a_1, a_2, ..., a_l$ are labeled in the order of the orientation of C. Since G is 2 - connected, $l \geq 2$. Set $A := \{a_1, a_2, ..., a_l\}$ and let b_i and d_i be the predecessor and successor respectively of a_i along C, $1 \leq i \leq l$. Set $B := \{b_1, b_2, ..., b_l\}$ and $D := \{d_1, d_2, ..., d_l\}$.

Next we will prove that for each i, $1 \leq i \leq l$, $b_i d_i \in E$. Suppose not, then there exists a k, $1 \leq k \leq l$, such that $b_k d_k \notin E$. Clearly, $d(h_k, d_k) = 2$ and $a_k \in N(h_k) \cap N(d_k)$. Since G is an L_1 - graph,

$$|N(h_k) \cap N(d_k)| \geq |N(a_k) - (N(h_k) \cup N(d_k))| - 1 \geq |\{b_k, d_k, h_k\}| - 1 = 2.$$

By the choice of C, we have $N(h_k) \cap N(d_k) \cap (V(G) - V(C)) = \emptyset$. Then there exists a vertex $a_j \in N(V(H)) \cap V(C)$ such that $a_j \in N(h_k) \cap N(d_k)$.

Let X be the set $N(h_k) \cap V(C) := \{x_1, x_2, ..., x_{l_1}\}$ with the x_i's ordered with increasing index in the direction of orientation of C. Then $X \subseteq A$ and $l_1 \geq 2$. Let s_i and t_i be the predecessor and successor respectively of x_i along C, for each i, $1 \leq i \leq l_1$. Set $S := \{s_1, s_2, ..., s_{l_1}\}$, $T := \{t_1, t_2, ..., t_{l_1}\}$. Clearly, $S \cup \{h_k\}$ is an independent set in G and $N(h_k) \cap N(s_i) \cap (V(G) - V(C)) = \emptyset$, for each i, $1 \leq i \leq l_1$. Moreover, for each i, $1 \leq i \leq l_1$, $d(h_k, s_i) = 2$ and $x_i \in N(h_k) \cap N(s_i)$, so by the hypothesis of Theorem 3, we have

$$|N(h_k) \cap N(s_i)| \geq |N(x_i) - (N(h_k) \cup N(s_i))| - 1.$$

Obviously, $N_S(x_i) \subseteq N(x_i) - (N(h_k) \cup N(s_i) \cup \{h_k\})$. Thus,

$|N_S(x_i)| \leq |N(x_i) - (N(h_k) \cup N(s_i))| - 1$. Therefore,

$|N_S(x_i)| \leq |N(h_k) \cap N(s_i)| = |N_X(s_i)|$. Hence,

$e(X, S) = \sum_{i=1}^{l_1} |N_S(x_i)| \leq \sum_{i=1}^{l_1} N_X(s_i)| = e(X, S)$.

It follows, for each i, $1 \leq i \leq l_1$, that

$$N(x_i) - (N(h_k) \cup N(s_i) \cup \{h_k\}) = N_S(x_i) \subseteq S. \tag{1}$$

Similarly, for each i, $1 \leq i \leq l_1$,

$$N(x_i) - (N(h_k) \cup N(t_i) \cup \{h_k\}) = N_T(x_i) \subseteq T. \tag{2}$$

We claim that there exists an i such that $s_{i+1} \neq t_i$, where $1 \leq i \leq l_1$ and s_{l_1+1} is regarded as s_1. Suppose not, then for each i, $1 \leq i \leq l_1$, $s_{i+1} = t_i$. Clearly, for each i, $1 \leq i \leq l_1$, $N(t_i) \cap V(H) = \emptyset$, otherwise C is not of maximum length, also for any pair of i, j, $1 \leq i, j \leq l_1$ and $i \neq j$, t_i, t_j do not have neighbors in the same component of the graph $G[V(G) - V(C) - V(H)]$, otherwise C is again not of maximum length. Therefore, $G - \{x_1, x_2, ..., x_{l_1}\}$ has at least $l_1 + 1$ components, contradicting the fact that G is 1 - tough.

