Generalized Bhaskar Rao Designs with Block Size 4 Signed over Elementary Abelian Groups

Gennian Ge*† Malcolm Greig‡ Jennifer Seberry§

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Abstract

de Launey and Seberry have looked at the existence of Generalized Bhaskar Rao designs with block size 4 signed over elementary Abelian groups and shown that the necessary conditions for the existence of a $(v, 4, \lambda; EA(g))$ GBRD are sufficient for $\lambda > g$ with 70 possible basic exceptions. This article extends that work by reducing those possible exceptions to just a $(9, 4, 18h; EA(9h))$ GBRD, where $\gcd(6, h) = 1$, and shows that for $\lambda = g$ the necessary conditions are sufficient for $v > 46$.

Key words: Generalized Bhaskar Rao design, Group divisible design.

AMS subject classification: 05B05.

1 Introduction

The existence problem for generalized Bhaskar Rao designs signed over elementary Abelian groups (our terminology and notation will be defined later in Section 2) was first considered for block size 3 by Lam and Seberry, and completed by Seberry who showed that the necessary conditions were sufficient [39, 52]. One of the less-obvious necessary conditions for $k = 3$ imposes no restriction for $k = 4$. (We will discuss this point in more detail in Section 3.) In [23], de Launey and Seberry looked at the existence problem

*Department of Mathematics, Suzhou University, Suzhou, 215006, P. R. China

†Researcher supported in part by YNSFC Grant 10001026

‡Greig Consulting, 317-130 East 11th St., North Vancouver, BC, Canada V7L 4R3

§School of Information Technology and Computer Science, University of Wollongong, Wollongong, NSW 2522, Australia.

for generalized Bhaskar Rao designs with block size 4 signed over elementary Abelian groups and showed that the necessary conditions were sufficient with the relatively short list of 70 possible exceptions when $\lambda > |G|$; they also provided a number of useful constructions for the case $\lambda = |G|$, but did not explore this case systematically. The main aim of this article is to extend that work and remove as many as we can of the possible exceptions when $\lambda > |G|$, and to study the $\lambda = |G|$ case. Our progress is assisted by several factors.

Firstly, more complete lists of PBDs are now available.

Secondly, Chaudhry et al. [16] showed how Wilson's fundamental construction could be adapted to deal with Bhaskar Rao designs and we continue this adaptation in Section 5. The value of the introduction of Wilson's fundamental construction was later demonstrated (along with a fair amount of computation) in tackling the EA(2) case [24, 30, 34]. One important incidental benefit of the introduction of Wilson's fundamental construction was that this provided a common structure to a number of apparently unrelated constructions that had appeared in the literature and pulling these constructions apart (reflected by our common use later of Theorem 5.2 followed by Theorem 5.4) allowed us to focus on the provision, and systematic use, of components for these constructions.

Thirdly, again along with a fair amount of computation, Mathon [43] had produced a very useful a $(45, 12, 3; Z_3)$ GBRD.

Lastly, we spend some effort in Section 6 in exploring the relationship of GBRDs with other designs, which has the benefit of allowing us some way of exploiting designs constructed for other purposes, for example, difference matrices which are discussed in Section 4. In fact, one of the earliest reasons for studying GBRDs was the relationship with GDDs [51, 53, 54], illustrated by Lemma 6.1, and a $(v, k, \lambda; G)$ GBRD can be regarded as a $(k, \lambda/|G|)$ GDD of type $|G|^v$ with G acting regularly on the points and semi-regularly on the lines [20]. This places GBRDs as intermediate between an unstructured GDD and a GDD which is given by a difference family. Difference families are well-known structures of considerable interest. Purely from the view of GBRD existence, one would like to have ways of exploiting these better studied difference families. Symbiotically, one could study GBRDs to provide more insight into recursive construction methods for difference families. There can also be practical benefits in direct constructions of GDDs if one proceeds via GBRDs, as then the problem can be broken down into the two steps of constructing the underlying BIBD and signing it. This can be a decided benefit if one step is easy or there already is software available for it.

We have already indicated that GBRDs have been studied as difference families. GBRDs have also proved useful in constructing RGDDs (here the underlying design was a *resolvable* BIBD). The GBRDs were given in a slightly different notation as "labelled" RBIBDs, see [44].

In Section 2 we give most of our basic definitions and notation. In Sections 3–6 we develop our general construction methods. Although we pay particular attention to $k = 4$, these constructions often have more general applicability. Finally, our constructions of $(v, 4, \lambda; EA(G))$ GBRDs are done in Sections 7–12, and our open cases are summarized in Section 13. We also correct some errors in [23] in an Appendix.

2 Terms and Notation

Some of the terminology we will use is quite standard in design theory; see [9]. For clarification of our notation (specifically how we indicate the standard parameters), we refer to pairwise balanced designs (PBDs), (including BIBDs), as (v, K, λ) designs, where K is a list of block sizes that possibly occur. The notation $K \cup \{h^*\}$ means we can identify one block of size h in the design, and the other blocks have sizes in K (more blocks of size h are allowed only if h is in K). A group divisible design is referred to as a (K, λ) GDD of group type $t_1^{g_1} \dots t_n^{g_n}$ if there are g_i groups of size t_i and transversal designs of order n as $\mathrm{TD}_\lambda(k, n)$, dropping the subscript when $\lambda = 1$; note that a $\mathrm{TD}_\lambda(k, n)$ is a (k, λ) GDD of group type n^k. The prefix "R" will denote a resolvable design; we say more about resolvability in Remark 4.2. All these designs may be represented by their v by b incidence matrices.

We should caution the reader that the "group" in a group divisible design is just a collection (of points), and group there has only its everyday meaning, not the special mathematical meaning that occurs when group is used to denote an algebraic structure. Unfortunately, we shall have to use group in both senses, but the meaning should be clear from the context.

To accomplish our aim of signing BIBDs we will also need to sign GDDs. The *generalized Bhaskar Rao versions* (over the group G) of these designs is given by "signing" the non-zero elements of the v by b incidence matrix of the design, say N, i.e., by replacing the non-zero elements of N by elements from the algebraic group G to give a new matrix W. If $\{i\}$ and $\{j\}$ were two points of the design, and $\sum_t n_{it} n_{jt} = \lambda_{ij}$, then in the signed version the list $w_{it} w_{jt}^{-1}$ for $t = 1, 2, \dots, b$ would contain $\lambda_{ij}/|G|$ copies of every element of G, plus some zeros (we use the convention that $0g = g0 = 0 = 0^{-1}$ for any $g \in G$, with 0 being a non-group element, unchanged from N). Note that, although it is more usual to represent the group operation as addition in Abelian groups, we will not follow this convention rigidly (especially for 2- and 3-groups and subgroups). More importantly, our definition covers non-Abelian signings which we will discuss in Section 6.

Such designs were introduced by Bhaskar Rao [10, 11] (under the name "balanced orthogonal design") and were initially just signed over Z_2, or equivalently, the 1's in N were replaced with ± 1's; the signing property in this case just amounts to row orthogonality using the standard inner product. Earlier, Butson [14, 15] had considered more general signings, but as generalized Hadamard designs, where the underlying BIBD is trivial. One usually refers to signings over the group of order 2 (usually considered in its multiplicative form as noted above) as Bhaskar Rao signings, and over other groups as *generalized* Bhaskar Rao signings; other authors have used "generalized" to include signings of designs other than BIBDs over the group of order 2. The order two group is implicit for Bhaskar Rao designs, but since we prefer to make it explicit here, we will use the term generalized for all our signed designs.

Our aim in this article is to consider signings of $(v, 4, \lambda)$ BIBDs over elementary Abelian groups. For clarification, by an elementary Abelian group of order n we mean the direct product of the cyclic groups Z_{p_i} for every prime p_i in the prime factorization of n; we denote this group as $EA(n)$, although for $n = 6$ we often use the equivalent group Z_n.

3 Some necessary conditions

Clearly, the balance condition for $(v, k, \lambda; G)$ GBRDs, namely that every element of G occurs equally often in the list $w_{it}w_{jt}^{-1}$ necessitates that $|G|$ divide λ.

Since we are concerned in this article with signings of $(v, 4, \lambda)$ BIBDs, it is worth noting that the necessary conditions for existence of the BIBD are sufficient [35], and those necessary conditions can be expressed as in Table 1.

Table 1: Necessary conditions on v for the existence of a $(v, 4, \lambda)$ BIBD

$\lambda \pmod 6$	v
1,5	$1, 4 \pmod{12}$
2,4	$1 \pmod 3$
3	$0, 1 \pmod 4$
0	any $v \geq 4$

There is another necessary condition known for GBRDs, but this introduces no fresh restrictions when $k = 4$. This condition was introduced by Street and Rodger in a more general setting [54], and later specialized by Seberry to BRDs (see [52, Theorem 1] for a proof) and GBRDs [39]. We now give a new, simplified proof of [52, Theorem 1].

Theorem 3.1 *If G has a normal subgroup of order 2, then it is necessary that $\lambda\binom{v}{2} \equiv 0 \pmod 4$ if k is odd.*

Proof: Consider the placement of the signings of the normal subgroup. Half the total pairs must have the same signing, and half opposite. If there are n_i blocks containing i of the identity in the subgroup, then we have:

$$\lambda\binom{v}{2} = 2\sum_i n_i i(k-i).$$

The theorem follows by noting that, if k is odd, $i(k-i)$ is always even since exactly one of the factors is. ∎

Remark 3.2 It is clear that the proof of [52, Theorem 1] is simplified, but the result claimed is apparently weaker. Now we want to establish our necessary conditions are the same as [52, Theorem 1]. Note that

$$\lambda\binom{v}{2} = b\binom{k}{2}$$

where b is the total number of blocks in the GBRD.

