On a Minimum Cutset of Strongly k-Extendable Graphs

N. Ananchuen
Department of Mathematics
Silpakorn University
Nakorn Pathom 73000 Thailand

Abstract. Let G be a simple connected graph on 2n vertices with a perfect matching. For a positive integer k, $1 \le k \le n - 1$, G is *k-extendable* if for every matching M of size k in G, there is a perfect matching in G containing all the edges of M. For an integer k, $0 \le k \le n - 2$, G is *strongly k-extendable* if G - {u, v} is k-extendable for every pair of vertices u and v of G. The problem that arises is that of characterizing k-extendable graphs and strongly k-extendable graphs. The first of these problems has been considered by several authors while the latter has been recently investigated. In this paper, we focus on a minimum cutset of strongly k-extendable graphs. For a minimum cutset S of a strongly k-extendable graph G, we establish that if $|S| = k + t$, for an integer $t \ge 3$, then the independence number of the induced subgraph G[S] is at most 2 or at least $k + 5 - t$. Further, we present an upper bound on a number of components of G – S.

1. Introduction

All graphs considered in this paper are finite, connected, loopless and have no multiple edges. For the most part our notation and terminology follows that of Bondy and Murty [3]. Thus G is a graph with vertex set V(G), edge set E(G), $\nu(G)$ vertices, $\varepsilon(G)$ edges, minimum degree $\delta(G)$, connectivity $\kappa(G)$ and independence number $\alpha(G)$. For $V' \subseteq V(G)$, G[V′] denotes the subgraph induced by V′. Similarly G[E′] denotes the subgraph induced by the edge set E′ of G. $N_G(u)$ denotes the neighbour set of u in G and $\overline{N}_G(u)$ the non-neighbours of u. Note that $\overline{N}_G(u) = V(G) \setminus (N_G(u) \cup \{u\})$. The *join* $G \vee H$ of disjoint graphs G and H is the graph obtained from $G \cup H$ by joining each vertex of G to each vertex of H.

A *matching* M in G is a subset of E(G) in which no two edges have a vertex in common. M is a *maximum matching* if $|M| \ge |M'|$ for any other matching M′ in G. A vertex v is *saturated* by M if some edge of M is incident to v; otherwise, v is said to be *unsaturated*. A matching M is *perfect* if it saturates every vertex of the graph. For simplicity we let V(M) denote the vertex set of the subgraph G[M] induced by M.

JCMCC **45** (2003), pp. 63-78

Let G be a simple connected graph on 2n vertices with a perfect matching. For a given positive integer k, $1 \le k \le n - 1$, G is *k-extendable* if for every matching M of size k in G, there exists a perfect matching in G containing all the edges of M. For convenience, a graph with a perfect matching is said to be 0-extendable. For an integer k, $0 \le k \le n - 2$, we say that G is *strongly k-extendable* or simply *k*-extendable* if for every pair of vertices u and v of G, G - {u, v} is k-extendable. A graph G is *bicritical* if G - {u, v} has a perfect matching for every pair of vertices u and v. Clearly, 0*-extendable graphs are bicritical and a concept of k*-extendable graphs is a generalization of bicritical graphs. Further, k*–extendable graphs are non–bipartite.

A number of authors have studied k-extendable graphs. Excellent surveys are the papers of Plummer [10, 11]. Lovasz [4], Lovasz and Plummer [5, 6] and Plummer [7] have studied k*-extendable graphs for k = 0 (bicritical graphs). For $k \ge 1$, k*-extendable graphs have been recently investigated by the author [1, 2]. In [1] we established a relationship between k-extendable and k*-extendable graphs. The results are:

Theorem 1.1: If G is a (k + 2)-extendable non-bipartite graph on 2n vertices, $0 \le k \le n - 3$, then G is k*-extendable. ❐

Theorem 1.2: If G is a k*-extendable graph on 2n vertices, $0 \le k \le n - 2$, then G is t-extendable for $0 \le t \le k + 1$. ❐

In [7] Plummer established the connectivity of a k-extendable graph. He proved the following result:

Theorem 1.3: Let G be a k-extendable graph on 2n vertices, $1 \le k \le n - 1$. Then
(i) G is (k - 1)-extendable;
(ii) G is (k + 1)-connected. ❐

He also established, in [9], that

Theorem 1.4: Suppose k is a positive integer. Suppose further that G is a k–extendable graph and S is a vertex cutset of G with $|S| = k + 1$, then
(i) $G = K_{k+1, k+1}$, or
(ii) if k = 1, then G – S has at least 2 even components, but no odd components, or exactly 2 odd components, but no even components;
(iii) if $k \ge 3$ and k is odd, then G – S has exactly 2 odd components and no even components, or exactly 2 even components and no odd components; or
(iv) if $k \ge 2$ and k is even, then G – S has exactly 2 components, one of which is odd and the other, even. ❐

A similar result to Theorem 1.3 for k*-extendable graphs was proved by Ananchuen [1, 2].

Theorem 1.5: If G is a k*-extendable graph on 2n vertices, $1 \leq k \leq n-2$, then

(i) G is (k - 1)*-extendable;

(ii) G is (k + 3)-connected. ❑

Theorems 1.1 and 1.2 indicate that k–extendable non–bipartite graphs and k*–extendable graphs are closely related. So we expect some similar results to Theorem 1.4 for k*–extendable graphs. Hence, in this paper, we focus our attention on a minimum cutset of k*-extendable graphs.

For a minimum cutset S of a k*-extendable graph G, we establish, in Section 2, that if $|S| = k + t$, for an integer $t \geq 3$, then the independence number of the induced subgraph G[S] is at most 2 or at least $k + 5 - t$. Further, in Section 3, we present some results concerning an upper bound on a number of components of G – S. In fact, we prove that if G is a k*-extendable graph on 2n vertices, $2 \leq k \leq n - 3$ and S is a minimum cutset with $|S| \leq 2k + 1$, then the number of components of G – S is at most $|S| - |M(S)| - k - 1$, where M(S) denotes a maximum matching in G[S].

We conclude this section by stating results that we make use of in our work. The following result is a very useful tool in establishing our results proved by Plummer [8].

