Partitioning of Trees Having Maximum Degree At Most Three

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Abstract

The vertices V of trees with maximum degree three and t degree two vertices are partitioned into sets R, B, and U such that the induced subgraphs $<V - R>$ and $<V - B>$ are isomorphic and $|U|$ is minimum. It is shown for $t \geq 2$ that there is such a partition for which $|U| = 0$ if t is even and $|U| = 1$ if t is odd. This extends earlier work by the authors which answered this problem when $t = 0$ or 1.

1 Introduction

Knisely, Wallis, and Domke [5] investigated *even* graphs, that is, graphs whose edges can be partitioned into sets R (red) and B (blue) in such a way that the subgraph induced by the R edges is isomorphic to the subgraph induced by the B edges. They proved that Fibonacci trees and binary heaps with an even number of edges are even. Heinrich and Horak [4] characterized even trees of maximum degree three in which no vertex of degree one is adjacent to a vertex of degree three.

Some time ago the authors initiated a study of a related problem in which the vertices of graph $G = (V, E)$, rather than the edges, are partitioned into sets R and B, as well as into a possibly empty third set U (uncolored) in such a way that the induced subgraphs $< V - R >$ and

JCMCC 45 (2003), pp. 79-93

$<V - B>$ are isomorphic. Such a partition is called a *splitting partition*, and the *splitting number* $\mu(G)$ is the size of $|R|$ (and $|B|$) for any splitting partition which maximizes $|R|$ (minimizes $|U|$). The values of $\mu(G)$ have been found when G is a path, cycle, complete graph, complete bipartite graph, Fibonacci tree, and spider [1, 2]. In [3], the value when G is a tree of maximum degree at most three with at most one degree two vertex is determined. The main results are shown in the following theorem.

Theorem 1 *Let T be a tree having p vertices and with maximum degree at most three.*

1. *If T has one vertex of degree two, $\mu(T) = (p-1)/2$.*

2. *If T has no vertices of degree two, then $\mu(T) = (p-2)/2$ if $p \equiv 0$ mod 4 and $\mu(T) = p/2$ if $p \equiv 2$ mod 4.*

O'Leary [6] has shown that $\mu(T) = p/2$ when T has exactly two degree two vertices. This paper extends Theorem 1 and O'Leary's result to all maximum degree three trees. Section 2 gives the value of $\mu(T)$ when the number of vertices is even and also provides insight into the odd case treated in Section 3. Note that the number of vertices in T and the number of vertices of T which are of degree two have the same parity.

2 The Value of $\mu(G)$ When the Number of Vertices is Even

It will be convenient to call a splitting partition for which $\mu(T) = p/2$ an *equal-partition (e-partition)*. Such a partition is not possible when p is odd. In this case we call a partition an *almost equal partition (ae-partition)* if there is a path $<a, b, c>$ such that (i) a and b are in R (or B) and, except for each other, are adjacent only to vertices in the opposite set, (ii) c is in B (or R) and is adjacent only to vertices in the opposite set, (iii) $U = \emptyset$, and (iv) $<B - \{a, b, c\}> \cong <R - \{a, b, c\}>$. Thus a and b induce a K_2 in $<R>$

(or <B>), c induces a K_1 in <B> (or <R>), and $T - \{a, b, c\}$ possesses an e-partition. We say a and b induce an *extra* K_2 and c induces an *extra* K_1.

The concept of a *terminal configuration* will be useful. If x is a vertex of a rooted tree, S_x is the subtree rooted at x. A terminal configuration is any one of the subtrees S_x illustrated in Figure 1 where the root x is the vertex at the bottom of the dotted edge, and the degree one vertices below the root are degree one in T. This dotted edge indicates how the configuration is connected to the rest of the graph if the graph is larger than the configuration. In this case the vertex at the top of the dotted edge is called the *connecting vertex* of the configuration. Lemma 2 indicates that these configurations are complete in some sense. Let T_7 be the complete binary tree with seven vertices.