Seberry's requirement for k odd is that $b(k-1) \equiv 0 \pmod 8$, or $b \equiv 0 \pmod 2$ if $k \equiv 5 \pmod 8$ (when we have $\binom{k}{2} \equiv 2 \pmod 4$), or $b \equiv 0 \pmod 4$ if $k \equiv 3 \pmod 4$ (when we have $\binom{k}{2} \equiv 1 \pmod 2$) and no restriction on b if $k \equiv 1 \pmod 8$ (when we have $\binom{k}{2} \equiv 0 \pmod 4$), so these are the same restrictions as ours when k odd.

Seberry's requirement for k even is that $b(k-4) \equiv 0 \pmod 4$, or $b \equiv 0 \pmod 2$ if $k \equiv 2 \pmod 4$ (when we have $\binom{k}{2} \equiv 1 \pmod 2$), and no restriction on b if $k \equiv 0 \pmod 4$ (when we have $\binom{k}{2} \equiv 0 \pmod 2$), so we should explain our lack of a restriction in the $k \equiv 2 \pmod 4$ case.

The answer is that in our GBRD we must have λ even, and this entails that b be even. Consider the underlying $(v, 4n+2, 2t)$ BIBD: the replication count is $r = 2t(v-1)/(4n+1)$, which is clearly even, and the block count is $b = vr/(4n+2)$. Rewriting this as

$$b = \frac{v(v-1)t}{(2n+1)(4n+1)},$$

we see that b must be even.

4 Difference and Generalized Hadamard Matrices

A $(|G|, k, \lambda)$ *difference matrix* is a $(k, k, \lambda|G|; G)$ GBRD, and a *generalized Hadamard matrix* $\mathrm{GH}(|G|, \lambda)$ is both a $(|G|, \lambda|G|, \lambda)$ difference matrix and a $(\lambda|G|, \lambda|G|, \lambda|G|; G)$ GBRD. A $(|G|, k, \lambda)$ difference matrix is represented by a k by $|G|$ matrix whose elements are entries of G; each column of this matrix can be taken as a base block and developed over G to yield an $\mathrm{RTD}_\lambda(k, |G|)$ on the point set $I_k \times G$; the first part of each element (i.e., the group identifier) is implicitly given by the row label. Since the development of every column spans the point set, we also have resolvability; the group set for the RTD is $\{i\} \times G$ for each $i \in I_k$. Deleting a row from a $(|G|, k, \lambda)$ difference matrix produces a $(|G|, k-1, \lambda)$ difference matrix. A difference matrix has no empty entries. One variant with some empty entries is a quasi-difference matrix.

A GH$(|G|, \lambda)$ yields an RTD with the maximum possible k. The following theorem is well-known; a proof can be found in [38, Proposition 3.1]; we will outline the proof, since it gives us a GBRD non-existence result.

Theorem 4.1 *If $g > 1$ and an $RTD_\lambda(k, g)$ exists, then $k \leq g\lambda$, and so no $(k, k, \lambda; G)$ GBRD exists if $k < \lambda$.*

Proof: It suffices to show no $\mathrm{RTD}_\lambda(g\lambda + 1, g)$ exists. Consider the points of one block; they have no more incidences in that parallel class, and even the most uniform placement in the remaining blocks yields too many pairs for the points of that block. ■

Remark 4.2 Normally, in a $(|G|, k, \lambda; G)$ difference matrix each column generates a parallel class and gives us an $\mathrm{RTD}_\lambda(k, |G|)$, and we can add a s points each to λ parallel classes to get a $(\{k, k+1\}, \lambda)$ GDD of type $|G|^k s^1$. However, if we have converted the difference matrix into a $(k, k, \lambda|G|; G)$ GBRD, then adding points to the parallel classes does not necessarily produce a GBRD: we have to sign the added points and for the GBRD balance, we would like not only every point to occur in the parallel class, but that every point occur with every possible signing λ times; in such a case we can add an arbitrarily signed point. We will call a set of blocks where each element occurs with every possible signing exactly once a *signed parallel class*.

Theorem 4.3 *If q is a prime power, then a $(q, q, 1; EA(q))$ difference matrix exists.*

Proof: The multiplication table for GF(q) gives the difference matrix. ■

Corollary 4.4 *If $q \geq 4$ is a prime power, then a $(q, 4, 1; EA(q))$ difference matrix exists.*

Jungnickel [9, Theorem VII.3.14], generalizing a result of Butson's [14], found the following result.

Theorem 4.5 *If q is a prime power, then a $(q, 2q, 2; EA(q))$ difference matrix exists.*

Any $\mathrm{RTD}_\lambda(k, 2)$ can be represented as a $(2, k, 2; Z_2)$ difference matrix over Z_2 by simply taking one representative of each parallel class as a column of the matrix. It is easy to verify that no $\mathrm{RTD}_\lambda(3, 2)$ exists if λ is odd. This is the key fact behind Drake's result [25].

Theorem 4.6 *A $(|G|, 3, \lambda; G)$ difference matrix does not exist if λ is odd and $|G| \equiv 2 \pmod 4$.*

For composite numbers that are not prime powers, $(|G|, k, 1; EA(G))$ difference matrices are known for a number of orders; for Abel's $|G| = 20$, see [34]; for $|G| \in \{24, 48, 55\}$ see [57]; for $|G| = 36$ we quote a result of Wojtas' [58]; for Mills' $|G| = 39$, see [45] or [9, Example VIII.3.19]; the rest can be found in [2].

Table 2: Known $(|G|, k, 1; EA(G))$ Difference Matrices

$\lvert G\rvert$	k	$\lvert G\rvert$	k	$\lvert G\rvert$	k	$\lvert G\rvert$	k	$\lvert G\rvert$	k	$\lvert G\rvert$	k
12	6	15	5	20	5	21	6	24	7	28	6
33	6	35	6	36	9	39	4	40	8	44	6
45	7	48	7	51	6	52	6	55	7	56	8

We can extend Table 2 using a result of Jungnickel's [37, Theorem 6].

Theorem 4.7 *If q is a prime power, and there exists r MOLS of order $q+1$, then there exists a $(q^2+q+1, r, 1; Z_{q^2+q+1})$ difference matrix.*

Corollary 4.8 *There exists a $(57, 7, 1; Z_{57})$ difference matrix.*

However, there is a general result due to Evans [27, Theorem 1], which we restate in an equivalent form.

Theorem 4.9 *If $n > 3$ is odd and not divisible by 9, then there exists an $(n, 4, 1; Z_n)$ difference matrix.*

To discuss $(4, 4, \lambda; G)$ we need a few basic results. One basic method of construction is the process we call *juxtaposition*, that is, the placing of several signed incidence matrices side-by-side to build an example with the desired index.

Lemma 4.10 *If a $(v, k, \lambda_1; G)$ GBRD and a $(v, k, \lambda_2; G)$ GBRD both exist, then a $(v, k, \lambda_1 + \lambda_2; G)$ GBRD exists also.*

We next need a couple of results that we expand on later in Sections 5 and 6, but which specialize to well-known results on difference matrices [17, Theorems IV.11.25–26].

Theorem 4.11 *If a* $(|G|, k, \lambda; G)$ *difference matrix and a* $(|H|, k, \mu; H)$ *difference matrix both exist, then a* $(|G| \cdot |H|, k, \lambda\mu; G \times H)$ *difference matrix exists.*

Theorem 4.12 *If a* $(|G|, k, \lambda; G)$ *difference matrix exists and* $H \lhd G$*, then a* $(|G|/|H|, k, \lambda|H|; G/H)$ *difference matrix exists.*

We can now state our results on $(4, 4, \lambda; EA(n))$ GBRDs.

Theorem 4.13 *A necessary condition for a a* $(4, 4, \lambda; EA(n))$ *GBRD to exist is that* n *divide* λ*. If* n *divides* λ*, then a* $(4, 4, \lambda; EA(n))$ *GBRD exists unless:*

a. n *is even, when we have the definite exception of* $n \equiv \lambda \equiv 2 \pmod 4$*,*

b. n *is odd, when we have the definite exception of* $n = \lambda = 3$.

Proof: The non-existence when $n = 3$ follows from Theorem 4.1, and from Theorem 4.6 for $n \equiv \lambda \equiv 2 \pmod 4$.

If the prime power decomposition of n contains no prime powers smaller than 4, then a $(4, 4, n; EA(n))$ GBRD may be constructed by repeated application of Theorem 4.11 to the designs given by Corollary 4.4.

If $n \equiv 2 \pmod 4$, a $(4, 4, 2n; EA(n))$ GBRD may be constructed by starting with a $(4, 4, 2h; EA(h))$ GBRD where $h = 2$ and this design is given by applying Theorem 4.12 to the $(4, 4, 4; EA(4))$ given by Corollary 4.4, or $h = 6$ and this design is given by applying Theorem 4.12 to the $(4, 4, 12; EA(12))$ given in Table 2, and then applying Theorem 4.11 using the $(4, 4, n/h; EA(n/h))$ GBRD constructed above.

If $n \equiv 0 \pmod{12}$, then a $(4, 4, n; EA(n))$ GBRD may be constructed by starting with a $(4, 4, h; EA(h))$ with $h = 12$ or 24 given in Table 2, and, by choice of h we know the prime power decomposition of n/h contains no prime powers smaller than 4, so again we may then apply Theorem 4.11 using the $(4, 4, n/h; EA(n/h))$ GBRD constructed above.

If $3 < n$ and $n \equiv 3 \pmod 6$ (so 3 is the highest power of 3 dividing n), we will suppose p is some other prime dividing n and let $h = 3p$. Then a $(4, 4, n; EA(n))$ GBRD may be constructed by starting with a $(4, 4, h; EA(h))$ GBRD with $h = 3p$ given by Theorem 4.9 and, by choice of h we know the prime power decomposition of n/h contains no prime powers smaller than 4, so again we may then apply Theorem 4.11 using the $(4, 4, n/h; EA(n/h))$ GBRD constructed above.

If $n = 3$, then a $(4, 4, 6; EA(3))$ GBRD is given by Theorem 4.5, and Theorem 4.12 gives a $(4, 4, 9; EA(3))$ GBRD.