Without loss of generality, assume that $s_1 \neq t_{l_1}$. Observe that $s_1 \in N(t_1)$, otherwise from (2), we have $s_1 \in T$, which is impossible. Since $s_1 t_1 \in E$, $s_2 \neq t_1$. Observe again that $s_2 \in N(t_2)$, otherwise from (2), we have $s_2 \in T$, which is also impossible. Repeating this process, we have $s_j t_j \in E$, for each j, $1 \leq j \leq l_1$. This implies that $b_k d_k \in E$, a contradiction. Hence, for each i, $1 \leq i \leq l$, $b_i d_i \in E$.

Let y_1 be the first vertex on $\overrightarrow{C}[d_1, b_2]$ such that $b_1 y_1 \notin E$. The existence of y_1 is guaranteed by the fact that $b_2 \notin N(b_1)$. Moreover, we have $y_1 \notin N(a_1)$. Suppose not, then, by the choice of C, we have $h_1 y_1 \notin E$ otherwise G has a longer cycle than C. Thus $d(h_1, y_1) = 2$ and $a_1 \in N(h_1) \cap N(y_1)$. Since G is an L_1 - graph,

$$|N(h_1) \cap N(y_1)| \geq |N(a_1) - (N(h_1) \cup N(y_1))| - 1 \geq |\{b_1, y_1, h_1\}| - 1 = 2.$$

By the choice of C, we have $N(y_1) \cap N(h_1) \cap (V(G) - V(C)) = \emptyset$. Then there exists a vertex $a_j \in N_C(H)$ such that $a_j \in N(h_1) \cap N(y_1)$. Thus, G has a cycle

$$h_1 a_j \overrightarrow{C}[y_1, b_j] \overrightarrow{C}[d_j, b_1] \overleftarrow{C}[y_1^-, a_1] h_1$$

which is longer than C, a contradiction. Let y_2 be the first vertex in $\overrightarrow{C}[d_2, b_3]$ such that $b_2 y_2 \notin E$. As before, y_2 indeed exists and $y_2 \notin N(a_2)$. Let q be any vertex in H. Then the choice of C implies that $\{q, y_1, y_2\}$ is an independent set in G. Set

$$V_1 = \overrightarrow{C}[y_1, b_2],$$

$$V_2 = \overrightarrow{C}[y_2, b_3],$$

$$V_3 = \vec{C}[a_3, b_4],$$

$$\dots$$

$$V_{l-1} = \vec{C}[a_{l-1}, b_l],$$

$$V_l = \vec{C}[a_l, b_1],$$

$$V_{l+1} = \vec{C}[a_1, y_1^-],$$

$$V_{l+2} = \vec{C}[a_2, y_2^-], \text{ and}$$

$$V_{l+3} = V(G) - V(C).$$

For each vertex w in G, we simplify some notation letting $N_i(w)$ replace $N_{V_i}(w)$, $d_i(w) = |N_i(w)|$ and $(N_i(w))^- = N_i^-(w)$.

Clearly, $d_1(q) = 0$. Also $N_1^-(y_1) \cap N_1(y_2) = \emptyset$, otherwise there is a cycle in G which is longer than C. Therefore,

$d_1(q) + d_1(y_1) + d_1(y_2) = |N_1^-(y_1)| + |N_1(y_2)| = |N_1^-(y_1) \cup N_1(y_2)| \leq |V_1 - \{ b_2 \}| = |V_1| - 1.$

Similarly, $d_2(q) + d_2(y_1) + d_2(y_2) \leq |V_2| - 1.$

For each i, $3 \leq i \leq l$, it follows that $y_j a_i \notin E$, $y_j d_i \notin E$, $y_j d_i^+ \notin E$, where $j = 1$ or 2, otherwise there is a cycle in G which is longer than C. For each i, $3 \leq i \leq l$, $N_i(y_1) \cap N_i^-(y_2) = \emptyset$, otherwise we can again find a cycle in G which is longer than C. Therefore, for each i, $3 \leq i \leq l$,