Theorem 1.6: Let G be k-connected, $k \geq 1$, let S be a minimum cutset in G, and let C be any component of G – S. Then given any subset $S' \subseteq S$, $S' \neq \emptyset$ and $|S'| \leq |V(C)|$, there exists a complete matching of S′ into V(C). ❑

Let M(S) denote a maximum matching in G[S]. The next result, established by the author [1], provides a necessary and sufficient condition for k*-extendable graphs.

Theorem 1.7: Let G be a graph on 2n vertices. For $0 \leq k \leq n - 2$, G is k*-extendable if and only if for all $S \subseteq V(G)$

$$o(G - S) \leq \begin{cases} |S| - 2t, & \text{for } |S| \leq 2k+1 \\ |S| - 2t - 2, & \text{for } |S| \geq 2k+2 \end{cases}$$

where $t = \min\{|M(S)|, k\}$. ❑

A characterization of (n – 2)*-extendable graphs on 2n vertices was given in [2] by the author.

Theorem 1.8: G is an (n – 2)*-extendable graph on $2n \geq 4$ vertices if and only if G is K_{2n} ❑

2. The independence number of a minimum cutset

In this section, we investigate the independence number of a minimum cutset of strongly k-extendable graphs. By Theorem 1.8, the only $(n-2)^*$-extendable graph on 2n vertices is K_{2n} which is clearly $(2n-1)$-connected. Hence, in the rest of this paper, we will restrict our attention to k^*-extendable graphs on 2n vertices for $0 \le k \le n-3$. It follows directly from the definition of bicritical graphs (0^*-extendable) that such graphs are 2-connected. A graph $K_2 \vee 2K_{2r}$ for any positive integer r is an example of a bicritical graph which is 2-connected. For $1 \le k \le n-3$, it follows from Theorem 1.5 (ii) that k^*-extendable graphs on 2n vertices are $(k+3)$-connected. Our first result establishes the independence number of a minimum cutset of k^*-extendable graphs.

Theorem 2.1: Let G be a k^*-extendable graph on 2n vertices with $0 \le k \le n-3$ and suppose $S \subseteq V(G)$ is a minimum cutset of G with $|S| = k + t$ for $t \ge 3$, then $\alpha(G[S]) \ge k + 5 - t$ or $\alpha(G[S]) \le 2$.

Proof: Suppose to the contrary that there is a minimum cutset S of G with $|S| = k + t$, $t \ge 3$ and $3 \le \alpha(G[S]) \le k + 4 - t$. Then $k \ge t - 1$ and $2|M(S)| \ge (k + t) - (k + 4 - t) = 2t - 4$. Thus $|S| \ge 2t - 1$ and $|M(S)| \ge t - 2$. Let M be a matching of size $t - 2$ in G[S] and let u and v be vertices of $S \setminus V(M)$. Such vertices exist since $|S| \ge 2t - 1$. Put

$$S_1 = S \setminus (V(M) \cup \{u, v\}).$$

Then $|S_1| = (k+t) - 2(t-2) - 2 = k - t + 2 \ge 1$.

Let $S_1 = \{x_1, x_2, \ldots, x_{k-t+2}\}$. Further, let $C_1, C_2, \ldots, C_r$ be components of $G - S$. We claim that $|V(C_i)| \le k - t + 1$ for all i, $1 \le i \le r$. Suppose to the contrary that there exists a component C_j with $|V(C_j)| \ge k - t + 2$. By Theorem 1.6, there is a matching M_1 which matches vertices of S_1 into $V(C_j)$. Let $M_1 = \{x_1y_1, x_2y_2, \ldots, x_{k-t+2}y_{k-t+2}\}$. Clearly, $M \cup M_1$ is a matching of size $(t-2) + (k-t+2) = k$. Since $G - \{u, v\}$ has a perfect matching containing all the edges of $M \cup M_1$, $C_j \setminus (V(M_1))$ is an even component of $G - (S \cup V(M_1))$. Now x_1 must be adjacent to some vertex $w_1 \in V(C_i)$ for some $i \ne j$ since S is a minimum cutset. Consider the matching $M_2 = (M \cup M_1 \cup \{x_1w_1\}) \setminus \{x_1y_1\}$. Clearly, $|M_2| = k$. Since M_2 covers $S \setminus \{u, v\}$ and $G - (S \cup V(M_2))$ contains $C_j \setminus V(M_2)$ as an isolated odd component, M_2 does not extend to a perfect matching in $G - \{u, v\}$, a contradiction. Hence, $|V(C_i)| \le k - t + 1$ for all i, $1 \le i \le r$.

Next we let $V(C_1) = \{w_1, w_2, \ldots, w_m\}$ where $m = |V(C_1)|$. By Theorem 1.6, there is a matching M_3 which matches vertices of $V(C_1)$ into S_1. Let this matching be $\{x_1w_1, x_2w_2, \ldots, x_mw_m\}$. Clearly, $|S_1 \setminus V(M_3)| = k - t + 2 - m \ge 1$. Suppose $\left|\bigcup_{i=2}^{r} V(C_i)\right| \ge k - t + 3 - m$. Then, in view of Theorem 1.6, there is a matching M_4 of size $k - t + 3 - m$ which matches vertices of $\{x_m, x_{m+1}, \ldots, x_{k-t+2}\}$

into $\bigcup_{i=2}^{r} V(C_i)$. Let $x_m z \in M_4$ where $z \in \bigcup_{i=2}^{r} V(C_i)$. Now $M_5 = (M \cup (M_3 \setminus \{x_m w_m\}) \cup M_4)$ is a matching of size $(t-2) + (m-1) + (k-t+3-m) = k$ in $G - \{u, v\}$ which does not extend to a perfect matching in $G - \{u, v\}$ since M_5 covers $S \setminus \{u, v\}$ and $G - (S \cup V(M_5))$ contains w_m as an isolated vertex. Thus $\left|\bigcup_{i=2}^{r} V(C_i)\right| \le k - t + 2 - m$. But then

$$2n = \nu(G) = |S| + \left|\bigcup_{i=1}^{r} V(C_i)\right| \le k + t + m + (k - t + 2 - m) = 2k + 2 \le 2n - 4,$$

a contradiction. This completes the proof of our theorem. ❑

Remark 2.1: Theorem 2.1 holds for 0*-extendable graphs with a minimum cutset of order 2.

The next result follows directly from the proof of Theorem 2.1.