Finally, we may juxtapose the above incidence matrices if necessary to achieve the appropriate index. ■

5 Basic Constructions

There are several basic ways to construct specific examples of GBRDs.

The simplest is the direct replacement of 1's in the incidence matrix of a BIBD with group elements. This is called signing and we give some examples of this method later.

However, for many of constructions, we will need to use the recursive techniques developed in [16, 34]. These basic methods were developed or adapted for signings over Z_2, and so need further adaptation for other groups. In Chaudhry et al. [16], the powerful recursive construction, known as Wilson's fundamental construction, or WFC, was adapted from group divisible designs, or GDDs, to the analogous Bhaskar Rao type designs, or BRGDDs, and we now need to define generalized Bhaskar Rao group divisible designs, or GBRGDDs.

Definition 5.1 *A $(K, \lambda; G)$ Generalized Bhaskar Rao GDD, or GBRGDD, is defined by its signed incidence matrix, W, in which every element takes on the non-group value zero or the value of some member of the group G, and which has the property that if we replace all the non-zero elements by ones then the resulting matrix, N, is the incidence matrix of a (K, λ) GDD. If $\{i\}$ and $\{j\}$ were two points of the design and $\sum_t n_{it}n_{jt} = \lambda_{ij}$, then in the signed version the list $w_{it}w_{jt}^{-1}$ for $t = 1, 2, \ldots, b$ would contain $\lambda_{ij}/|G|$ copies of every element of G. We use the group type of the underlying GDD as the group type of the GBRGDD.*

Note that the incidence matrix, N, of a (K, λ) GDD has the property that every off-diagonal element of NN^T is either λ or zero; (the diagonal need not be constant).

Our adaptation of Wilson's fundamental construction, given below as Theorem 5.2, together with the use of Theorem 5.4, gives a uniform framework to several other constructions that have appeared in the literature (see [23, 39, 50, 51]). Although the authors in [16, 34] were only concerned with the group Z_2, others dealing with simpler direct products of Abelian groups [23, 39] and the more complex case of non-Abelian groups [19, 47] have given the special case of "breaking the blocks" (Theorem 5.3 below). Although the primary concern in this article is with elementary Abelian groups here, we have given some of our constructions for more general groups.

We now give the variant of WFC for GBRGDDs.

Theorem 5.2 *Let H be a normal subgroup of G. Suppose we have a master $(K', \lambda_1; G/H)$ GBRGDD with group type $\mathcal{G} = (|G_1|, \ldots, |G_g|)$. Suppose $w(x)$ is a positive weighting function defined for each point of the master design. Also, we have an ingredient $(K, \lambda_2; H)$ GBRGDD with a group type vector of $W(B) = (|w(b_1)|, \ldots, |w(b_{k'})|)$ for each block $B = \{b_1, \ldots, b_{k'}\}$. Then there is a $(K, \lambda_1\lambda_2; G)$ GBRGDD with group type*

$$W(G) = \left(\sum_{x \in G_1} w(x), \ldots, \sum_{x \in G_g} w(x) \right).$$

Proof: If we ignore the signing aspect, this is simply the WFC for GDDs [55], so the only aspect we need deal with is the signing.

We note that the elements of G/H are actually the cosets, which we will denote by a system of representatives, say $S = \{s_1, \ldots, s_{|G|/|H|}\}$. Now the property of being signed over G/H means that between any two points from different groups, say i and j, the products $s_i \cdot s_j^{-1}$ form $\lambda|H|/|G|$ systems of representatives considering the totality of such products over the blocks common to both points.

Now we replace the signing Hs_i by s_i. The signing rule we use is that when we are looking at a master block containing b_i with a sign of $s(b_i)$, and in a block of the appropriate ingredient design we have $w_j(b_i)$ with a sign of $t(w_j(b_i))$, then in the resultant design we give the point a sign of $s(b_i) \cdot t(w_j(b_i))$.

Now consider a master block containing the pair of points i and j with the signing s_i and s_j; for each ingredient design on this pair the point pair $(w_m(b_i), w_n(b_j))$ has a signing (t_m, t_n) in the ingredient design with the property that, over this whole design, $t_m \cdot t_n^{-1}$ gives $\lambda_2/|H|$ copies of H. Hence the signing we generate has the property in the resultant design that the products $s_i t_m \cdot (s_j t_n)^{-1} = s_i(t_m \cdot t_n^{-1})s_j^{-1}$ contain $\lambda_2/|H|$ copies of $s_i H s_j^{-1} = H(s_i s_j^{-1})$ from this block, and so, noting that $s_i s_j^{-1}$ is part of a system of representatives by the GBR property of the master design, considering all the blocks in the master design, we have $(\lambda_1|H|/|G|) \cdot (\lambda_2/|H|)$ systems of representatives, each multiplied by the coset H; i.e., the resultant design will contain $\lambda_1\lambda_2/|G|$ copies of all cosets, as required. ■

Theorem 5.3 is a very useful special case of Theorem 5.2. It describes a construction that is is known as "breaking the blocks"; in fact, it is so useful that we prefer to invoke its use by referring to the more descriptive phrase "breaking the blocks," rather than as "Theorem 5.3".

Theorem 5.3 *Let H be a normal subgroup of G. Suppose we have a master $(v, K, \lambda_1; G/H)$ GBRD, and for every block size $j \in K$, we have a $(j, k, \lambda_2; H)$ GBRD. Then there is a $(v, k, \lambda_1\lambda_2; G)$ GBRD.*

Proof: Treating the master GBRD as a GBRGDD with group type 1^v, we give every point of the master GBRGDD a weight of one in Theorem 5.2. ■

We next look at filling in the groups of the GBRGDD.

Theorem 5.4 *Let us suppose that we have a $(k, \lambda; H)$ GBRGDD with group type $\mathcal{G} = (|G_1|, |G_2|, \ldots, |G_g|)$, and for the first group, we have a $(|G_1| + w, k, \lambda; H)$ GBRD, and for the remaining groups we have a $(|G_i| + w, k, \lambda; H)$ GBRD that is missing a $(w, k, \lambda; H)$ GBRD subdesign, then we have a $(v + w, k, \lambda; H)$ GBRD, where $v = \sum_i |G_i|$, which contains a $(|G_1| + w, k, \lambda; H)$ GBRD subdesign.*

Proof: Augment the point set of the GBRGDD with w new points, (sometimes called the infinite points), and use the $(|G_i| + w, k, \lambda; H)$ GBRD missing a $(w, k, \lambda; H)$ GBRD subdesign, to fill the i-th group (for $i > 1$), ensuring that the missing subdesign is aligned on the w new points. Finally, use the $(|G_1| + w, k, \lambda; H)$ GBRD to complete the design. ■

Remark 5.5 Every $(v, k, \lambda; H)$ GBRD has a $(w, k, \lambda; H)$ GBRD subdesign if $w = 0$ or 1: these trivial subdesigns contain no blocks.

We can also derive the following variant of a result originally due to Lam and Seberry [39]; note that we do not require that the missing subdesign exist.

Theorem 5.6 *If there exists a $(v, k, \lambda; G)$ GBRD, and a $(u + w, k, \lambda; G)$ GBRD missing a $(w, k, \lambda; G)$ GBRD, and if further there is a TD(k, u); then there exists a $(uv + w, k, \lambda; G)$ GBRD missing a $(w, k, \lambda; G)$ GBRD. If there also exists a $(u + w, k, \lambda; G)$ GBRD or a $(w, k, \lambda; G)$ GBRD, then there exists a $(uv + w, k, \lambda; G)$ GBRD containing a $(u + w, k, \lambda; G)$ GBRD subdesign and a $(v, k, \lambda; G)$ GBRD subdesign.*

Proof: Take the $(v, k, \lambda; G)$ GBRD as the master in Theorem 5.2, and give each point a weight of u. The TD provides the ingredient, and generates a $(k, \lambda; G)$ GBRGDD of type u^v. Then fill the groups using Theorem 5.4, to get the result. ■

6 More Constructions

In this section we provide a link which allows us to exploit several published designs that are important for our subsequent existence arguments.

The first result, and its proof, is taken from [31, p. 124].

Lemma 6.1 *If there exists a $(v, k, \lambda; G)$ GBRD on the point set I_v containing b blocks then there exists a $(k, \lambda/|G|)$ GDD of type $|G|^v$.*

Proof: For any $g \in G$, let P_g denote the permutation matrix corresponding to the development of g over G. Now, by replacing every occurrence of g in the incidence matrix of the GBRD by P_g and every 0 by the $|G|$ by $|G|$ zero matrix, we get the incidence matrix of the GDD. ■

It is clear that this result can be strengthened to yield the following theorem; this strengthening proved useful in helping mine the literature for direct constructions of GDDs that could be converted into GBRDs.

Theorem 6.2 *There exists a $(v, k, \lambda; G)$ GBRD on the point set I_v containing b blocks iff there exists a $(k, \lambda/|G|)$ GDD of type $|G|^v$ that is given by a difference family containing b blocks developed over G with the point set $I_v \times G$.*

Corollary 6.3 *If $v \in \{41, 61, 81\}$, then there is a BRD$(v, 5, 2)$.*

Proof: See [59, Lemmas 2.1 and 2.7]; their constructions of a $GDD(5, 2^v)$ are by base blocks developed over $Z_v \times Z_2$. ■

Given a (K, λ) GDD of type $ng_1, ng_2, \ldots, ng_m$ we may form a $(K, n^2\lambda)$ GDD of type $g_1, g_2, \ldots, g_m$ by simply collapsing points in an arbitrary n to 1 fashion, so long as we collapse n old points from the same group into a new point in the corresponding group.

However, if a (K, λ) GDD of type $(ng)^v$ is given by a difference family over $I_v \times G_n$ where $|G_n| = ng$, and $N \lhd G_n$ with $|N| = n$, and we collapse within the cosets of N, then we may form a $(K, n\lambda)$ GDD of type g^v; here the point set will be $I_v \times G_n/N$.