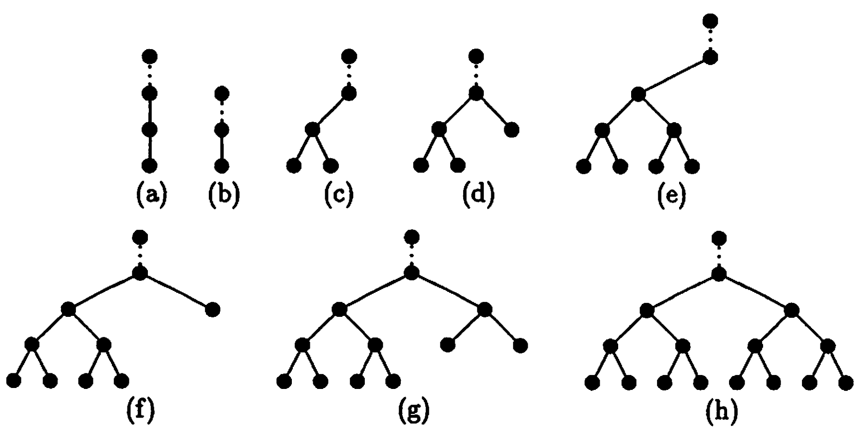

Figure 1: Terminal configurations

Lemma 2 *Any tree T with maximum degree at most three and with at least five vertices is either T_7 or possesses a vertex x such that S_x is isomorphic to a terminal configuration of Figure 1.*

Proof: Observe that, as one moves from Configuration (a) to Configuration (h), none of the previous configurations occur. All that remains is to show that one of the subtrees must appear, under the assumption that T is not T_7. Consider a vertex x such that S_x has height two. If neither Configurations

(a) nor (b) occur, S_x must either be T_7 or one of Configurations (c) or (d). If $S_x \cong T_7$, Configurations (e), (f), (g,), and (h) represent the only possibilities. □

The following lemma leads to Corollary 4 which is the main result of this section. The degree of a vertex v is indicated by d_v. Recall that the number of degree two vertices has the same parity as the order of the maximum degree three tree T.

Lemma 3 *Let T be a tree on p vertices with maximum degree at most three and at least one vertex of degree two. Then T has an e-partition if p is even and an ae-partition if p is odd.*

Proof: We proceed by induction on p, the result being easily checked when $p = 3$ or 4. Assume, therefore, that T has $p > 4$ vertices.

Suppose first that p is odd. If T contains a path $<x, y, z>$ where $d_x = 1$ and $d_y = 2$, form $T' = T - \{x, y\}$. Since T' has an odd number of vertices, it has an ae-partition by the inductive hypothesis. This can be extended to an ae-partition of T by placing x in the same set as z, and y in the other set. Otherwise T must have a path of the form $<a, b, c>$ where $d_a = d_c = 1$ and $d_b = 3$, and where z is the remaining neighbor of b. Let $T' = T - \{a, b, c\}$, so T' has $p' = p - 3$ vertices and, in it, vertex z has degree one or two. If T' contains a degree two vertex or $p' \equiv 2 \ mod\ 4$, T' has an e-partition by either the inductive hypothesis or Theorem 1 which can be extended to an ae-partition of T by placing c in the same set as z and a and b in the other. The only remaining case is if T' has no degree two vertices and $p' \equiv 0 \ mod\ 4$. This can happen only if $d_z = 1$ in T' and z is the only vertex of degree two in T. One possibility is T is T_7. Figure 2 shows an ae-partition for T_7, where labels r and b indicate a vertex's membership in R and B, respectively (later we also will use u to represent membership in U). Otherwise T contains a second $K_{1,2}$ terminating structure whose removal leaves a tree T' in which $d_z = 2$. The earlier argument then shows T has an ae-partition.

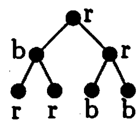

Figure 2: An ae-partition of complete binary tree with seven vertices

Suppose next that p is even. We discuss cases depending on the type of configurations of Figure 1 which appear in T. The argument for each case assumes previously considered cases do not occur.