Corollary 2.2: Let G be a k*-extendable graph on 2n vertices with $2 \le k \le n - 3$ and suppose $S \subseteq V(G)$ is a minimum cutset of G with $|S| = k + t$ for $3 \le t \le k + 1$, then $|M(S)| \le t - 3$. ❑

As a consequence of Theorem 1.5 (ii) and Corollary 2.2, we have the following corollary:

Corollary 2.3: Let G be a k*-extendable graph on 2n vertices; $2 \le k \le n - 3$. If $S \subseteq V(G)$ is a cutset of G with $|S| = k + 3$, then S is independent. ❑

Theorem 1.1 together with Theorem 2.1 yields the following corollary:

Corollary 2.4: Let G be a k-extendable non–bipartite graph on 2n vertices with $2 \le k \le n - 1$ and suppose $S \subseteq V(G)$ is a minimum cutset of G with $|S| = k + t - 2$ for $t \ge 3$, then $\alpha(G[S]) \ge k + 3 - t$ or $\alpha(G[S]) \le 2$. ❑

We conclude this section by establishing a necessary condition, in terms of connectivity, for k*-extendable graphs which are locally connected. A graph G is said to be *locally connected* if for every vertex u of G, the induced subgraph $G[N_G(u)]$ is connected.

Theorem 2.5: Let G be a k*-extendable graph on 2n vertices with $2 \le k \le n - 3$. If G is locally connected, then G is (k+4)-connected.

Proof: Suppose to the contrary that G is not (k + 4)- connected. By Theorem 1.5(ii), $\kappa(G) = k + 3$. Let S be a cutset of order k + 3 of G. Then S is independent by Corollary 2.3. But then $G[N_G(u)]$ is disconnected for any vertex

$u \in S$, contradicting the locally connected of G. Hence, G is $(k + 4)$-connected as required. ❑

Remark 2.2: (1) For an odd integer $n \geq 5$, $G_1 = K_2 \vee 2K_{n-1}$ and $G_2 = K_4 \vee (K_{n-1} \cup K_{n-3})$ are k^*-extendable for $k = 0$ and 1, respectively. Clearly, G_1 and G_2 are locally connected but $\kappa(G_1) = 2 < 4$ and $\kappa(G_2) = 4 < 5$. Hence, the lower bound on k in Theorem 2.5 is best possible.

(2) Theorem 2.5 is best possible in the sense that there exists a graph G on 2n vertices which is k^*-extendable, locally connected and $\kappa(G) = k + 4$. Such a graph is $(K_1 \vee \overline{K}_{k+3}) \vee (K_{2k} \cup K_{k+4})$.

3. The structure of G – S

In this section, we establish some results concerning an upper bound on the number of components of $G - S$, denoted by $\omega(G - S)$, when S is a minimum cutset of a k^*-extendable graph G. We begin with a minimum cutset of order at most $2k + 1$.

Theorem 3.1: Let G be a k^*-extendable graph on 2n vertices with $2 \leq k \leq n - 3$ and let S be a minimum cutset of G and M(S) a maximum matching in G[S]. If $|S| \leq 2k + 1$, then $2 \leq \omega(G - S) \leq |S| - |M(S)| - k - 1$.

Proof: Clearly, since S is a cutset, $\omega(G - S) \geq 2$. Now we suppose to the contrary that $\omega(G - S) \geq |S| - |M(S)| - k$. Since G is k^*-extendable and S is a minimum cutset, by Corollary 2.2, $|M(S)| \leq |S| - k - 3 \leq (2k + 1) - k - 3 = k - 2$. Thus, $|S| - 2|M(S)| \geq k - |M(S)| + 3$. Let $x, y \in V(G) \setminus S$. Since $S \setminus V(M(S))$ is independent and G is k^*-extendable, $\nu(G - (S \cup \{x, y\})) \geq |S \setminus V(M(S))| = |S| - 2|M(S)|$. Thus $\nu(G - S) \geq |S| - 2|M(S)| + 2$. Now let $C_1, C_2, \ldots, C_r$ be components of $G - S$. Clearly, $r \geq |S| - |M(S)| - k \geq 3$.

We claim that there is a subset X of $\bigcup_{i=1}^{r} V(C_i)$ of cardinality $k - |M(S)| \geq 2$ which $G - (S \cup X)$ contains at least $|S| - |M(S)| - k - 1 \geq 2$ odd components. Suppose there is no such subset. Among subsets of $\bigcup_{i=1}^{r} V(C_i)$ with cardinality $k - |M(S)|$, let A be a subset of $\bigcup_{i=1}^{r} V(C_i)$ with $|A| = k - |M(S)|$ and $o(G - (S \cup A))$ is as large as possible. Notice that $\nu(G - (S \cup A)) \geq |S| - 2|M(S)| + 2 - (k - |M(S)|) = |S| - |M(S)| - k + 2$. Suppose $\omega(G - (S \cup A)) = 1$. This implies that $G - (S \cup A)$ is connected and then there exists a component of $G - S$, C_1 say, which $V(C_1) \setminus A \neq \varnothing$ and $V(C_i) \cap A = V(C_i)$; $2 \leq i \leq r$. Since $\nu(G - (S \cup A)) \geq |S| - |M(S)| - k + 2$, $|V(C_1) \setminus A| \geq |S| - |M(S)| - k + 2$. Let $x_1, x_2, \ldots, x_{|S| - |M(S)| - k - 1} \in V(C_1) \setminus A$ and $y_i \in V(C_i) \cap A$, $2 \leq i \leq |S| - |M(S)| - k$. Put

$$A_1 = (A \cup \{x_1, x_2, \ldots, x_{|S| - |M(S)| - k - 1}\}) \setminus \{y_2, y_3, \ldots, y_{|S| - |M(S)| - k}\}.$$