Now, using the relationship given in Theorem 6.2, we could derive the following result of Gibbons and Mathon [32, Theorem 2]. We have also given their proof, as it is so brief.

Theorem 6.4 *If there exists a $(v, k, \lambda; G)$ GBRD and $N \lhd G$. Then there exists a $(v, k, \lambda; G/N)$ GBRD.*

Proof: Use the homomorphism from G to G/N with kernal N to obtain the new design from the hypothesized one. It is easily verified that the new design is a GBRD. ■

In the literature, it seems that whenever points are collapsed over a normal subgroup the information contained in that signing within the normal subgroup is discarded, presumably as no one noticed a use for it. Now we indicate how that signing information can be retained in a GBRGDD setting.

We have already seen, in Theorem 6.2, that replacing the signing group element g in a GBRD by its corresponding permutation matrix P_g gives a GDD. If (x_{it}, y_{jt}) is the list of the signings for points $\{x\}$ and $\{y\}$ in the t-th block where both points are present, the generalized Bhaskar Rao property is that every element of the group has $\lambda/|G|$ representations in the list $x_{it} \cdot y_{jt}^{-1}$; the corresponding property for the permutation matrix representation is that $\sum P_{x_{it}} P_{y_{jt}}^{-1} = \lambda/|G|J$ where J is the all-ones matrix of order $|G|$.

Let $N \lhd G$. What we want now is a signed permutation matrix presentation for the points, where the permutation matrix is for the development of g over G/N, and the signings are from N. We already know, from Theorem 6.4, that we have a representation such that $\sum P_{x_{it}} P_{y_{jt}}^{-1} = \lambda|N|/|G|J$ where J is the all-ones matrix of order $|G|/|N|$, and where P_g here is the unsigned matrix for the development of g over G/N. So now we want to sign this matrix over N so that each $\lambda|N|/|G|$ entry in $\lambda|N|/|G|J$ is replaced by a collection of $\lambda/|G|$ copies of each element of N.

Theorem 6.5 *Let G be a finite Abelian group, with $N \times H$, i.e., G is the direct product of N and $H = G/N$. If there exists a $(v, K, \lambda; G)$ GBRD, then there exists a $(K, \lambda; N)$ GBRGDD of type $|H|^v$.*

Proof: Here the group operation is given by $(a, b) \cdot (x, y) = (ax, by)$. If $g = (n, h) = (n, 1) \times (1, h)$ we take the permutation representation of h over H, and sign every element with n. Clearly these signed permutation matrices are an isomorphic form of the group. Their group operation is matrix multiplication of the matrices, with the product of two non-zero elements being combined by the original group operation in N. (Technically, matrix multiplication is defined over a field rather than a group, and we only have multiplication, not addition, in a group; however, as we only use permutation matrices, the question of addition doesn't really arise, by adopting the convention that the non-group elements (the nominal zeros) in the permutation matrix have an empty group product with both group and non-group elements).

Finally, in the signed incidence matrix of the GBRD, replacing (n, h) by nP_h produces the signed incidence matrix of the GBRGDD. ■

Example 6.6 An example with $Z_2 \lhd Z_2 \times Z_2$. The replacement matrices are given below.

$$00 \mapsto \begin{pmatrix} 0 & \\ & 0 \end{pmatrix} \quad 01 \mapsto \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \quad 10 \mapsto \begin{pmatrix} & 0 \\ 0 & \end{pmatrix} \quad 11 \mapsto \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$$

Theorem 6.7 *Let Z_{mn} be a finite Abelian group, with $Z_n \lhd Z_{mn}$, and take $Z_{mn}/Z_n \simeq Z_m$. If there exists a $(v, K, \lambda; Z_{mn})$ GBRD, then there exists a $(K, \lambda; Z_n)$ GBRGDD of type m^v.*

Proof: Let C be the circulant of order m whose only non-empty element in the first row is in the last column, and let (the unsigned) $M_i = C^i$ for $i = 0, 1, \ldots, m-1$.

Now sign these matrices with the group element 0 on or below the main diagonal, and with 1 above the diagonal.

Compute M_{am+i} for $a = 1, 2, \ldots, n-1$ by adding $a \pmod n$ to the non-empty elements of M_i for $i = 0, 1, \ldots, m$.

As in Theorem 6.5, these M's form a group that is isomorphic to Z_{mn} and, in the signed incidence matrix of the GBRD, replacing g by M_g produces the signed incidence matrix of the GBRGDD. ■

Example 6.8 An example with $Z_2 \lhd Z_4$. The replacement matrices are given below.

$$0 \mapsto \begin{pmatrix} 0 & \\ & 0 \end{pmatrix} \quad 1 \mapsto \begin{pmatrix} & 1 \\ 0 & \end{pmatrix} \quad 2 \mapsto \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \quad 3 \mapsto \begin{pmatrix} & 0 \\ 1 & \end{pmatrix}$$

Example 6.9 An example with $Z_4 \lhd Z_8$. The replacement matrices are given below.

$$0 \mapsto \begin{pmatrix} 0 & \\ & 0 \end{pmatrix} \quad 1 \mapsto \begin{pmatrix} & 1 \\ 0 & \end{pmatrix} \quad 2 \mapsto \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \quad 3 \mapsto \begin{pmatrix} & 2 \\ 1 & \end{pmatrix}$$

$$4 \mapsto \begin{pmatrix} 2 & \\ & 2 \end{pmatrix} \quad 5 \mapsto \begin{pmatrix} & 3 \\ 2 & \end{pmatrix} \quad 6 \mapsto \begin{pmatrix} 3 & \\ & 3 \end{pmatrix} \quad 7 \mapsto \begin{pmatrix} & 0 \\ 3 & \end{pmatrix}$$

Example 6.10 An example with $Z_3 \lhd Z_9$. The replacement matrices are given below.

$$0 \mapsto \begin{pmatrix} 0 & & \\ & 0 & \\ & & 0 \end{pmatrix} \quad 1 \mapsto \begin{pmatrix} & & 1 \\ 0 & & \\ & 0 & \end{pmatrix} \quad 2 \mapsto \begin{pmatrix} & 1 & \\ & & 1 \\ 0 & & \end{pmatrix}$$

$$3 \mapsto \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \quad 4 \mapsto \begin{pmatrix} & & 2 \\ 1 & & \\ & 1 & \end{pmatrix} \quad 5 \mapsto \begin{pmatrix} & 2 & \\ & & 2 \\ 1 & & \end{pmatrix}$$

$$6 \mapsto \begin{pmatrix} 2 & & \\ & 2 & \\ & & 2 \end{pmatrix} \quad 7 \mapsto \begin{pmatrix} & & 0 \\ 2 & & \\ & 2 & \end{pmatrix} \quad 8 \mapsto \begin{pmatrix} & 0 & \\ & & 0 \\ 2 & & \end{pmatrix}$$

Now we quote the Frobenius-Stickleberger theorem on the structure of finite Abelian groups (see e.g., [49, Theorem 4.2.6]).

Theorem 6.11 *An Abelian group is finite if and only if it is a direct product of finitely many cyclic groups with prime power orders.*

Using Theorem 6.11 with Theorems 6.5 and 6.7, and noting that the Kronecker product of permutation matrices yields a permutation representation for the corresponding direct product group, we immediately obtain a general result for the Abelian case.

Theorem 6.12 *Let G be a finite Abelian group. If a $(v, K, \lambda; G)$ GBRD exists and $N \lhd G$, then there exists a $(K, \lambda; N)$ GBRGDD of type $(|G|/|N|)^v$.*

Example 6.13 A non-Abelian example with $Z_n \lhd D_n$. where D_n is the dihedral group $\langle a, b : a^n = b^2 = (ab)^2 = 1\rangle$. The replacement matrices are given below.

$$a^i \mapsto \begin{pmatrix} a^i & \\ & a^{-i} \end{pmatrix} \quad a^i b \mapsto \begin{pmatrix} & a^i \\ a^{-i} & \end{pmatrix}$$

We had intended to conjecture that Theorem 6.12 would hold for non-Abelian groups also, so although any subgroup in an Abelian group is obviously normal, we included normality in the hypotheses of the above theorems as an indication of what we think the non-Abelian version ought to be.

However, when we looked at the dihedral group we found we could give a signed matrix representation when we chose to sign over one of the basic normal subgroups ($\langle a : a^n = 1\rangle$) as shown in Example 6.13, but could not over the other ($\langle b : b^2 = 1\rangle$), nor over $D_m \lhd D_{mn}$. We don't intend to prove this here, but the reader might infer the truth from consideration of $D_3 \simeq S_3$. There are only six permutation matrices of order 3, so we would take an asymmetric one to represent a in the normal subgroup $\langle a : a^3 = 1\rangle$, and a^2 would be the other, and I_3 the identity; now we'ld like to sign these over $\langle d : b^2 = 1\rangle$ to distinguish the elements of D_3, but we must have $(a^i b)^2 = 1$, so $a^i b$ must be represented by a symmetric permutation matrix, since $P^{-1} = P^T$ for any permutation matrix P.

It is not clear to us what governs when we can sign over a normal subgroup, and when we can't.

One particularly useful application of Theorem 6.12 is to the difference matrices of Theorem 4.3.

Theorem 6.14 *Let $0 \le n \le N$. If p is a prime power, and $k \le p^N$, then there exists a $(k, p^{N-n}; EA(p^{N-n})$ GBRGDD of type $(p^n)^k$.*

Example 6.15 A construction of a $(8, 4; EA(4))$ GBRGDD of type 2^8 is constructed from the multiplication table of $GF(2^3)$ generated by a root of the primitive equation $x^3 + x + 1 = 0$. The original elements of the form $Cx^2 + Bx + A$ are replaced by elements of the form C_{AB} where the subscript represents the signing over $EA(4)$; the 16 blocks of the design are formed by developing C over $Z_2 \simeq GF(8)/GF(4)$. Each column generates two blocks, and the group label of each point is implicitly given by the row label.