Case 1: Configuration (a) occurs. Let $<a, b, c>$ be the path with $d_a = 1$, $d_b = d_c = 2$, and z the other neighbor of c. We know $T' = T - \{a, b, c\}$ has an ae-partition by the inductive hypothesis. Without loss of generality the extra K_2 can be assumed to have its vertices in R so the extra K_1 is in B. Extend this to an e-partition of T as follows. If $z \in R$, place a in R and b and c in B. If $z \in B$, place a and b in B and c in R.

Case 2: Configuration (b) occurs. Let $<a, b>$ be the path with $d_a = 1$, $d_b = 2$, and z the other neighbor of b. Vertex z must have degree three in T for otherwise we would have an instance of Configuration (a). Then $T' = T - \{a, b\}$ has an e-partition by the inductive hypothesis. Extend it to an e-partition of T by placing a in the same set as z and b in the other set.

Case 3: Configuration (c) occurs. Two situations must be examined. Suppose, first, there are two vertex disjoint type (c) configurations. If T is the graph shown in Figure 3 (a), the vertices can be assigned to sets R and B as shown. Otherwise, T', the tree obtained by removing both configurations, has an even number of vertices. Suppose either T' has degree two vertices or $|V(T')| \equiv 2 \ mod \ 4$. By the inductive hypothesis or Theorem 1, T' has an e-partition which can be extended to one for T as shown in (b) or (c) of Figure 3, depending upon the two possible assignments in T' of the connecting vertices of the structures (which may be the same vertex).

Notice that, when two connecting vertices receive different assignments and they lead to isomorphic configurations, the vertices of each configuration can always be assigned to the set opposite to that of the connecting vertex. For that reason, such situations will not be mentioned explicitly below. Furthermore, "opposite" assignments will not be mentioned. Thus Figure 3 does not show the case when both connecting vertices are in B since all that need be done in such a case is to switch the assignments shown in Figure 3 (b).

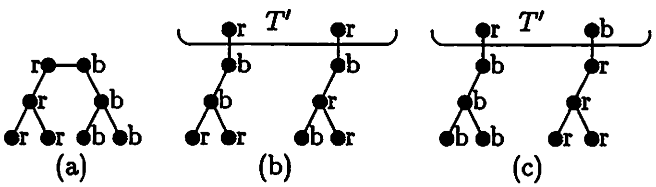

Figure 3: Two instances of Configuration (c) occur

Suppose next that $|V(T')| \equiv 0 \ mod \ 4$ and T' has no vertices of degree two. If the connecting vertices z_1 and z_2 are distinct, they must be of degree two in T. Remove the connecting vertices along with the two structures, let x_1 and x_2 be the new connecting vertices, and set $T'' = T' - \{z_1, z_2\}$. Note that $|V(T'')| \equiv 2 \ mod \ 4$ so T'' has an e-partition which can be extended to T as in Figure 4(a), even if $x_1 = x_2$. If the two structures have the same connecting vertex z, form $T'' = T' - \{z\}$. The number of vertices in T'' is odd and there are at least three of them. By the inductive hypothesis, T'' has an ae-partition. Without loss of generality, assume the partition has an extra K_2 in R and an extra K_1 in B. Then the ae-partition can be extended to an e-partition of T as shown in Figure 4 (b) or (c).