Clearly, $|A_1| = |A|$ and $G - (S \cup A_1)$ contains at least $|S| - |M(S)| - k - 1 \geq 2$ odd components. This contradicts the choice of A. Hence, $\omega(G - (S \cup A)) \geq 2$. Now we suppose that $G - (S \cup A)$ contains only odd components. Since $o(G - (S \cup A)) \leq |S| - |M(S)| - k - 2$, there are at least 2 components of $G - S$, C_j and $C_{j'}$ say, with $V(C_i) \cap A = V(C_i)$ for $i = j, j'$. Further, there exists an odd component of $G - (S \cup A)$, H_1 say, which $v(H_1) \geq 3$. Let $a_1, a_2 \in V(H_1)$, $b_1 \in V(C_j)$ and $b_2 \in V(C_{j'})$. Put $A_2 = (A \cup \{a_1, a_2\}) \setminus \{b_1, b_2\}$. Clearly, $|A_2| = |A|$ and $o(G - (S \cup A_2)) = o(G - (S \cup A)) + 2$, a contradiction. Thus $G - (S \cup A)$ contains at least one even component. Suppose there is a component of $G - S$, $C_{j''}$ say, with $V(C_{j''}) \cap A = V(C_{j''})$. Let $w \in V(C_{j''})$ and $z \in V(H_j)$ for some an even component H_j of $G - (S \cup A)$. Then $A_3 = (A \cup \{z\}) \setminus \{w\}$ has the same cardinality with A and $o(G - (S \cup A_3)) = o(G - (S \cup A)) + 2$, a contradiction. Hence, $V(C_j) \setminus A \neq \varnothing$ for all j, $1 \leq j \leq r$. Consequently, $\omega(G - (S \cup A)) = \omega(G - S) = r$ and $G - (S \cup A)$ contains at least 2 even components.

Let $W_1, W_2, \ldots, W_t$ be odd components of $G - (S \cup A)$ and $W_{t+1}, W_{t+2}, \ldots, W_r$ be even components of $G - (S \cup A)$ where $t \leq |S| - |M(S)| - k - 2$. Without any loss of generality, we may assume that $V(W_i) = V(C_i) \setminus A$; $1 \leq i \leq r$. Suppose $V(C_{t+1}) \cap A \neq \varnothing$. Let $w' \in V(C_{t+1}) \cap A$ and $z' \in V(W_{t+2})$. Put $A_4 = (A \cup \{z'\}) \setminus \{w'\}$. Then $|A_4| = |A|$ and $o(G - (S \cup A_4)) = o(G - (S \cup A)) + 2$, contradicting the choice of A. Thus, $V(C_{t+1}) \cap A = \varnothing$. Similarly, $V(C_i) \cap A = \varnothing$, $t + 2 \leq i \leq r$. This implies that $V(W_i) = V(C_i)$; $t + 1 \leq i \leq r$. Now we will show that $|V(C_i) \cap A| \leq 1$, $1 \leq i \leq t$. Suppose there is an odd component W_j, $1 \leq j \leq t$, which $|V(C_j) \cap A| \geq 2$. Let $w_1, w_2 \in V(C_j) \cap A$, $z_1 \in V(W_{t+1})$, $z_2 \in V(W_{t+2})$. Then $A_5 = (A \cup \{z_1, z_2\}) \setminus \{w_1, w_2\}$ has the same cardinality with A and $o(G - (S \cup A_5)) = o(G - (S \cup A)) + 2$, a contradiction. Hence, $|V(C_i) \cap A| \leq 1$, $1 \leq i \leq t$. Now $k - |M(S)| = |A| = \sum_{i=1}^{t} |V(C_i) \cap A| \leq t \leq |S| - |M(S)| - k - 2$. Thus $|S| \geq 2k + 2$. This contradicts our assumption on $|S|$ and proves our claim.

Now let B be a subset of $\bigcup_{i=1}^{r} V(C_i)$ with $|B| = k - |M(S)|$ and $o(G - (S \cup B)) \geq |S| - |M(S)| - k - 1$. Since $|S| - 2|M(S)| \geq k - |M(S)| + 3$, in view of Theorem 1.6, there is a complete matching F of size $k - |M(S)|$ joining vertices of B to vertices of $S' \subseteq S \setminus V(M(S))$. Clearly, $|S| - (2|M(S)| + |S'|) \geq 3$. Let $c_1, c_2 \in S \setminus (V(M(S)) \cup S')$. Then $F \cup M(S)$ is a matching of size $k - |M(S)| + |M(S)| = k$ which does not extend to a perfect matching in $G - \{c_1, c_2\}$ since $S'' = S \setminus (V(M(S)) \cup S' \cup \{c_1, c_2\}) \subseteq V(G - (V(M(S) \cup F) \cup \{c_1, c_2\})$ of order $|S| - (2|M(S)| + k - |M(S)| + 2) = |S| - |M(S)| - k - 2$ and $G - (V(M(S) \cup F) \cup \{c_1, c_2\} \cup S'') = G - (S \cup B)$ contains at least $|S| - |M(S)| - k - 1$ odd components. This contradicts the k*-extendablity of G and completes the proof of our theorem. ❑

Corollary 3.2: Let G be a k*-extendable graph on 2n vertices with $2 \leq k \leq n - 3$. Let S be a minimum cutset of order at most $2k + 1$ which S is independent. Then

$$o(G-S) \leq \begin{cases} |S| - k - 2, & \text{for k is even} \\ |S| - k - 1, & \text{for k is odd.} \end{cases}$$

Proof: By Theorem 3.1, $o(G-S) \leq \omega(G-S) \leq |S| - k - 1$. Thus we only need to prove the case k is even. Suppose k is even and

$$o(G-S) = |S| - k - 1.$$

Since $\nu(G)$ is even, $|S|$ and $|S| - k - 1$ must have the same parity. This implies that $k + 1$ is even and hence k is odd, a contradiction. This completes the proof of our corollary. ❑

Remark 3.1: Let s and k be positive integers with $k + 3 \leq s \leq 2k + 1$. Let $G_1 = \overline{K}_s \vee (s-k-1)K_{2s+1}$ for an odd $k \geq 3$ and $G_2 = \overline{K}_s \vee (K_{2s} \cup (s-k-2)K_{2s+1})$ for an even $k \geq 2$. It is not difficult to show that G_1 and G_2 are both k^*-extendable. Clearly, $V(\overline{K}_s)$ is a cutset of G_i, $i = 1, 2$ and $G_1 - S$ and $G_2 - S$ contain exactly $s - k - 1$ and $s - k - 2$ odd components, respectively. Thus Corollary 3.2 is best possible.

The next corollary follows immediately from Theorem 3.1, corollaries 2.3 and 3.2.

Corollary 3.3: Let G be a k^*-extendable graph on 2n vertices with $2 \leq k \leq n - 3$. Suppose S is a cutset of G with $|S| = k + 3$. Then $G - S$ contains exactly 2 components. Further,

(i) If k is odd, then both components of $G - S$ are odd or even.