$d_i(q) + d_i(y_1) + d_i(y_2) \leq 1 + |N_i(y_1)| + |N_i^-(y_2)| = 1 + |N_i(y_1) \cup N_i^-(y_2)| \leq |V_i - \{ d_i \}| = |V_i| - 1.$

Clearly, $N(y_2) \cap V_{l+1} = \emptyset$ and $N(q) \cap V_{l+1} \subseteq \{ a_1 \}$, for otherwise a cycle longer than C can be found. Therefore,

$d_{l+1}(q) + d_{l+1}(y_1) + d_{l+1}(y_2) \leq |V_{l+1}|.$

Similarly, $d_{l+2}(q) + d_{l+2}(y_1) + d_{l+2}(y_2) \leq |V_{l+2}|.$

Notice that $q \notin N_{l+3}(q) \cup N_{l+3}(y_1) \cup N_{l+3}(y_2)$. Also notice that $N_{l+3}(q) \cap N_{l+3}(y_1) = \emptyset$, $N_{l+3}(y_1) \cap N_{l+3}(y_2) = \emptyset$, and $N_{l+3}(y_2) \cap N_{l+3}(q) = \emptyset$, otherwise there are again cycles in G which are longer than C. Therefore,

$$d_{l+3}(q) + d_{l+3}(y_1) + d_{l+3}(y_2) \leq |V_{l+3}| - 1.$$

Hence, $n - 2 \leq d(q) + d(y_1) + d(y_2) \leq \sum_{i=1}^{l+3}(d_i(q) + d_i(y_1) + d_i(y_2))$ $\leq n - l - 1$. A final contradiction. QED.

Proof of Theorem 4. Suppose G is a graph satisfying the conditions in Theorem 4 that is not traceable. Let P be a longest path in G with end–vertices a, b and the orientation of P is specified from a to b. Since G is not traceable, $V(G) - V(P) \neq \emptyset$. Assume that H is a connected component of the graph $G[V(G) - V(P)]$ and that $N(V(H) \cap V(P) := \{a_1, a_2, ..., a_l\}$ with $h_i a_i \in E$, where $h_i \in V(H)$ for $1 \leq i \leq l$. We also assume that $a_1, a_2, ..., a_l$ are labeled in the order of the orientation of P. Set $A := \{a_1, a_2, ..., a_l\}$. Clearly, $l \geq 1$ and a, $b \notin A$. Let b_i and d_i be the predecessor and successor respectively of a_i along P, $1 \leq i \leq l$. Set $B := \{b_1, b_2, ..., b_l\}$ and $D := \{d_1, d_2, ..., d_l\}$.

If $l = 1$, then $b_1 \in N(d_1)$. Otherwise since $d(h_1, d_1) = 2$, $a_1 \in N(h_1) \cap N(d_1)$ and G is an L_1 – graph, we have

$$|N(h_1) \cap N(d_1)| \geq |N(a_1) - (N(h_1) \cup N(d_1))| - 1 \geq |\{d_1, h_1, b_1\}| - 1 = 2.$$

By the choice of P, we have $N(h_1) \cap N(d_1) \cap (V(G) - V(P)) = \emptyset$. Thus $N(h_1) \cap N(d_1) \cap V(P) \neq \emptyset$, contradicting to the assumption of $l = 1$. Since P is longest path in G, $b_1 b \notin E$, otherwise G has a path which is longer than P. Let v be the first vertex on $\overrightarrow{P}[a_1, b]$ such that $b_1 v \notin E$.

Next we will count the degree sum of vertices a, v and h_1 and arriving at a contradiction. Set

$$V_1 = \overrightarrow{P}[a, b_1],$$

$$V_2 = \overrightarrow{P}[a_1, v^-],$$

$$V_3 = \overrightarrow{P}[v, b],$$

$$V_4 = V(G) - V(P).$$

Since $l = 1$, we have $ah_1 \notin E$ and $vh_1 \notin E$. Moreover, $av \notin E$, otherwise G has a path which is longer than P. Thus $\{a, v, h_1\}$ is an independent set in G.