0_e	0_e	0_e	0_e	0_e	0_e	0_e	0_e
0_e	0_a	0_b	1_e	0_{ab}	1_b	1_{ab}	1_a
0_e	0_b	1_e	0_{ab}	1_b	1_{ab}	1_a	0_a
0_e	1_e	0_{ab}	1_b	1_{ab}	1_a	0_a	0_b
0_e	0_{ab}	1_b	1_{ab}	1_a	0_a	0_b	1_e
0_e	1_b	1_{ab}	1_a	0_a	0_b	1_e	0_{ab}
0_e	1_{ab}	1_a	0_a	0_b	1_e	0_{ab}	1_b
0_e	1_a	0_a	0_b	1_e	0_{ab}	1_b	1_{ab}

Another useful application of Theorem 6.12 is to Bose's relative difference set for the punctured AG$(2, q)$ [12]. We now state an instance of Bose's construction.

Theorem 6.16 *If $GF(q^2)$ is the extension field of $GF(q)$, with a typical element of $GF(q^2)$ being $ax + b$ with $a, b \in GF(q)$, then the set of discrete logarithms*

$$D = \{\log(ax + 1) : a \in GF(q)\}$$

forms a relative difference set over Z_{q^2-1} for a $(q, 1)$ GDD of type $(q-1)^{q+1}$. The groups are those points having the same residue modulo $(q+1)$ in Z_{q^2-1}. This design can be completed to $AG(q)$ with the short block

$$\{\infty, 0, (q+1), 2(q+1), \ldots, (q-2)(q+1)\}.$$

Corollary 6.17 *If q is a prime power, then a $(q+1, q, q-1; Z_{q-1})$ GBRD exists.*

Proof: If D is the set given by Theorem 6.16, and we represent a point x of Z_{q^2-1} in the new point set $I_{q+1} \times Z_{q-1}$ by

$$x \mapsto (x \pmod{(q+1)}, \lfloor x/(q+1) \rfloor).$$

Now consider the development of the relative difference set in Z_{q^2-1}. This can be done by writing down the first $(q+1)$ translates, then successively adding $(q+1)$ to this set. Mapping this process into the new point set, it is clear that the GDD can be represented by these $(q+1)$ base blocks in the new point set (developed over Z_{q-1}) and the result now follows by Theorem 6.1. ■

Corollary 6.18 *If q is a prime power, and n is a factor of $q-1$, then a $(q+1, n; Z_n)$ GBRGDD of type $((q-1)/n)^{q+1}$ exists.*

Proof: Apply Theorem 6.12 to Corollary 6.17. ∎

Example 6.19 Using tables (e.g., [5, 33]) we can apply Bose's construction of Theorem 6.16 to compute a relative difference set over Z_{24} for a $(5,1)$ GDD of type 4^6 as $\{0, 17, 21, 8, 22\}$. We write down the first six translates as columns, apply the mapping $x \mapsto (x \pmod 6), \lfloor x/6 \rfloor)$, and so form the (essentially unique [32]) $(6, 5, 4; Z_4)$ GBRD given by the following incidence matrix.

$$\begin{array}{cccccc} 0 & 18 & 0 & 0 & 12 & - \\ - & 1 & 19 & 1 & 1 & 13 \\ 8 & - & 2 & 20 & 2 & 2 \\ 21 & 9 & - & 3 & 21 & 3 \\ 22 & 22 & 10 & - & 4 & 22 \\ 17 & 23 & 23 & 11 & - & 5 \end{array} \quad \mapsto \quad \begin{array}{cccccc} 0 & 3 & 0 & 0 & 2 & - \\ - & 0 & 3 & 0 & 0 & 2 \\ 1 & - & 0 & 3 & 0 & 0 \\ 3 & 1 & - & 0 & 3 & 0 \\ 3 & 3 & 1 & - & 0 & 3 \\ 2 & 3 & 3 & 1 & - & 0 \end{array}$$

We can now use the replacement scheme illustrated in Example 6.8 to get a $(5, 2; Z_2)$ GBRGDD of type 2^6.

Example 6.20 A similar example, but where the resulting GBRGDD is not signed over an *elementary* Abelian group is given by the punctured $AG(9)$. Using tables (e.g., [33]) we find $f(x) = x^2 + y^7x + y^7$ is a primitive polynomial in $GF(81)$ as an extension of $GF(9)$; y is a constant in $GF(81)$ and y is a root of $y^2 = y+1$ in GF(9). Applying Boses's construction of Theorem 6.16, we get a $(9,1)$ GDD of type 8^{10} as $\{0, 32, 39, 57, 63, 66, 65, 28, 44\}$ and, as in Example 6.19, form a $(10, 9, 8; Z_8)$ GBRD given by the following incidence matrix.

$$\begin{array}{cccccccccc} 0 & 4 & 3 & 6 & 7 & 7 & 5 & 7 & 4 & - \\ - & 0 & 4 & 3 & 6 & 7 & 7 & 5 & 7 & 4 \\ 3 & - & 0 & 4 & 3 & 6 & 7 & 7 & 5 & 7 \\ 6 & 3 & - & 0 & 4 & 3 & 6 & 7 & 7 & 5 \\ 4 & 6 & 3 & - & 0 & 4 & 3 & 6 & 7 & 7 \\ 6 & 4 & 6 & 3 & - & 0 & 4 & 3 & 6 & 7 \\ 6 & 6 & 4 & 6 & 3 & - & 0 & 4 & 3 & 6 \\ 5 & 6 & 6 & 4 & 6 & 3 & - & 0 & 4 & 3 \\ 2 & 5 & 6 & 6 & 4 & 6 & 3 & - & 0 & 4 \\ 3 & 2 & 5 & 6 & 6 & 4 & 6 & 3 & - & 0 \end{array}$$

We can now use the replacement scheme illustrated in Example 6.9 to get a $(9, 4; Z_4)$ GBRGDD of type 2^{10}.

We could also generate a $(9, 2; Z_2)$ GBRGDD of type 4^{10} as in the proof of Theorem 6.7.

The GBRGDDs produced by Theorem 6.14 and Corollary 6.18 prove to be very useful. The application of Theorem 6.12 to the difference matrices of Table 2 also give helpful GBRGDDs. The two remaining constructions in this section were constructed directly.

There are two interesting signings over Z_3 of symmetric BIBDs. The first is Baker's $(15,7,3;Z_3)$ GBRD [6]. Only one of the $(15,7,3)$ BIBDs is signable, and this signing is unique (up to isomorphism).

Example 6.21 We construct a $(15,7,3;Z_3)$ GBRD on the point set $(I_2 \times Z_7) \cup \{\infty\}$.

$$(\infty_0, (0,0)_0, (0,4)_2, (0,6)_1, (1,0)_0, (1,4)_1, (1,6)_2)$$
$$((0,0)_0, (0,4)_1, (0,6)_2, (1,1)_0, (1,2)_2, (1,3)_2, (1,5)_2)$$
$$((0,0)_0, (0,1)_0, (0,2)_0, (0,3)_0, (0,4)_0, (0,5)_0, (0,6)_0)$$

Only the first two blocks are developed over Z_7.

The second interesting signing is Mathon's $(45,12,3;Z_3)$ GBRD [43].

Example 6.22 Here we give the incidence matrix of the $(45,12,3)$ BIBD that Mathon was able to sign. We build this matrix by stages. Let

$$a = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad b = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad c = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Now let U be given by the Kronecker product $U = I \otimes J$, where I and J have their usual meanings as 3 by 3 matrices. Let

$$V = \begin{pmatrix} c & c & c \\ c & c & c \\ c & c & c \end{pmatrix} \quad W = \begin{pmatrix} I & a & b \\ a & b & I \\ b & I & a \end{pmatrix} \quad X = \begin{pmatrix} b & a & I \\ a & I & b \\ I & b & a \end{pmatrix}.$$

We can now give the incidence matrix of a signable $(45,12,3)$ BIBD.

$$\begin{pmatrix} U & 0 & V & W & X \\ X & U & 0 & V & W \\ W & X & U & 0 & V \\ V & W & X & U & 0 \\ 0 & V & W & X & U \end{pmatrix}$$

Lemma 6.23 *The following GBRDs exist:*

a. *a* $(v, \{9, 10\}, 3; Z_3)$ *GBRD for* $33 \leq v \leq 34$.

b. *a* $(v, \{6, 7, 9^*\}, 3; Z_3)$ *GBRD for* $v = 24$.

c. *a* $(v, \{9, 10, 11, 12\}, 3; Z_3)$ *GBRD for* $36 \leq v \leq 45$;

d. *a* $(v, \{6, 7, 9, 10, 11, 12\}, 3; Z_3)$ *GBRD for* $27 \leq v \leq 30$;

e. *a* $(v, \{5, 6, 7, 8, 9\}, 3; Z_3)$ *GBRD for* $24 \leq v \leq 26$;

f. *a* $(v, \{4, 5, 6, 7, 8, 9\}, 3; Z_3)$ *GBRD for* $21 \leq v \leq 23$;

Proof: For part (a), note that deleting all (or all but one) of the points of any block of the BIBD (i.e., residualization) still gives us a signed matrix. For part (b), we may delete all of the points in the first two blocks in Example 6.22.

For the remaining GBRDs, delete rows of the signed incidence matrix, starting from the top. ■

7 Powers of 2

In this section we will only consider signings over EA(2^n). Since λ will be even for all GBRDs, the admissible v are $v \equiv 1 \pmod 3$ if $3 \not| \lambda$, and any $v \geq 4$ if $3 \mid \lambda$ subject to the restrictions of Theorem 4.6 when $v = 4$.