(ii) If k is even, then one of components of $G - S$ is odd and the other is even. ❑

We make an observation here that $k + 3$ is the smallest order of a cutset of k^*-extendable graphs for $1 \leq k \leq n - 3$. Corollary 3.3 presents the number of components of $G - S$ when S is a cutset of order $k + 3$ of a k^*-extendable graph G for $2 \leq k \leq n - 3$. Our next two lemmas concern a similar result for $k = 0$ and 1. Note that 0^*-extendable graphs are 2 connected and 1^*-extendable graphs are 4-connected.

Lemma 3.4: Let G be a 0^*-extendable graph on $2n \geq 4$ vertices. Suppose S is a cutset of G with $|S| = 2$. Then $G - S$ contains at least 2 even components and no odd components.

Proof: It follows directly from the definition of 0^*-extendable graphs and the fact that $|S|$ is even. ❑

Lemma 3.5: Let G be a 1^*-extendable graph on $2n \geq 6$ vertices. Suppose S is a cutset of G with $|S| = 4$.

(i) If G[S] contains an edge, then G – S contains at least 2 even components but no odd components.

(ii) If S is an independent set, then G – S contains exactly 2 odd components and no even components or at least 2 even components but no odd components.

Proof: Let S = {a, b, c, d} be a cutset of G. Without any loss of generality, we may assume that ab ∈ E(G). If G – S contains an odd component, then the edge ab does not extend to a perfect matching in G – {c, d}. This contradicts 1*-extendability of G. Hence, G – S has no odd components. Since S is a cutset of G, G – S contains at least 2 even components but no odd components. This proves (i).

Now we suppose that S is independent and G – S contains an odd component (and hence, by parity, at least 2 odd components). Further, we suppose that G – S contains H_0 as an even component. Since |S| = 4, by Theorem 1.5(ii), S is a minimum cutset. Thus there exists an edge e = xy joining a vertex x of S to a vertex y of H_0. Without any loss of generality, we may assume that x = a. Then the edge ay does not extend to a perfect matching in G – {b, c} since the odd components of G – S together with $H_0 \setminus y$ form at least 3 odd components of (G – (S ∪ {y}) and |S \ {a, b, c}| = |{d}| = 1, a contradiction. Hence, G – S contains only odd components. It follows from Theorem 1.7 that G – S contains exactly 2 odd components and no even components. If G – S has no odd components, then G – S contains at least 2 even components as S is a cutset. This completes the proof of our lemma. ❑

Remark 3.2: (1) For n ≥ 3, a graph $K_2 \vee (n-1)K_2$ is 0*-extendable which satisfies Lemma 3.4.

(2) For n ≥ 4, a graph $K_4 \vee (n-2)K_2$ is 1*-extendable which satisfies Lemma 3.5 (i) and for 2n ≥ 12 graphs $\overline{K}_4 \vee (K_1 \cup K_{2n-5})$ and $\overline{K}_4 \vee (n-2)K_2$ are both 1*-extendable which satisfy Lemma 3.5 (ii).

Theorem 1.1 together with Theorem 3.1 yields the following corollary:

Corollary 3.6: Let G be a k-extendable non–bipartite graph on 2n vertices with $4 \le k \le n-1$ and let S be a minimum cutset of G and M(S) a maximum matching in G[S]. If $|S| \le 2k - 3$, then $2 \le \omega(G - S) \le |S| - |M(S)| - k + 1$. ❑

Theorem 3.1 gives an upper bound on a number of components of G – S when S is a minimum cutset of order at most 2k + 1 of a k*-extendable graph G. One might expect a similar result for |S| ≥ 2k + 2 but this is not the case. For non-negative integers s and t, a graph $G_1 = (K_{2k} \cup \overline{K}_{t+2}) \vee (s+t+2)K_{2k+4}$ for t is even and a graph $G_2 = (K_{2k} \cup \overline{K}_{t+2}) \vee [(s+t+1)K_{2k+4} \cup$

$K_{2k+3}]$ for t is odd are k*-extendable with a minimum cutset $S = V(K_{2k} \cup \overline{K}_{t+2})$. Clearly, $\omega(G_i - S) = s + t + 2 \geq 2$ for $i = 1, 2$. However, if the number of odd components of $G - S$ is sufficiently large, then an upper bound on a number of even components of $G - S$ can be given with some restriction on the size of $M(S)$. Our next result establishes this.

Theorem 3.7: Let G be a k*-extendable graph on 2n vertices with $1 \leq k \leq n - 3$ and let S be a minimum cutset of G with $|S| \geq 2k + 2$ and $M(S)$ a maximum matching in $G[S]$. Suppose $o(G - S) = |S| - 2|M(S)| - 2 - r$ for some non-negative integer r. If $2|M(S)| + r \leq 2k - 2$, then the number of even components of $G - S$ is at most $|M(S)| + \left\lfloor \frac{r}{2} \right\rfloor$.

Proof: Let $\eta(G - S)$ be the number of even components of $G - S$. Suppose to the contrary that $\eta(G - S) \geq |M(S)| + \left\lfloor \frac{r}{2} \right\rfloor + 1 = t$. Let $H_1, H_2, \ldots, H_t$ be even components of $G - S$. Choose $x_i \in V(H_i)$, $1 \leq i \leq t$. Since $2|M(S)| + r \leq 2k - 2$, $t = |M(S)| + \left\lfloor \frac{r}{2} \right\rfloor + 1 \leq k$ and $|S| \geq 2k + 2 \geq t + 2$. Let $y_1, y_2, \ldots, y_t, y_{t+1}, y_{t+2} \in S$. In view of Theorem 1.6, there is a matching M' of size t joining vertices of $\{x_1, x_2, \ldots, x_t\}$ to vertices of $\{y_1, y_2, \ldots, y_t\}$. Clearly, $G - (V(M') \cup S)$ contains $|S| - 2|M(S)| - 2 - r + t = |S| - |M(S)| - \left\lceil \frac{r}{2} \right\rceil - 1$ odd components. Further $|S \setminus (V(M') \cup \{y_{t+1}, y_{t+2}\})| = |S| - (t + 2) = |S| - |M(S)| - \left\lfloor \frac{r}{2} \right\rfloor - 3$. If M' extended to a perfect matching in $G - \{y_{t+1}, y_{t+2}\}$, then each odd component of $G - (V(M') \cup S)$ would be joined to at least one vertex of $S \setminus (V(M') \cup \{y_{t+1}, y_{t+2}\})$. But this is impossible since $o(G - (V(M') \cup S)) = |S| - |M(S)| - \left\lceil \frac{r}{2} \right\rceil - 1$ while $|S \setminus (V(M') \cup \{y_{t+1}, y_{t+2}\})| = |S| - |M(S)| - \left\lfloor \frac{r}{2} \right\rfloor - 3$. Hence, $\eta(G - S) \leq |M(S)| + \left\lfloor \frac{r}{2} \right\rfloor$ as required. ❑

Our next result concerns an upper bound on a number of odd components of $G - S$ when S is an independent cutset of a k*-extendable graph G with $|S| \geq 2k + 2$.