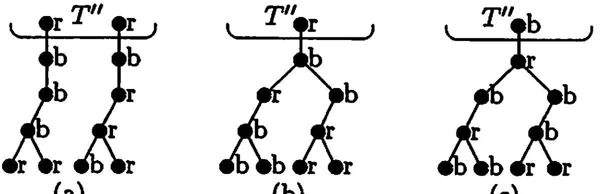

Figure 4: Further instances when two Configuration (c)s occur

Finally, suppose there is no second copy of Configuration (c). Observe first that there is no maximum degree three tree T which has an even number of at least six vertices and has no instance of Configuration (b) and has every vertex of T at distance two or less from the root of the Configuration (c). Thus, we may assume there is an instance of at least one of Configurations (d) through (h) or a T_7 which is vertex disjoint from the Configuration (c). If a Configuration (d) exists, remove all but its root vertex, create an e-partition for the resulting T', and extend it to one for T, as shown in Figure 5(a). The remaining possible configurations all involve a complete binary tree on seven vertices. Let tree T' be obtained by removing the Configuration (c) and all but the root vertex of this complete subtree, so that T' has an even number of vertices. If T' contains vertices of degree two or $|V(T')| \equiv 2 \mod 4$, it has an e-partition which can be extended to one for T, as shown in Figure 5 (b) or (c) depending on the assignments given in T' to the connecting vertex of the Configuration (c) and the root of the T_7. If T' does not have degree two vertices and $|V(T')| \equiv 0 \mod 4$, then the only such vertices in T must be the root of the Configuration (c) and its connecting vertex. Define T'' to be the tree obtained by removing Configuration (c), its connecting vertex, and the entire complete subtree on seven vertices. Then $|V(T'')| \equiv 2 \mod 4$ and, thus, has an e-partition which can be extended to one for T as shown in Figure 6 (a) and (b).

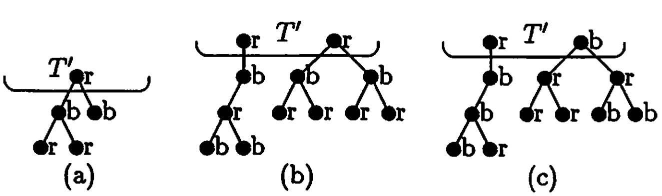

Figure 5: Instances involving one occurrence of Configuration (c)

Case 4: Configuration (d) occurs. This situation was covered in Case 3 and the extension is as shown in Figure 5 (a).

Case 5: Configuration (e) occurs. In addition to the Configuration (e), there must be, outside that configuration, two leaves with a single

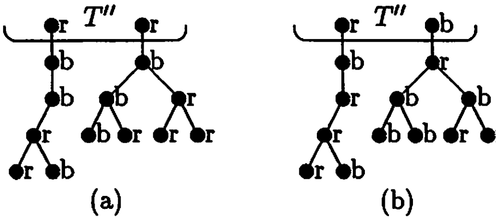

Figure 6: Still more instances involving Configuration (c)

parent, for otherwise Configuration (a) or (b) would be present. Removing Configuration (e) and the two extra leaves results in a tree T' which has an even number of vertices. Any e-partition of T' can be extended to T by the assignment shown in Figure 7 (a) or (b). If T' has no e-partition, T has exactly two vertices of degree two: the root of Configuration (e) and its connecting vertex. Removing the configuration and the connecting vertex leaves a tree T' with an odd number of vertices and hence an ae-partition, with an extra K_2 in R and an extra K_1 in B. This can be extended to an e-partition of T as shown in Figure 7 (c) or (d).

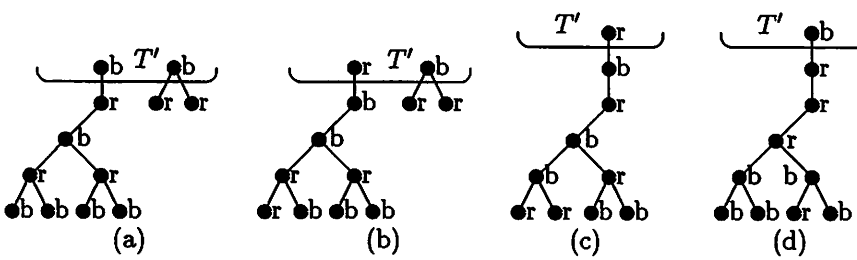

Figure 7: Instances involving Configuration (e)

Case 6: Configuration (f) occurs. Removing all but the root vertex of a Configuration (f) leaves a tree with an e-partition which can be extended to one for T as shown in Figure 8 (a).

