# A Note on Adjoint Polynomials and Uniquely Colorable Graphs

Haixing Zhao[1,2] Ruying Liu [2]
[1]Department of Computer Science and Engineering
Northwestern Polytechnical University
Xi'an, Shaanxi 710072, P. R. China
[2]Department of Mathematics, Qinghai Normal University
Xining, Qinghai, 810008, P. R. China

**Abstract**

In this note, we present many uniquely $n$–colorable graphs with $m$ vertices and new constructing ways of uniquely colorable graph by using the theory of adjoint polynomials of graphs. We give new constructing ways of two uniquely colorable graphs which are chromatically equivalent also.

**Key Word:** uniquely colorable graph; adjoint polynomial; chromatically equivalent.

**AMS Subject Classifications**: 05C15, 05C50

# 1 Introduction

All graphs considered are finite and simple. Undefined notation and terminology will conform to those in [1]. Let $G$ be a graph. We denote by $V(G)$, $E(G)$, $\overline{G}$, $p(G)$ and $q(G)$ the set of vertex , the set of edge, the complement, the number of vertices and the number of edges of $G$, respectively. Let $\lambda$ be a positive integer. An $\lambda$–coloring of $G$ is a partition of $V(G)$ into $\lambda$ color class such that the vertices in the same color class are not adjacent. By $\chi(G)$ we denote the chromatic number of $G$, i.e., $\chi(G) = min\{\lambda\}$. If every $\chi(G)$–coloring of $G$ gives the same partition of $V(G)$, then $G$ is said to be *a uniquely $\chi(G)$-colorable graph.* Let $P(G,\lambda)$ denote chromatic polynomial of $G$, two graphs $G$ and $H$ are said to be *chromatically equivalent* if and only if $P(G,\lambda) = P(H,\lambda)$.

JCMCC 45 (2003), pp. 123-128

In [2–4], the unique $n$–colorablily of graphs was studied. Some results of the unique $n$–colorablity of graphs were obtained. In this paper, we obtain some new results on the unique colorablily of graphs by using new theory of the adjoint polynomials of graphs (see[8]), which was introduced by Liu in [7]. In the search for chromatically unique graph, it turned out that many new results could be obtained by applying the adjoint polynomials of graphs (see[8]).

By $K_m$ and $P_n$ we denote the complete graph with $m$ vertices and the path with $n$ vertices, respectively. $D_n (n \geq 4)$ denotes the graph obtained from $K_3$ and $P_{n-2}$ by identifying a vertex of $K_3$ with a vertex of degree 1 of $P_{n-2}$. $T(l_1, l_2, l_3)$ denotes the tree with a vertex $v$ of degree 3 such that $T(l_1, l_2, l_3) - v = P_{l_1} \cup P_{l_2} \cup P_{l_3}$, where $l_i \geq 1$, $i = 1, 2, 3$.

# 2 Definitions and Basic Lemmas

Let $G$ be a graph with $p$ vertices. If $G_0$ is a spanning subgraph of $G$ and each component of $G_0$ is complete, then $G_0$ is called *an ideal subgraph* of $G$. By $b_i(G)$ we denote the number of ideal subgraph with $p - i$ components, where $0 \leq i \leq p - 1$. By Theorem 15 in [9], we can easily obtain the following formula

$$P(\overline{G}, \lambda) = \sum_{i=0}^{p-1} b_i(G)(\lambda)_{p-i},$$

where $(\lambda)_i = \lambda(\lambda - 1)(\lambda - 2)\cdots(\lambda - i + 1)(\lambda \geq 1)$ (see [8]).

**Definition 1.** [8]. If the chromatic polynomial of $\overline{G}$ is

$$P(\overline{G}, \lambda) = \sum_{i=0}^{p-1} b_i(G)(\lambda)_{p-i},$$

then the polynomial

$$h(G, x) = \sum_{i=1}^{p-1} b_i(G)x^{p-i}$$

is called *the adjoint polynomial* of $G$.

In [6], the $\sigma$–polynomial of $G$ can be written as follows:

$$\sigma(G, x) = \sum_{i=0}^{k} a_i x^{k-i},$$

where $k = p(G) - \chi(G)$ and $a_i$ is the number of ways of partition $V(G)$ into $p - i$ disjoint independent sets.

Let $h(G,x) = x^{\alpha(G)}h_1(G,x)$ such that $h_1(G,x)$ is a polynomial in $x$ with a nonzero constant. It is easy to show that $h_1(G,x) = \sigma(\overline{G},x)$. In particular, $\overline{\sigma}(G,x) = h_1(G,x)$ in [5], where $\overline{\sigma}(G,x) = \sigma(\overline{G},x)$.

Let $t(G)$ be the lowest term of $h(G,x)$. Lemmas 1 and 2 can follow from the above argument and Definition 1.