Corollary 3.8: Let G be a k*-extendable graph on 2n vertices with $2 \leq k \leq n - 3$ and let S be a minimum cutset of G with $|S| \geq 2k + 2$. If S is independent, then $o(G - S) \leq |S| - 4$. Further, if $k \geq 3$ and $|S| - 5 \leq o(G - S)$, then $G - S$ has no even components.

Proof: Suppose to the contrary that $o(G - S) \geq |S| - 3$. It follows from Theorem 3.7 that $G - S$ has no even components. Let $C_1, C_2, \ldots, C_t$ be odd components of $G - S$. If $|V(C_i)| = 1$; $1 \leq i \leq t$, then G is bipartite which is impossible since G is k*-extendable. Hence, there is a component of $G - S$, C_1 say, with $|V(C_1)| \geq 3$. Let $x, y \in V(C_1)$ and $a, b, c, d \in S$. In view of Theorem 1.6, there is a matching M of size 2 joining vertices of $\{x, y\}$ to vertices of $\{a, b\}$. But then M does not extend to a perfect matching in $G - \{c, d\}$ since $G - (S \cup \{x, y\})$ contains at least $|S| - 3$ odd components while $|S \setminus \{a, b, c, d\}| = |S| - 4$. This contradicts the k*-extendability of G and proves that $o(G - S) \leq |S| - 4$.

Further, we assume that $k \geq 3$ and $|S| - 5 \leq o(G - S)$. Since $\nu(G)$ is even, $|S|$ and $o(G - S)$ have the same parity. This implies that $o(G - S) = |S| - 4$. By Theorem 3.7, $G - S$ has at most one even component.

Suppose H is an even component of $G - S$. We will show that $\nu(H) = 2$. Suppose to the contrary that $\nu(H) \geq 4$. Let $z_1, z_2, z_3 \in V(H)$ and $w_1, w_2, w_3, w_4, w_5 \in S$. By Theorem 1.6, there is a matching M_1 of size 3 joining vertices of $\{z_1, z_2, z_3\}$ to vertices of $\{w_1, w_2, w_3\}$. Applying a similar argument used above establishes that M_1 does not extend to a perfect matching in $G - \{w_4, w_5\}$, a contradiction. Hence, $\nu(H) = 2$. Since G has a perfect matching and S is independent, $\nu(G - S) \geq |S|$. Because $\nu(H) = 2$ and $o(G - S) = |S| - 4$, there is an odd component of $G - S$, C say, with $\nu(C) \geq 3$. Now let $a_1, a_2 \in V(C)$ and $b \in V(H)$. Then, in view of Theorem 1.6, there is a matching M_2 of size 3 joining vertices of $\{a_1, a_2, b\}$ to vertices of $\{w_1, w_2, w_3\}$. Again, M_2 does not extend to a perfect matching in $G - \{w_4, w_5\}$, a contradiction. This proves that $G - S$ has no even components and completes the proof of our corollary. ❑

Remark 3.3: For a positive integer $s \geq 4$, a graph $G_1 = \overline{K_s} \vee (s - 2)K_{2s+1}$ is 1*-extendable containing $V(\overline{K_s})$ as a minimum cutset. Clearly, $G_1 - V(\overline{K_s})$ contains $s - 2$ odd components. Further, for a positive integer $s \geq 5$, a graph $G_2 = \overline{K_s} \vee [(s - 4)K_{2s+1} \cup K_{2s}]$ is 2*-extendable which $V(\overline{K_s})$ is a minimum cutset and $G_2 - V(\overline{K_s})$ contains $s - 4$ odd components and an even component. Thus the bound on k in Corollary 3.8 is best possible.

Our next result concerns a minimum cutset of a k*-extendable graph which its induced subgraph has a small independence number. We begin with the following lemma.

Lemma 3.9: Let G be a simple graph with $\alpha(G) \leq 2$ and M a maximum matching in G. Then $|M| = \frac{\nu(G)-1}{2}$ for $\nu(G)$ is odd and $|M| \geq \frac{\nu(G)}{2} - 1$ for $\nu(G)$ is even.

Proof: Let $\nu(G)$ be odd. Suppose $|M| < \frac{\nu(G)-1}{2}$. Clearly, $|M| \leq \frac{\nu(G)-3}{2}$ and $G - V(M)$ is independent since M is a maximum matching. Then $G - V(M)$ contains at least $\nu(G) - 2|M| \geq 3$ independent vertices, contradicting the fact that $\alpha(G) \leq 2$. Hence, $|M| = \frac{\nu(G)-1}{2}$. Similarly, $|M| \geq \frac{\nu(G)}{2} - 1$ for $\nu(G)$ is even. ❑

Theorem 3.10: Let G be a k*-extendable graph on 2n vertices with $0 \leq k \leq n - 3$ and let $S \subseteq V(G)$ be a minimum cutset of G. Suppose $\alpha(G[S]) \leq 2$. Then $|S| \geq 2k + 2$ and $o(G - S) \leq |S| - 2k - 2$.

Proof: By Theorem 1.7 and the fact that 0*-extendable graphs are 2-connected, our theorem follows immediately for $k = 0$. So we only need to consider the case $k \geq 1$. Since G is $(k + 3)$-connected, $|S| \geq k + 3 \geq 4$. Suppose $|S| \leq 2k + 1$. Let M be a maximum matching in $G[S]$. We will show that $G - S$ contains only even components.