Case 7: Configuration (g) occurs. Because T has degree two vertices, there must be a vertex outside the configuration which has two neighbors of degree one. Removing these neighbors and all vertices of the Configuration (g) except the root leaves a graph with an e-partition which can be extended to one for T as shown in Figure 8 (b) and (c).

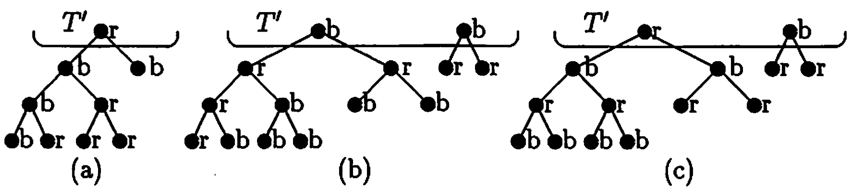

Figure 8: Instances involving Configurations (f) and (g)

Case 8: Configuration (h) occurs. Proceeding as in the previous case we obtain an e-partition of T by the assignment shown in Figure 9 (a) or (b).

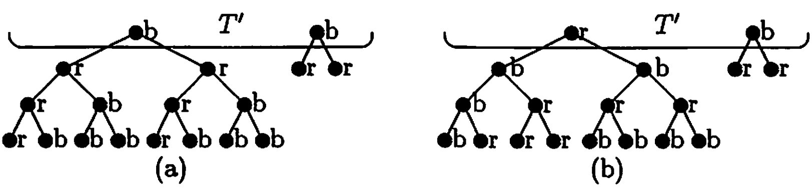

Figure 9: Instances involving Configuration (h)

The following corollary summarizes the previous results when the number of vertices is even and some have degree two.

Corollary 4 *Let T be a tree of maximum degree three, p vertices, and a positive even number of vertices of degree two. Then $\mu(T) = p/2$.*

3 The Value of $\mu(G)$ When the Number of Vertices is Odd

In this section we show, for maximum degree three trees T having an odd number p of vertices, that $\mu(T) = (p-1)/2$, that is, there need be only one vertex in U. We may assume that there are at least three vertices of degree two since Theorem 1 provides the result when there is only one. Once again terminal configurations will play a central role and, here, will consist of the set of all subtrees of height exactly two, as shown in Figure 10. Notice that Configurations (a) through (d) have an odd number of vertices and Configurations (e) through (g) have an even number. Every tree with at least three vertices contains at least one such subtree as a terminal configuration.

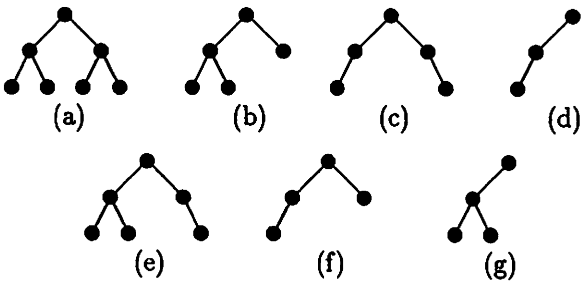

Figure 10: Terminal configurations when the tree has an odd number of vertices

Most trees will contain two of the above configurations that are vertex disjoint. This will be the case if the connecting vertex roots a subtree of height at least two which is vertex disjoint from the configuration. Thus the only trees which have an odd number of vertices, and contain one of the configurations but not two, are Configurations (a) through (d) and trees obtained from any of the configurations by appending the connecting vertex with zero, one, or two children. The choice of the number of children allowed is restricted so that the total number of vertices is odd. It is straightforward to demonstrate set assignments showing $\mu(T) = (p-1)/2$ for such trees.

Lemma 5 *If a maximum degree three tree T with an odd number of vertices contains any of Configurations (a) through (d),* $\mu(T) = (p-1)/2$.

Proof: We assume trees with the various configurations are examined in order, that is all trees with Configuration (a) are dealt with before any with Configuration (b), and so forth. Let tree T' be T with the appropriate configuration removed. Since T and the configuration both have an odd number of vertices, T' has an even number. If T' possesses an e-partition, it can be extended to an optimum partition of T as shown in Figure 11.