7.1 The Group of Order 2

The case where we sign over Z_2 (or equivalently, the multiplicative group $\{+1, -1\}$) is the classical case studied by Bhaskar Rao, and the existence of $(v, 4, \lambda; Z_2)$ GBRD has previously been fairly well studied. de Launey and Seberry solved the existence problem for $\lambda > 2$ with just 7 exceptions on the values $v \in \{28, 34, 39\}$. Deleting three non-concurrent lines of PG$(2, 7)$ gives a $(36, \{5, 6\}, 1)$ PBD, then restoring 3 deleted non-collinear points, one from each of the three deleted lines, yields a $(39, \{5, 6, 7\}, 1)$ PBD. We can break the blocks of this PBD to get a $(39, 4, 6; Z_2)$ GBRD, as noted in [34]. A direct construction of a $(v, 4, 6; Z_2)$ GBRD for $v = 28$ and 34 is known [34] and by juxtaposing with one or two copies of a $(v, 4, 4; Z_2)$ GBRD, we can also deal with $\lambda = 10$ and 14. Existence for $(v, 4, 2; Z_2)$ GBRDs with $v \equiv 1 \pmod 6$ was also shown in [34], and for $(v, 4, 2; Z_2)$ GBRDs with $v \equiv 4 \pmod 6$ was studied in [24], where existence was shown with at most 28 possible exceptions, plus two definite exceptions. These possible exceptions were all constructed by Ge and Lam [30]. For $v = 4$, it follows from Theorem 4.6 that no design exists unless $\lambda \equiv 0 \pmod 4$. Also, no $(10, 4, 2; Z_2)$ GBRD exists [21]. We may summarize these results.

Theorem 7.1 *A* $(4,4,\lambda;Z_2)$ *GBRD exists iff* $\lambda \equiv 0 \pmod 4$. *For* $v > 4$, *the necessary conditions for a* $(v,4,\lambda;Z_2)$ *GBRD, namely that* $\lambda \equiv 0 \pmod 2$ *and* $\lambda(v-1) \equiv 0 \pmod 3$, *are sufficient with the definite exception of a* $(10,4,2;Z_2)$ *GBRD.*

7.2 The Group of Order 4

For EA(4), de Launey and Seberry had shown the necessary conditions were sufficient for a $(v,4,\lambda;EA(4))$ GBRD when $\lambda > 4$ with the possible exception of a $(v,4,12;EA(4))$ GBRD for $v \in \{15,23\}$. For $(v,4,4;EA(4))$ GBRDs, de Launey and Seberry were able to break the blocks of a $(v,4,1)$ BIBD with a $(4,4,4;EA(4))$ GBRD to deal with $v \equiv 1,4 \pmod{12}$, but found no way to generally exploit their example of a $(19,4,4;EA(4))$ GBRD to deal with the other cases where $v \equiv 1 \pmod 3$. It is known that no $(7,4,4;EA(4))$ GBRD exists [21].

We now deal with some of these missing cases by direct construction or by using a known PBD result.

Example 7.2 We construct a $(31,4,4;EA(4))$ GBRD on point set Z_{31}.

$(0_{00},7_{00},14_{11},22_{00})$	$(0_{00},3_{00},6_{10},10_{01})$	$(0_{00},1_{01},3_{01},5_{11})$
$(0_{00},1_{11},2_{01},10_{00})$	$(0_{00},2_{11},16_{01},19_{10})$	$(0_{00},11_{00},20_{10},30_{00})$
$(0_{00},7_{10},13_{10},19_{01})$	$(0_{00},5_{00},11_{01},16_{11})$	$(0_{00},4_{00},13_{01},17_{00})$
	$(0_{00},5_{01},13_{11},21_{11})$	

We construct a $(43,4,4;EA(4))$ GBRD on point set Z_{43}.

$(0_{00},5_{00},22_{00},33_{00})$	$(0_{00},3_{00},6_{11},10_{10})$	$(0_{00},4_{00},8_{10},19_{01})$
$(0_{00},12_{10},25_{01},34_{00})$	$(0_{00},11_{10},16_{00},34_{11})$	$(0_{00},14_{00},26_{01},41_{10})$
$(0_{00},1_{00},24_{10},27_{11})$	$(0_{00},1_{01},30_{00},32_{01})$	$(0_{00},1_{10},3_{10},5_{01})$
$(0_{00},14_{11},19_{00},37_{00})$	$(0_{00},6_{01},12_{11},19_{11})$	$(0_{00},7_{11},14_{10},22_{01})$
$(0_{00},8_{00},16_{01},26_{10})$	$(0_{00},1_{11},10_{01},23_{00})$	

We construct a $(55,4,4;EA(4))$ GBRD on point set Z_{55}.

$(0_{00},25_{00},32_{00},44_{00})$	$(0_{00},11_{11},24_{00},38_{00})$	$(0_{00},17_{11},20_{00},52_{10})$
$(0_{00},7_{11},14_{10},22_{01})$	$(0_{00},8_{00},16_{10},42_{01})$	$(0_{00},12_{11},18_{00},45_{00})$
$(0_{00},2_{00},11_{10},20_{10})$	$(0_{00},17_{10},36_{01},51_{10})$	$(0_{00},14_{01},26_{00},42_{00})$
$(0_{00},5_{10},10_{10},16_{11})$	$(0_{00},24_{01},31_{11},37_{11})$	$(0_{00},5_{01},30_{10},51_{00})$
$(0_{00},1_{10},3_{00},5_{11})$	$(0_{00},1_{00},2_{01},17_{01})$	$(0_{00},1_{11},23_{11},26_{10})$
$(0_{00},23_{01},27_{10},45_{11})$	$(0_{00},6_{10},14_{11},26_{01})$	$(0_{00},9_{11},19_{10},40_{01})$

Example 7.3 We construct a $(15,4,12;EA(4))$ GBRD on the point set Z_{15}.

$$B_0 = (0_e, 1_a, 5_e, 10_e) \quad B_1 = (0_e, 5_b, 7_b, 10_{ab}),$$
$$B_2 = (0_e, 1_a, 5_a, 10_b) \quad B_3 = (0_e, 5_{ab}, 10_a, 13_{ab}),$$
$$C_0 = (1_w, 2_x, 4_y, 8_z).$$

Now in C_0 we replace (w, x, y, z) by:

$$(e,e,e,e) \quad (e,a,b,ab) \quad (e,b,ab,a) \quad (e,ab,a,b) \quad (e,e,ab,ab)$$
$$(e,ab,b,a) \quad (e,b,e,b) \quad (e,e,e,a) \quad (e,ab,a,ab) \quad (e,b,ab,e).$$

These 10 base blocks together with $B_0, \ldots, B_3$ generate our design.

Remark 7.4 Abel and Ling [4, Lemma 2.2.1] give some direct constructions that can be interpreted, via Theorem 6.2, as $(v, \{4,5\}, 4; EA(4))$ GBRDs having u signed parallel classes on the blocks of size 4; for $v = 23$ and 27, we have $u = 4$, and for $v = 63$ and 97, we have $u = 7$.

From an Abel and Ling design we get a $(23, \{4,5\}, 4; EA(4))$ GBRD; now break the blocks with $(k, 4, 3)$ BIBDs for $k = 4$ and 5 for a $(23, 4, 12; EA(4))$ GBRD.

We now state a result of Rees and Stinson [48, Theorem 8.28], incorporating Drake and Larson's [26] bound for the "only if" part. Using this result with $t = 19$, and breaking the blocks of the PBD will later help establish Theorem 7.6.

Theorem 7.5 *Let $v \equiv 7, 10 \pmod{12}$ and let $t \equiv 7, 10 \pmod{12}$; then a $(v, \{4, t^*\}, 1)$ PBD exists if and only if $v \geq 3t + 1$.*

We summarize our EA(4) results.

Theorem 7.6 *The necessary conditions for a $(v, 4, \lambda; EA(4))$ GBRD, i.e., that $\lambda \equiv 0 \pmod 4$ and $\lambda(v-1) \equiv 0 \pmod 3$, are sufficient with the definite exception of the non-existing $(7, 4, 4; EA(4))$ GBRD and the possible exception of $(v, 4, 4; EA(4))$ GBRDs for $v \in \{10, 22, 34, 46\}$.*

Proof: We can break the blocks of a $(v, 4, 1)$ BIBD to deal with the $1, 4 \pmod{12}$ cases, and get the $7, 10 \pmod{12}$ cases from the $(19, 4, 4; EA(4))$ GBRD and Theorem 7.5 if $v \geq 58$. The designs for $v = 31$ and 43 are given in Example 7.2. ∎

7.3 The Groups of Order 8 or More

We now look at EA(n) where $n = 2^r \geq 8$. Here de Launey and Seberry found the necessary conditions were sufficient for $\lambda > n$; this follows essentially from the $(v, 4, \lambda; Z_2)$ GBRD result, since we can use a $(4, 4, n/2; EA(n/2)$ GBRD to break the blocks of the $(v, 4, \lambda; Z_2)$ GBRD. However, now we have a stronger result for EA(2), we can also apply this construction for $\lambda = n$ and state a stronger result.

Theorem 7.7 *Let $n \geq 8$ be a power of 2. Then the necessary conditions for a $(v, 4, \lambda; EA(n))$ GBRD, namely that $\lambda \equiv 0 \pmod n$ and $\lambda(v-1) \equiv 0 \pmod 3$, are sufficient with the possible exception of a $(10, 4, n; EA(n))$ GBRD.*

8 Powers of 3

In this section we will only consider signings over EA(3^n). Since λ will be a multiple of three for all GBRDs, the admissible v are $v \equiv 0, 1 \pmod 4$ if λ is odd and any $v \geq 4$ if λ is even.

8.1 The Group of Order 3

For $\lambda > 3$, de Launey and Seberry solved the existence problem for signings over Z_3. For $\lambda = 3$, no $(4, 4, 3; Z_3)$ GBRD exists, but de Launey and Seberry showed $v \equiv 1 \pmod 4$ was sufficient, and gave examples of $(v, 4, 3; Z_3)$ GBRDs for $v \equiv 0 \pmod 4$ with $4 < v \leq 28$. We will use these to deal with the $\lambda = 3$ case more generally.

We first state a result of Abel et al. [1, Theorem 4.5].