Suppose to the contrary that $G - S$ contains an odd component. Assume that $G - S$ contains exactly one odd component. Then $|S|$ is odd by the fact that $\nu(G)$ is even. Further, since S is a cutset, $G - S$ contains an even component, H say. By Lemma 3.9, $|M| = \frac{|S|-1}{2} \leq k$. Let $x \in S \setminus V(M)$ and $y \in V(H)$. Then M does not extend to a perfect matching in $G - \{x, y\}$ since $G - (V(M) \cup \{x, y\})$ contains $o(G - S) + 1 = 2$ isolated odd components, a contradiction. Hence, $G - S$ contains at least 2 odd components. Clearly, $|S|$ is odd otherwise G is not k*-extendable since $\frac{|S|}{2} - 1 \leq |M| \leq k$ and $|S \setminus V(M)| = 0$ or 2. Consequently, $G - S$ contains at least 3 odd components. Let C_1 be an odd component of $G - S$ and let $z \in V(C_1)$. By Lemma 3.9, $|M| = \frac{|S|-1}{2} \leq k$ and there is a vertex $x \in S \setminus V(M)$. Now M does not extend to a perfect matching in $G - \{x, z\}$ since $G - (V(M) \cup \{x, z\})$ contains $o(G - S) - 1 \geq 2$ isolated odd components, again a contradiction. This proves that $G - S$ contains only even components. Consequently, $|S|$ is even and $|S| \leq 2k$. Further, $G - S$ contains at least two even components, H_1 and H_2 say. By Lemma 3.9, $\frac{|S|}{2} - 1 \leq |M| \leq k$. Let $a \in V(H_1)$ and $b \in V(H_2)$. If $|M| = \frac{|S|}{2} \leq k$, then M does not extend to a perfect matching in $G - \{a, b\}$ since $G - (V(M) \cup \{a, b\})$ contains $H_1 - a$ and $H_2 - b$ as isolated odd components. This contradicts the fact that G is k*-extendable. Thus $|M| = \frac{|S|}{2} - 1 \geq 1$ since $|S| \geq 4$.

Let $a_1b_1 \in M$, a_2 and b_2 belong to $S \setminus V(M)$. Since S is a minimum cutset, in view of Theorem 1.6, there is a matching $M_1 = \{a_1x_1, b_1x_2 \mid x_1 \in V(H_1)$ and $x_2 \in V(H_2)\}$. Then $M_2 = (M \cup M_1) \setminus \{a_1b_1\}$ is a matching of size $(\frac{|S|}{2} - 1) + 2 - 1 = \frac{|S|}{2} \le k$. Clearly M_2 does not extend to a perfect matching in $G - \{a_2, b_2\}$ since $G - (V(M_2) \cup \{a_2, b_2\})$ contains $H_1 - x_1$ and $H_2 - x_2$ as isolated odd components. This contradiction proves that $|S| \ge 2k + 2$. It follows immediately from Theorem 1.7 that $o(G - S) \le |S| - 2k - 2$. This completes the proof of our theorem. ❑

Remark 3.4: Theorem 3.10 is best possible in the sense that there is a k*-extendable graph G with a cutset S satisfying the conditions of the theorem and $G - S$ contains a number of odd components up to $|S| - 2k - 2$.

Let $G_1 = K_{2k+2+r} - \{\text{an edge}\}$, $G_2 = \bigcup_{i=1}^{q} K_{2a_i+1}$ and $G_3 = \bigcup_{j=1}^{m} K_{2b_j}$ where r, q, m, a_i, b_j are non-negative integers, $q + m \ge 2$, $q \le r$ and $q \equiv r \pmod 2$. Put $G = G_1 \vee (G_2 \cup G_3)$. Figure 3.1 depicts the graph G. Throughout the paper we adopt the convention that a double line in our diagram denotes the join between the corresponding graphs. It is not too difficult to show that G is k*-extendable

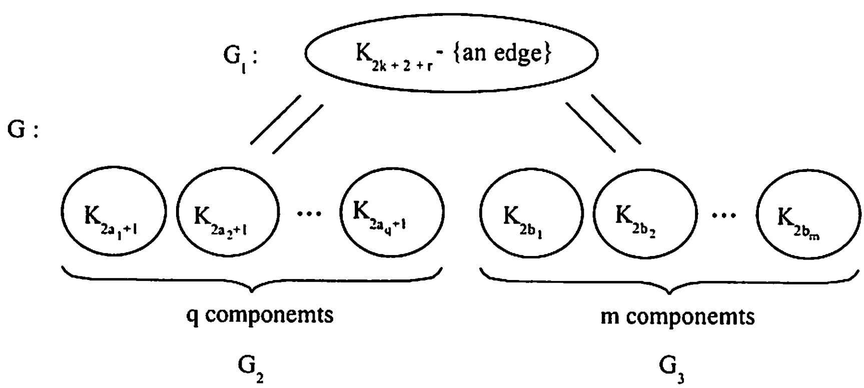

Figure 3.1

containing $V(G_1)$ as a cutset of order $2k + 2 + r$. Notice that the number of components of $G - V(G_1)$ can be any integer which is at least 2.

Theorem 1.1 together with Theorem 3.10 yields the following corollary:

Corollary 3.11: Let G be a k-extendable non–bipartite graph on 2n vertices with $2 \le k \le n - 1$ and let S be a minimum cutset of G. Suppose $\alpha(G[S]) \le 2$. Then $|S| \ge 2k - 2$ and $o(G - S) \le |S| - 2k + 2$. ❑

We conclude our paper by establishing a lower bound on an order of k*-extendable graphs in terms of an order of a minimum cutset.

Theorem 3.12: Let G be a k*-extendable graph on 2n vertices with $2 \le k \le n - 3$ and let S be a minimum cutset of G and M(S) a maximum matching in G[S]. If

(i) $|S| \le 2k + 1$, or

(ii) $|S| \ge 2k + 2$ and $|M(S)| \le k$,

then $2n \ge 2|S| + 2k - 2|M(S)| + 2$.