Clearly, $d_1(h_1) = 0$. Also $N_1^-(a) \cap N_1(v) = \emptyset$, otherwise there is a path in G which is longer than P. Therefore,

$d_1(a) + d_1(v) + d_1(h_1) = |N_1^-(a)| + |N_1(v)| = |N_1^-(a) \cup N_1(v)| \leq |V_1 - \{ b_1 \}| = |V_1| - 1.$

Clearly, $d_2(h_1) = 1$. Also $d_2(a) = 0$, otherwise G has a path which is longer than P. Notice that $a_1 \notin N(v)$. Otherwise since $d(h_1, v) = 2$, $a_1 \in N(h_1) \cap N(v)$ and G is an L_1 - graph, we have

$$|N(h_1) \cap N(v)| \geq |N(a_1) - (N(h_1) \cup N(v))| - 1 \geq |\{ v, h_1, b_1 \}| - 1 = 2.$$

By the choice of P, we have $N(h_1) \cap N(v) \cap (V(G) - V(P)) = \emptyset$. Thus $N(h_1) \cap N(v) \cap V(P) \neq \emptyset$, contradicting again to the assumption of $l = 1$. Therefore,

$$d_2(a) + d_2(v) + d_2(h_1) \leq |V_2|.$$

Clearly, $d_3(h_1) = 0$. Notice that $b \notin N(a)$, otherwise G has a path which is longer than P. Moreover, $N_3(a) \cap N_3^-(v) = \emptyset$, otherwise G again has a path which is longer than P. Therefore,

$d_3(a) + d_3(v) + d_3(h_1) = |N_3(a)| + |N_3^-(v)| = |N_3(a) \cup N_3^-(v)| \leq |V_3 - \{ b \}| = |V_3| - 1.$

Clearly, $N_4(a)$, $N_4(v)$, and $N_4(h_1)$ are pairwise disjoint and $h_1 \notin N_4(a) \cup N_4(v) \cup N_4(h_1)$. Therefore,

$$d_4(a) + d_4(v) + d_4(h_1) \leq |V_4| - 1.$$

Hence, $n-2 \leq d(a)+d(v)+d(h_1) \leq \sum_{i=1}^{4}(d_i(a)+d_i(v)+d_i(h_1)) \leq n-3$, a contradiction.

If $l \geq 2$, using arguments similar to that in the proof of Theorem 3, we have, for each i, $1 \leq i \leq l$, that $b_i d_i \in E$. Furthermore, there exists a vertex v in $\overrightarrow{P}[d_1, b_2]$ such that $v \notin N(b_1)$, $v \notin N(a_1)$, and $\overrightarrow{P}[a_1, v^-] \subseteq N(b_1)$. Set

$$V_1 = \overrightarrow{P}[a, b_1],$$

$$V_2 = \overrightarrow{P}[a_1, v^-],$$

$$V_3 = \overrightarrow{P}[v, b],$$

$$V_4 = V(G) - V(P).$$

Clearly, $ah_1 \notin E$ and $vh_1 \notin E$. Moreover, $av \notin E$, otherwise G has a path which is longer than P. Thus $\{a, v, h_1\}$ is an independent set in G.

By similar arguments as in the case of $l = 1$, we have

$d_1(a) + d_1(v) + d_1(h_1) \leq |V_1| - 1.$

$d_2(a) + d_2(v) + d_2(h_1) \leq |V_2|.$

$d_4(a) + d_4(v) + d_4(h_1) \leq |V_4| - 1.$

Clearly, $b \notin N(a)$. We also have $N_3(a) \cap N_3^-(v) = \emptyset$, $N_3^-(v) \cap N_3(h_1) = \emptyset$, and $N_3(h_1) \cap N_3(a) = \emptyset$. Otherwise there are paths in G which are longer than P. Therefore,

$d_3(a)+d_3(v)+d_3(h_1) = |N_3(a)|+|N_3^-(v)|+|N_3(h_1)| = |N_3(a) \cup N_3^-(v) \cup N_3(h_1)| \leq |V_3 - \{b\}| = |V_3| - 1.$

Hence, $n-2 \leq d(a)+d(v)+d(h_1) \leq \sum_{i=1}^{4}(d_i(a)+d_i(v)+d_i(h_1)) \leq n-3$, a contradiction, which completes the proof. QED.

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