Theorem 8.1 *Let $H_{0,1(4)} = \{n : n \equiv 0, 1 \pmod 4\} \cap \{n : n \geq 8\}$, and let $H_{4,5(8)} = \{n : n \equiv 4, 5 \pmod 8\} \cap \{n : n \geq 8\}$. Also let $A = H_{0,1(4)} \cap \{n : n \leq 56\}$, $B = H_{4,5(8)} \cap \{n : 60 \leq n \leq 93\}$ and $K = A \cup B \cup \{88, 101\}$. Then K is a PBD basis for $H_{0,1(4)}$ i.e., a $(v, K, 1)$ PBD exists for every $v \in H_{0,1(4)}$. All the elements of K are essential with the possible exception of 101.*

Bennett et al. [7] examined the PBD closure of $\{5, 8, 9\}$, and determined that 12, 13, 16, 17, 20, 24, 28, 29, 32, 33, and 44 were definite exceptions, and the possible exceptions were 52, 60, 68, 84, 92, 96, 100, 104,…,308, 312. For our purposes, it would suffice to exhibit a $(v, 4, 3; Z_3)$ GBRD for the values $v = 5$, 8 and 9, and all values in the intersection of Abel et al.'s basis set and Bennett et al.'s exception set.

Theorem 8.2 *The necessary conditions for a* $(v, 4, \lambda; Z_3)$ *GBRD, namely that* $\lambda \equiv 0 \pmod 3$ *and* $\lambda v(v-1) \equiv 0 \pmod 4$ *are sufficient with the definite exception of a* $(4, 4, 3; Z_3)$ *GBRD.*

Proof: We need to establish a $(v, 4, 3; Z_3)$ GBRD exists for $v \in \{32, 44, 52, 60, 68, 84, 92, 96\}$. For $v = 32$, 52 or 68: there are 4 by g difference matrices known over EA(g) for $g = 24$, 39 or 51. Applying Theorem 6.12, we get $(4, 3; Z_3)$ GBRGDDs of types 8^4, 13^4 or 17^4, which we may then fill by Theorem 5.4. For $v = 44$, a solution (as a $(4, 1)$ GDD of type 3^{44}) is given in [28, Lemma 2.1]. For $v = 60$ or 96: we may truncate a TD of order 12 or 11 to get a GDD of type 12^5 or $11^8 7^1$ and then fill the groups, possibly using an extra point, to get a $(v, \{5, 8, 9, 12\}, 1)$ PBD; we can break the blocks of these PBDs with $(k, 4, 3; Z_3)$ GBRDs. For $v = 84$: take a 4-GDD of type 4^7 (i.e., a $(28, 4, 1)$ RBIBD missing a parallel class) and give points a weight of 3 using a $(4, 3; Z_3)$ GBRGDD of type 3^4 as ingredient in Theorem 5.2 to get a $(4, 3; Z_3)$ GBRGDD of type 12^7 which we may then fill by Theorem 5.4. For $v = 92 = 13(8-1) + 1$: apply Theorem 5.6.

The existence for $\lambda > 3$ was shown in [23]. ■

8.2 The Group of Order 9

For EA(9), de Launey and Seberry had shown the necessary conditions were sufficient for a $(v, 4, \lambda; EA(9))$ GBRD when $\lambda > 9$ with the possible exception of a $(v, 4, 18; EA(9))$ GBRD for $v \in \{6, 14, 15, 18, 23, 26, 27, 38, 42, 47\}$; note we included $v = 14$ and 15. For $(v, 4, 9; EA(9))$ GBRDs, de Launey and Seberry were able construct $v \in \{4, 5, 9\}$.

Example 8.3 We construct an $(8, 4, 9; EA(9))$ GBRD on point set $Z_7 \cup \{\infty\}$.

$$\begin{array}{lll}
(0_{00}, 1_{00}, 2_{01}, 4_{12}) & (0_{00}, 1_{12}, 2_{02}, 4_{02}) & (0_{00}, 1_{11}, 4_{00}, 6_{20}) \\
(\infty_{00}, 0_{10}, 1_{01}, 3_{20}) & (\infty_{00}, 0_{00}, 2_{12}, 3_{11}) & (\infty_{00}, 0_{22}, 2_{02}, 3_{21})
\end{array}$$

We construct a $(12, 4, 9; EA(9))$ GBRD on point set $Z_{11} \cup \{\infty\}$.

$$\begin{array}{lll}
(0_{00}, 3_{00}, 5_{00}, 7_{01}) & (0_{00}, 2_{12}, 3_{01}, 4_{00}) & (0_{00}, 1_{21}, 5_{10}, 8_{01}) \\
(0_{00}, 2_{22}, 4_{21}, 5_{11}) & (0_{00}, 2_{11}, 7_{20}, 8_{20}) & (0_{00}, 4_{11}, 5_{20}, 6_{01}) \\
(\infty_{00}, 0_{10}, 3_{22}, 8_{20}) & (\infty_{00}, 0_{00}, 1_{01}, 3_{11}) & (\infty_{00}, 0_{21}, 6_{02}, 7_{12})
\end{array}$$

After a preliminary lemma, we will deal with some of these missing cases. Lemma 8.4.a is claimed in [8, Table II.3.17], citing [9], and is claimed in [46, Result 3.2], where the result seems to be attributed to Wilson, although the citation is [9]. Lemma 8.4.c is due to Hanani [36] and is cited by Wilson in [56]. Beth et al. give a proof of Lemma 8.4.c (only) in [9, Proposition IX.4.6].

Lemma 8.4 *Let* $v \equiv 0, 1 \pmod 4$.

a. If $v \notin \{8, 9, 12\}$, *then a* $(v, \{4, 5\}, 1)$ *PBD exists.*

b. If $v \notin \{5, 8, 9, 12\}$, *then a* $(v, \{4, 5, 4^*\}, 1)$ *PBD exists.*

c. A $(v, \{4, 5, 8, 9, 12\}, 1)$ *PBD exists.*

Proof: For part (a), we can use a $(v, 4, 1)$ BIBD to deal with $v \equiv 1, 4 \pmod{12}$, and adding 1, 4, 5 or 20 points to a $(v, 4, 1)$ RBIBD deals with the other residue classes modulo 12, but that only deals with $v \equiv 0 \pmod{12}$ for $v \geq 84$. For $v = 24$, 36 or 60, we can delete $d = 1$, 5 or 1 collinear points from a $(v + d, 5, 1)$ BIBD. Lamken et al. construct directly a $(48, \{4, 5\}, 1)$ with the property that its 4-blocks form nine parallel classes [40]. For $v = 72$, we fill in the groups of a $TD(4, 17)$ using 4 extra points; the filling $(21, \{4, 5, 4^*\}, 1)$ PBD can be constructed by deleting 4 collinear points from $AG(2, 5)$.

For $v = 21$, our original construction amounts to completing AG(2, 4) to PG(2, 4), but we have given an alternative construction which exhibits a block of size 4. Apart from $v = 21$, adding points to the RBIBD always leaves some blocks of size four, so we have already proved part (b).

Part (c) follows immediately from part (a). ■

Theorem 8.5 *The necessary conditions for a* $(v, 9, \lambda; EA(9))$ *GBRD, i.e., that* $\lambda \equiv 0 \pmod 9$ *and* $\lambda v(v-1) \equiv 0 \pmod 4$, *are sufficient with the possible exception of a* $(18, 4, 18; EA(9))$ *GBRD.*

Proof: The result for $\lambda = 9$ follows from Lemma 8.4.a and Example 8.3 and breaking the blocks of the PBD.

For $\lambda > 9$, we only have to deal with de Launey and Seberry's possible exceptions for $\lambda = 18$ noted above.

For all but $v = 23$ (and $v = 18$), we have a $(v, 4, 18; EA(18))$ GBRD (given below in Theorem 9.10); collapse points over the normal subgroup of order 2, using Theorem 6.4. For $v = 23$, a $(v, K, 3; Z_3)$ GBRD (with $K \subset \{4, 5, 6, 7, 8, 9, 10, 11, 12\}$) is given by deleting points from Mathon's design; we may break their blocks using $(k, 4, 6; Z_3)$ GBRDs. ■

8.3 The Groups of Order 27 or More

We now look at EA(n) where $27 \leq n = 3^r$. Here de Launey and Seberry found the necessary conditions were sufficient for $\lambda > n$; this follows essentially from the $(v, 4, \lambda; Z_3)$ GBRD result, since we can break the blocks of the $(v, 4, \lambda; Z_3)$ GBRD with a $(4, 4, n/3; EA(n/3))$ GBRD. However, de Launey and Seberry did not discuss the case $\lambda = n$.

Theorem 8.6 *Let $n \geq 27$ be a power of 3. Then the necessary conditions for a $(v, 4, \lambda; EA(n))$ GBRD, namely that $\lambda \equiv 0 \pmod{n}$ and $\lambda v(v-1) \equiv 0 \pmod{4}$, are sufficient.*

Proof: It only remains to deal with $\lambda = n$. We have a $(4, 4, n; EA(n))$ GBRD and a $(4, 4, n/3; EA(n/3))$ GBRD. We may use the latter design to break the blocks of a $(v, 4, 3; Z_3)$ GBRD for $v = 5$, 8, 9 or 12, and so have the necessary ingredients to break a $(v, \{4, 5, 8, 9, 12\}, 1)$ PBD, thence the result follows by Lemma 8.4. ∎

9 Some 2 mod 4 cases

Here we consider EA(n) groups of order $n = 2 \cdot 3^s \geq 6$. The only general restriction on v is given by Theorem 4.6 when $v = 4$.

9.1 The Group of Order 6

For the group EA(6) (or equivalently, Z_6), de Launey and Seberry solved the existence problem for $\lambda > 6$ with 21 exceptions in the range $8 \leq v \leq 34$. de Launey and Seberry constructed a number of small designs with $\lambda = 6$, in fact, all values of $v \leq 20$, except 4, 5, 6, 8, 17 and 19. For $v = 4$, it is known that no design exists unless $\lambda \equiv 0 \pmod{4}$. Also, no $(5, 4, 6; Z_6)$ GBRD exists [21].