Proof: Clearly, by the assumption on |S| and Corollary 2.2, $|S| - 2|M(S)| \ge 2$. Let x and y be vertices of $S \setminus V(M(S))$. Since G is k*-extendable, there is a perfect matching F in $G - \{x, y\}$ containing all the edges of M(S). Put

$$F_1 = \{ab \in F \mid a \in S \setminus (V(M(S)) \cup \{x, y\}), b \notin S\}$$

and

$$F_2 = \{ab \in F \mid a, b \notin S\}.$$

Then

$$|F_1| = |S| - 2|M(S)| - 2 \ge 0$$

and

$$\begin{aligned}|F_2| &= \frac{1}{2}[2n - |S| - |F_1|] \\ &= \frac{1}{2}[2n - |S| - (|S| - 2|M(S)| - 2)] \\ &= n - |S| + |M(S)| + 1.\end{aligned}$$

If $|F_2| = 0$, then M(S) does not extend to a perfect matching in G since $G - V(M(S))$ contains $S \setminus V(M(S))$ as an independent set of order $|S| - 2|M(S)|$ and $\nu(G - V(M(S))) = |S| - 2|M(S)| + (|S| - 2|M(S)| - 2) = 2|S| - 4|M(S)| - 2$, contradicting the k*-extendability of G. Thus $|F_2| \ge 1$. Let $zw \in F_2$. Suppose $|F_2| \le k + 1$. Then $F_2 \setminus \{zw\}$ does not extend to a perfect matching in $G - \{z, w\}$ since $G - V(F_2)$ contains $S \setminus V(M(S))$ as an independent set of order $|S| - 2|M(S)|$ and $\nu(G - (S \cup V(F_2)) = |F_1| = |S| - 2|M(S)| - 2$, again a contradiction. Hence, $n - |S| + |M(S)| + 1 = |F_2| \ge k + 2$. Thus $2n \ge 2|S| + 2k - 2|M(S)| + 2$ as required. This completes the proof of our theorem. ❑

As a corollary we have:

Corollary 3.13: Let G be a k-extendable non–bipartite graph on 2n vertices with $4 \le k \le n - 1$ and let S be a minimum cutset of G and M(S) a maximum matching in G[S]. If

(i) $|S| \le 2k - 3$, or

(ii) $|S| \geq 2k - 2$ and $|M(S)| \leq k - 2$
then $2n \geq 2|S| + 2k - 2|M(S)| - 2$. ❑

Remark 3.5: Theorems 3.1 and 3.12 are best possible in the sense that for $k \geq 2$ there is a k*-extendable graph G on $2n \geq 2|S| + 2k - 2|M(S)| + 2$ vertices containing a minimum cutset S of order at most $2k + 1$ with $2 \leq \omega(G - S) \leq |S| - |M(S)| - k - 1$. For non-negative integers k, s, t, q, r, m with

(i) $k + 3 \leq s \leq 2k + 1$
(ii) $0 \leq t \leq s - k - 3$
(iii) $0 \leq 2q + r \leq s - t - k - 3$,

let $G = (K_{2t} \cup \overline{K}_{s-2t}) \vee [K_{s-2q} \cup K_{2k+2-2r-2t+2m} \cup (2q)K_1 \cup rK_2]$. Figure 3.2 illustrates the graph G. It is not too difficult to show that G is k*-extendable. Clearly, $S = V(K_{2t} \cup \overline{K}_{s-2t})$ is a cutset of order s, $v(G) = 2s + 2k - 2t + 2 + 2m$ and $2 \leq \omega(G - S) = 2q + r + 2 \leq s - t - k - 1$.

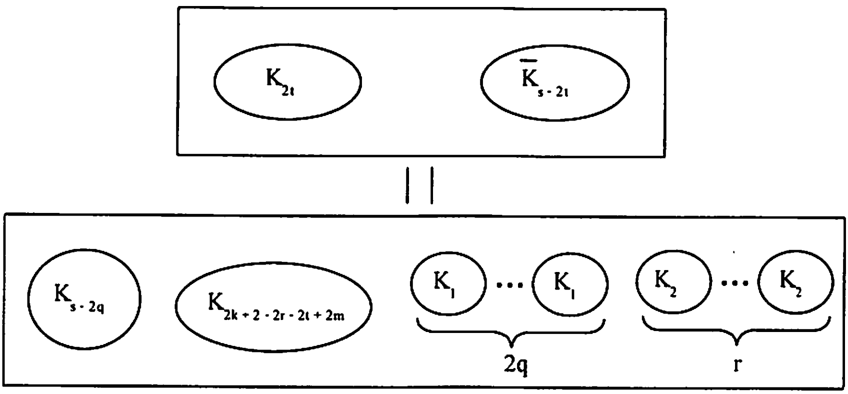

Figure 3.2

Acknowledgement

This work has been supported by a grant (RSA 3980009) from The Thailand Research Fund.

References

[1] N. Ananchuen, On Strongly k-Extendable Graphs, *Journal of Combinatorial Mathematics and Combinatorial Computing*, **38** (2001), 3–19.

[2] N. Ananchuen, On Minimum Degree of Strongly k-Extendable Graphs, *Journal of Combinatorial Mathematics and Combinatorial Computing*, **38** (2001), 149–159.

[3] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, The Macmillan Press, London, (1976).

[4] L. Lovasz, On the Structure of Factorization graphs, *Acta Mathematica Academiae Scientiarum Hungaricae*, **23** (1972), 179-195.

[5] L. Lovasz and M. D. Plummer, On Bicritical graphs, *Colloquia Mathematica Societatis Janos Bolyai*, **10** (1973), 1051-1079.

[6] L. Lovasz and M. D. Plummer, On A Family Of Planar Bicritical graphs, *Proc. London Math. Soc.*, **30** (1975), 160 - 176.

[7] M. D. Plummer, On n-Extendable graphs, *Discrete Mathematics,* **31** (1980) 201 – 210.

[8] M. D. Plummer, Matching Extension and Connectivity in Graphs, *Congressus Numerantium*, **63** (1988) 147 – 160.

[9] M. D. Plummer, Matching Extension and Connectivity in Graphs ii, in Y. Alavi, G. Chartrand, O.R. Oellermann and A. J. Schwenk (eds), *Graph Theory, Combinatorics, and Applications Vol 2* (1991) 651 - 664.

[10] M. D. Plummer, Extending Matchings in Graphs: a survey, *Discrete Mathematics*, **127** (1994), 227-292.

[11] M. D. Plummer, Extending Matchings in Graphs: An update, *Congressus Numeratium*, **116** (1996), 3-32.