It is conceivable that removal of the configuration can eliminate all degree two vertices from T. If this occurs and $|V(T')| \equiv 0 \; mod \; 2$, T' will not have an e-partition. This cannot happen with Configurations (a)

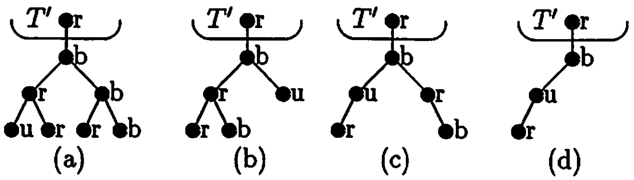

Figure 11: Configurations having an odd number of vertices

and (b) since they have no degree two vertices and hence the only possible degree two vertex which could be lost by their removal is the connecting vertex. Thus there will be always at least two degree two vertices in T'. However, removal of Configuration (c) or (d) can reduce the number of degree two vertices by three. Unless T is one of the small trees already considered, it must have a second configuration that is vertex disjoint from the first. Since we have already handled trees with Configurations (a) and (b), the second configuration must be one of (c) through (g). Each of these, however, has a degree two vertex (the root in (g)), contradicting the fact that T' does not. Thus we may assume T' has an e-partition and the above argument is sufficient. □

We now turn attention to the remaining configurations of Figure 10, all of which have an even number of vertices.

Lemma 6 *If a maximum degree three tree T with an odd number of vertices contains any of Configurations (e) through (g), $\mu(T) = (p-1)/2$.*

Proof: By earlier remarks, it is possible to assume a second configuration exists in T which is vertex disjoint from the first. We consider all of the six possible pairs of the remaining configurations which can exist together. The method in each case is to remove a part, possibly all, of each of the configurations so that an odd number of vertices are removed to leave a tree T' having an even number of vertices. This is illustrated in Figure 12 (a) and (b) which, when the two configurations are both of form (e), show extensions from an e-partition of T' to an optimum partition of T.

Each of the two copies of Configuration (e) has a degree two vertex. Moreover, the connecting vertex to the root of the leftmost copy could

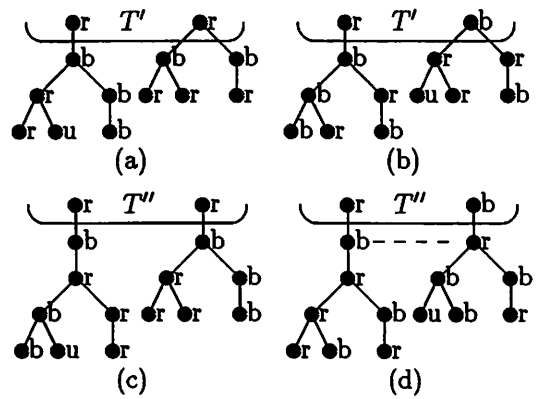

Figure 12: Two instances of Configuration (e) occur

change from degree two in T to degree one in T'. It thus is conceivable that T' would have no degree two vertices and $|V(T')| \equiv 0 \; mod \; 4$ and so wouldn't have an e-partition. In this case we form a new T'' by removing from T all that was removed before plus the connecting vertices. Now $|V(T'')| \equiv 2 \; mod \; 4$ so T'' has an e-partition which can be extended to an ae-partition of T as shown in Figure 12 (c) and (d). It is possible that T'' formed in this way is empty. Then the tree must be the one illustrated in Figure 12 (d) where the vertices shown in T'' are not present and the dotted edge connects the two subtrees. The set assignment given there applies to this tree also.

Figures 13 to 17 illustrate appropriate set assignments for the other possible combinations of Configurations (e) through (g). In these figures, assignments (a) and (b) are to be employed when T' has an e-partition while (c) and (d) are to be used when it does not. □

4 Summary and Concluding Comments

The complete result for trees with maximum degree three follows immediately from Theorem 1, Corollary 4, and Lemmas 5 and 6.