**Lemma 1.** Let $G$ be a graph. Then $\overline{G}$ is uniquely $n$–colorable if and only if $t(G) = x^n$.

**Lemma 2.** Let $m$ be a positive integer. Then $t(K_m) = x$.

**Lemma 3.** [8]. Let $G$ be a graph with $k$ components $G_1, G_2, \ldots, G_k$. Then

$$h(G,x) = \prod_{i=1}^{k} h(G_i,x).$$

**Lemma 4.** [8]. Let $G$ be a graph with edge $uv$. If $uv$ is not an edge of any triangle of $G$, then

$$h(G,x) = h(G-uv,x) + xh(G-\{u,v\},x).$$

**Definition 2.** [5]. Let $G$ be a graph with vertex $v$, and $N_G(v) = A \cup B$ and $A \cap B = \emptyset$. $H = (G,v,A,B)$ is the graph defined as follows: $V(H) = (V(G) - \{v\}) \cup \{v_1,v_2\}(v_1,v_2 \notin V(G))$ and $E(H) = \{e \in E(G) | e$ is not incident with $v\} \cup \{v_1u | u \in A\} \cup \{v_2u | u \in B\}$. Then $H$ is called *the graph obtained from $G$ by splitting vertex $v$*, and write $H = G|_v$. $H$ is said to be *a vertex splitting graph* of $G$ if $H$ is obtained from $G$ by a sequence of vertex splitting.

**Definition 3.** [5]. Let $G$ be a graph and $A, B \subseteq V(G)$. $A$ and $B$ is said to be *adjacent in* $G$ if for any $x \in A$ and $y \in B$, we have $xy \in E(G)$.

**Lemma 5.** [5]. Let $G$ be a graph with vertex $v$ and $H = (G,v,A,B)$. Then $h_1(H,x) = h_1(G,x)$ if and only if $A$ and $B$ are adjacent in $G$.

We consider a graph $G$ containing $K_3$ as a subgraph. Let $\{u,v,w\} = V(K_3) \subset V(G)$. If there exists a vertex of degree 2 in $V(K_3)$ , without loss of generality, say $d_G(v) = 2$. Let $A = \{u\}$ and $B = \{w\}$. It is clear that $A$ and $B$ are adjacent in $G$. By Lemma 5, we know that $h_1(G,x) = h_1(G|_v,x)$. In particular, $h_1(D_n,x) = h_1(T(1,2,n-3),x)$.

# 3 Uniquely $n$–colorable graph

Denote by $G(K_m,P_s)$ the graph obtained from $K_m$ and $P_s$ by identifying a vertex of $K_m$ with a vertex of degree 1 of $P_s$. Clearly $G(K_m,P_1) = K_m$.

**Lemma 6.** Let $s \geq 3$. Then

(1) $h(G(K_m, P_s), x) = x(h(G(K_m, P_{s-1}), x) + h(G(K_m, P_{s-2}), x))$;

(2) $t(G(K_m, P_s)) = \begin{cases} x^{\frac{s+1}{2}}, & \text{if } s \text{ is odd,} \\ \frac{s+2}{2} x^{\frac{s+2}{2}}, & \text{if } s \text{ is even.} \end{cases}$

**Proof:** (1) Let $uv \in E(G(K_m, P_s))$ such that $d(v) = 1$, and $d(u) = 2$ and $uv \in E(P_s)$. When $s \geq 3$, by Lemma 4 we have

$$\begin{aligned} h(G(K_m, P_s), x) &= h(G(K_m, P_s) - uv, x) + xh(G(K_m, P_s) - \{u, v\}, x) \\ &= x(h(G(K_m, P_{s-1}), x) + h(G(K_m, P_{s-2}), x)). \end{aligned}$$

(2) Proof by induction on $s$.

From Lemma 4, we have

$$h(G(K_m, P_2), x) = x(h(K_m, x) + h(K_{m-1}, x))$$

and

$$h(G(K_m, P_3), x) = x(h(G(K_m, P_2), x) + h(K_m, x)).$$

Hence, by Lemma 2 we know that

$$t(G(K_m, P_2)) = 2x^2 \text{ and } t(G(K_m, P_3)) = x^2.$$

Suppose that (2) holds when $s < k$, where $k \geq 4$. From the first part of the proof, we have

$$h(G(K_m, P_k), x) = x(h(G(K_m, P_{k-1}), x) + h(G(K_m, P_{k-2}), x)).$$

If $k$ is even, then $t(G(K_m, P_{k-1})) = x^{\frac{k}{2}}$ and $t(G(K_m, P_{k-2})) = \frac{k}{2} x^{\frac{k}{2}}$ by the induction hypothesis. Hence $t(G(K_m, P_k)) = \frac{k+2}{2} x^{\frac{k+2}{2}}$.

If $k$ is odd, then $t(G(K_m, P_{k-1})) = \frac{k+1}{2} x^{\frac{k+1}{2}}$ and $t(G(K_m, P_{k-2})) = x^{\frac{k-1}{2}}$ by the induction hypothesis. Therefore $t(G(K_m, P_k)) = x^{\frac{k+1}{2}}$.

This completes the proof of the theorem.

By Lemmas 1 and 6, Theorem 1 and Corollary 1 is easily proved.