Theorem 7 *Let T be a tree having p vertices and maximum degree at most three.*

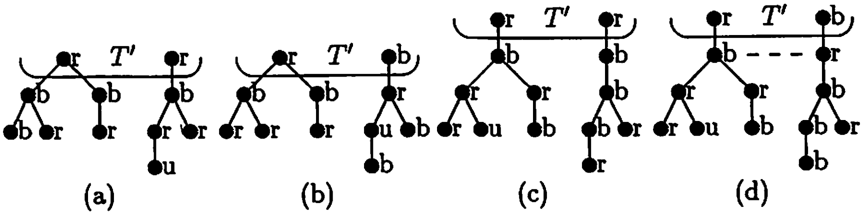

Figure 13: Configurations (e) and (f) are present

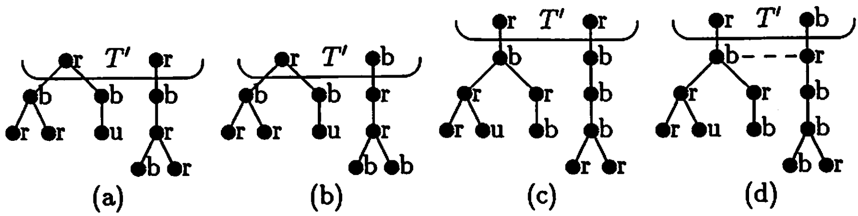

Figure 14: Configurations (e) and (g) are present

1. *If T has no vertices of degree two, $\mu(T) = (p-2)/2$ if $p \equiv 0$ mod 4 and $\mu(T) = p/2$ if $p \equiv 2$ mod 4.*

2. *If T has an odd number of vertices of degree two, $\mu(T) = (p-1)/2$.*

3. *If T has a positive even number of vertices of degree two, $\mu(T) = p/2$.*

It is natural to ask similar questions about arbitrary trees, but this seems to be a difficult problem, even if restricted to trees with maximum degree 4.

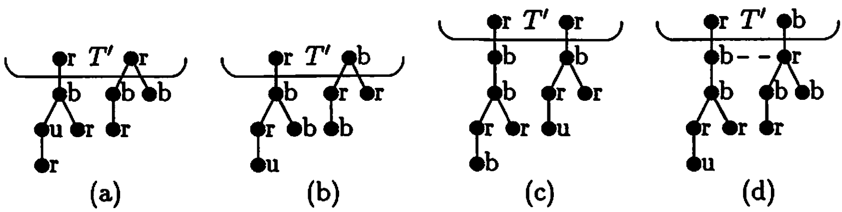

Figure 15: Two instances of Configuration (f) occur

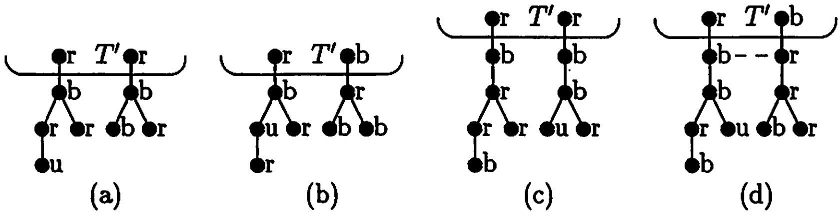

Figure 16: Configurations (f) and (g) are present

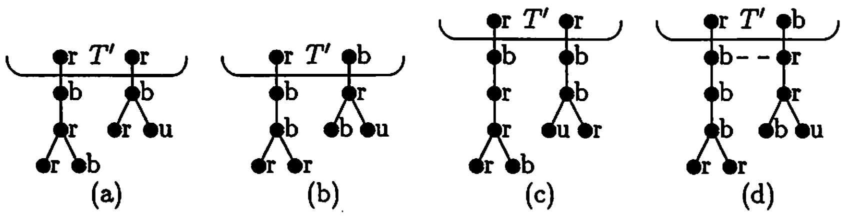

Figure 17: Two instances of Configuration (g) occur

5 Acknowledgement

The authors wish to express their appreciation to the referee for many valuable suggestions.

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