**Theorem 1.** Let $s$ be an odd integer. Then $\overline{G(K_m, P_s)}$ is uniquely $\frac{s+1}{2}$-colorable graph with $m + s - 1$ vertices.

**Corollary 1.** For any $n \geq 1$ and $m \geq 2$, we have

(1) $\overline{P_{2n}}$ is uniquely $n$-colorable graph with $2n$ vertices,

(2) $\overline{D_{2m+1}}$ is uniquely $n$-colorable graph with $2m + 1$ vertices.

**Theorem 2.** Let $G$ be a graph with $k$ components $G_1, G_2, \cdots, G_k$. Then $\overline{G}$ is uniquely $n$-colorable if and only if each complement $\overline{G_i}$ is uniquely

$m_i$–colorable and $n = \sum_{i=1}^{k} m_i$, where $i = 1, 2, \cdots, k$.

**Proof:** By Lemma 3, $t(G) = \prod_{i=1}^{k} t(G_i)$. The theorem can be easily proved by Lemma 1.

By applying Theorems 1 and 2, we can find many families of uniquely $n$–colorable graphs with $m$ vertices, where $n \geq 3$ and $m \geq 3$. Hence we have the following corollary.

**Corollary 2.** There exist infinite uniquely $n$–colorable graphs with $m$ vertices, where $n \geq 3$, $m \geq 3$ and $m \geq n$.

Let $v \in V(G(K_m, P_s))$ such that $v \in V(K_m)$ and $d(v) = m - 1$. By $G'(K_m, P_s)$ we denote the graph obtained from $G(K_m, P_s)$ by splitting vertex $v$ (see Definition 2).

**Theorem 3.** Let $s$ be an odd integer. Then $\overline{G'(K_m, P_s)}$ and $\overline{G(K_m, P_s) \cup K_1}$ is uniquely $\frac{s+3}{2}$–colorable graph which are chromatically equivalent.

**Proof:** Choose $A$ and $B$ such that $A \cap B = \emptyset$ and $A \cup B = V(K_m) \backslash v$, where $A \neq \emptyset$ and $B \neq \emptyset$. Note that $A$ and $B$ are adjacent in $G(K_m, P_s)$. By Lemma 5, we have $h_1(G'(K_m, P_s), x) = h_1(G(K_m, P_s), x)$. Since

$$p(G'(K_m, P_s)) = p(G(K_m, P_s) \cup K_1),$$

we have

$$h(G'(K_m, P_s), x) = h(G(K_m, P_s) \cup K_1, x).$$

From Lemma 6, we know that

$$t(G'(K_m, P_s)) = \begin{cases} x^{\frac{s+3}{2}}, & \text{if } s \text{ is odd,} \\ \frac{s+4}{2} x^{\frac{s+4}{2}}, & \text{if } s \text{ is even.} \end{cases}$$

Theorem 3 holds by Lemma 1.

Note that there are many ways of choosing $A$ and $B$ such that $A$ and $B$ are adjacent in $G(K_m, P_s)$ . Hence there exist many graphs which are chromatically equivalent with $\overline{G(K_m, P_s) \cup K_1}$. Let $v$ be splitted into two vertices $v_1$ and $v_2$. If $d(v_1) \neq 1$ or $d(v_2) \neq 1$, then $v_1$ or $v_2$ can be splitted in $G'(K_m, P_s)$. Hence, we can obtain many graphs which are chromatically equivalent with $\overline{G(K_m, P_s) \cup 2K_1}$. By repeating the above process and Theorem 2, we can obtain the following corollary.

**Corollary 3.** Let $n \geq 3$. There exist infinite uniquely $n$–colorable graphs which are chromatically equivalent.

## Acknowledgements

This work is supported by National Science Foundation of China (No. 10061003) and Science Foundation of the Ministry of Education. The authors would like to express their sincere thanks to the referees for their helpful comments.

## References

[1] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, North–Holland, Amsterdam, 1976.

[2] C. Y. Chao and Z. Chen, On uniquely 3–colorable graph, *Discrete Math.* **112** (1993), 21-27.

[3] C. Y. Chao and Z. Chen, On uniquely 3–colorable graph(II), *Discrete Math.* **189** (1998), 259-265.

[4] G. L. Chia, On graphs uniquely colorable the action of their automorphism groups, *Discrete Math.* **162** (1996), 281–184.

[5] Q. Du, On $\sigma$–Equivalence and $\chi$–Equivalence of Graphs, *J. Graph Theory.* **21** (1996), 211-217.

[6] R. Korfhage, $\sigma$–polynomial and graph coloring, *J. Combin. Theory (B)* **24** (1978), 137–153.

[7] R. Liu, A new method for finding chromatic polynomial of graph and its applications, *Kexue Tongbao* **32** (1987), 1508–1509.

[8] R. Liu, Adjoint polynomials and chromatically unique graphs, *Discrete Math.* **172** (1997), 85-92.

[9] R. C. Read, An introduction of chromatic polynomials, *J. Combin. Theory* **4** (1968), 52-71.