Our first step is to augment de Launey and Seberry's direct constructions of $(v, 4, 6; Z_6)$ GBRDs. By exploiting Theorem 6.2, we obtained Examples 9.1 and 9.4 from [13, Lemmas 6.14–15].

Example 9.1 We construct a $(6, 4, 6; Z_6)$ GBRD on the point set $I_2 \times Z_3$.

$$(00_0, 01_3, 10_0, 11_1) \quad (00_0, 01_1, 11_0, 12_5) \quad (00_0, 01_5, 10_1, 12_4)$$
$$(00_0, 01_2, 02_0, 10_3) \quad (00_0, 10_2, 11_2, 12_0)$$

Example 9.2 We construct a $(17, 4, 6; Z_6)$ GBRD on the point set Z_{17}. This example was adapted from [18, Lemma 2.4].

$$(7_0, 10_0, 11_1, 6_4) \quad (9_0, 8_0, 2_1, 15_4) \quad (13_0, 4_0, 3_1, 14_4) \quad (12_0, 5_0, 1_1, 16_4)$$
$$(14_0, 3_0, 5_1, 12_4) \quad (16_0, 1_0, 13_1, 4_4) \quad (6_0, 11_0, 9_1, 8_4) \quad (2_0, 15_0, 7_1, 10_4)$$

Example 9.3 We construct a $(29, 4, 6; Z_6)$ GBRD on the point set Z_{29}.

$$B_0 = (1_0, 2_0, 3_1, 5_4) \quad C_0 = (1_0, 3_0, 0_1, 28_4)$$

Multiply B_0 and C_0 by 1, 16, 24, 7, 25, 23 and 20 to generate 14 base blocks.

Example 9.4 Let $v \equiv 3 \bmod 4$ be a prime power. A $(v,4,6;Z_6)$ GBRD may be constructed on the point set $GF(v)$. Let x be a primitive generator, and let α be chosen so that $x^{2\alpha}-1$ is not a square; (exactly one of $\alpha = 1$ or $\alpha = q-2$ will work since $x^2-1=-x^2(x^{-2}-1)$.) Our base blocks are:

$$(0_0, x_3^{\alpha+2i}, x_2^{2i}, -x_2^{2i}) \qquad \text{for } i = 0,1,\ldots,(q-3)/2.$$

Example 9.5 We construct a $(21,4,6;Z_6)$ GBRD on point set Z_{21}.

$(0_0,1_0,2_1,7_0)$ $(0_0,3_0,9_1,14_4)$ $(0_0,1_3,8_2,10_1)$ $(0_0,8_0,11_1,14_3)$
$(0_0,4_5,8_3,15_1)$ $(0_0,4_0,8_1,12_4)$ $(0_0,1_2,2_0,3_5)$ $(0_0,2_4,5_1,10_3)$
$(0_0,2_2,5_0,14_5)$ $(0_0,4_2,9_0,15_2)$

We construct a $(22,4,6;Z_6)$ GBRD on point set $Z_{21}\cup\{\infty\}$.

$(0_0,3_0,6_1,9_0)$ $(0_0,4_0,8_2,13_2)$ $(0_0,4_5,8_3,15_1)$ $(0_0,6_2,12_5,17_3)$
$(0_0,6_4,11_0,16_5)$ $(0_0,2_2,4_1,7_5)$ $(0_0,1_1,2_0,3_2)$ $(0_0,1_3,2_3,9_3)$
$(0_0,1_4,9_5,12_2)$ $(\infty_0,0_0,2_4,10_3)$ $(\infty_0,0_1,7_2,14_5)$

We construct a $(26,4,6;Z_6)$ GBRD on point set $Z_{25}\cup\{\infty\}$.

$(0_0,5_0,10_1,17_0)$ $(0_0,7_0,12_2,19_3)$ $(0_0,7_2,12_1,18_2)$ $(0_0,2_0,4_3,7_3)$
$(0_0,1_2,2_5,3_4)$ $(0_0,1_0,2_4,10_3)$ $(0_0,9_1,10_2,19_1)$ $(0_0,8_3,10_4,19_0)$
$(0_0,4_4,8_4,13_2)$ $(0_0,4_1,10_5,13_1)$ $(0_0,3_5,6_2,14_4)$ $(\infty_0,0_0,3_1,14_2)$
$(\infty_0,0_3,4_5,8_4)$

We construct a $(28,4,6;Z_6)$ GBRD on point set $Z_{27}\cup\{\infty\}$.

$(0_0,2_0,10_0,16_0)$ $(0_0,4_0,16_1,20_2)$ $(0_0,3_0,15_2,20_3)$ $(0_0,5_2,13_1,26_3)$
$(0_0,2_2,3_3,8_3)$ $(0_0,4_2,20_1,21_3)$ $(0_0,4_5,14_3,19_2)$ $(0_0,8_1,10_2,11_2)$
$(0_0,6_4,7_2,25_2)$ $(0_0,3_5,5_4,9_1)$ $(0_0,1_5,15_0,18_4)$ $(0_0,4_4,7_0,16_3)$
$(\infty_0,0_0,2_3,9_4)$ $(\infty_0,0_5,5_2,14_1)$

We construct a $(32,4,6;Z_6)$ GBRD on point set $Z_{31}\cup\{\infty\}$.

$(0_0,10_0,11_0,12_1)$ $(0_0,8_0,13_0,22_1)$ $(0_0,8_1,11_1,29_0)$
$(0_0,18_1,25_0,26_4)$ $(0_0,4_5,17_2,23_1)$ $(0_0,7_4,9_3,10_5)$
$(0_0,6_2,15_4,18_2)$ $(0_0,15_5,18_4,19_3)$ $(0_0,2_3,19_4,22_0)$
$(0_0,10_2,15_0,27_5)$ $(0_0,2_4,6_4,7_1)$ $(0_0,5_1,11_4,16_3)$
$(0_0,4_2,21_2,25_5)$ $(0_0,7_2,9_4,24_0)$ $(\infty_0,0_0,3_3,11_5)$
$(\infty_0,0_2,16_1,23_4)$

We construct a $(34,4,6;Z_6)$ GBRD on point set $Z_{33}\cup\{\infty\}$.

$(0_0,1_0,16_0,19_1)$ $(0_0,10_0,12_0,22_1)$ $(0_0,1_1,2_3,12_2)$
$(0_0,1_4,2_1,3_0)$ $(0_0,2_4,4_3,7_0)$ $(0_0,3_2,6_1,20_3)$
$(0_0,3_4,14_0,23_2)$ $(0_0,4_1,8_1,12_3)$ $(0_0,4_5,8_3,13_0)$
$(0_0,6_2,12_5,26_3)$ $(0_0,7_2,15_2,26_5)$ $(0_0,9_0,16_4,25_1)$
$(0_0,7_5,15_3,24_2)$ $(0_0,5_0,11_0,21_2)$ $(0_0,5_1,15_4,20_2)$
$(\infty_0,0_0,14_1,27_2)$ $(\infty_0,0_5,5_4,11_3)$

We construct a $(35,4,6;Z_6)$ GBRD on point set Z_{35}.

$(0_0,3_0,16_0,22_1)$	$(0_0,7_0,9_0,14_1)$	$(0_0,1_0,27_1,34_3)$
$(0_0,4_0,21_0,32_1)$	$(0_0,10_0,13_2,14_4)$	$(0_0,8_0,12_3,33_4)$
$(0_0,7_3,24_2,34_5)$	$(0_0,5_2,9_4,13_3)$	$(0_0,9_3,18_5,32_2)$
$(0_0,2_4,20_0,32_5)$	$(0_0,13_4,23_2,29_0)$	$(0_0,6_3,21_4,24_1)$
$(0_0,2_1,8_3,24_0)$	$(0_0,15_4,20_1,30_0)$	$(0_0,9_1,25_5,27_4)$
$(0_0,1_5,7_4,12_2)$	$(0_0,1_4,12_0,16_1)$	

We construct a $(38,4,6;Z_6)$ GBRD on point set $Z_{37}\cup\{\infty\}$.

$(0_0,6_0,13_0,30_1)$	$(0_0,1_0,7_1,13_3)$	$(0_0,9_0,17_0,25_1)$
$(0_0,10_0,17_3,28_1)$	$(0_0,21_0,25_2,29_2)$	$(0_0,9_4,19_0,33_2)$
$(0_0,1_2,22_4,35_5)$	$(0_0,1_3,2_4,12_2)$	$(0_0,2_5,19_3,22_3)$
$(0_0,3_2,4_1,19_1)$	$(0_0,11_1,14_5,23_2)$	$(0_0,3_5,9_2,29_3)$
$(0_0,5_2,11_0,23_0)$	$(0_0,5_0,10_3,15_1)$	$(0_0,14_3,24_2,29_1)$
$(0_0,30_2,32_5,33_3)$	$(0_0,2_0,6_5,21_4)$	$(\infty_0,0_0,14_1,21_3)$
	$(\infty_0,0_2,2_4,11_5)$	

([29]): We construct a $(44,4,6;Z_6)$ GBRD on point set $Z_{43}\cup\{\infty\}$.

$(13_1,20_1,29_1,35_1)$	$(6_1,16_4,21_0,36_2)$	$(15_1,22_3,25_1,33_1)$
$(13_1,24_3,28_4,30_0)$	$(6_1,8_1,9_1,27_2)$	$(10_1,19_0,35_5,40_4)$
$(10_1,14_1,23_2,39_3)$	$(4_1,20_4,23_1,37_2)$	$(3_1,38_2,41_1,1_3)$
$(3_1,14_1,15_4,25_2)$	$(7_1,12_4,19_2,32_4)$	$(2_1,41_5,43_2,1_0)$
$(2_1,11_5,16_3,31_4)$	$(7_1,12_2,18_0,33_4)$	$(18_1,30_5,34_3,42_5)$
$(5_1,11_3,34_2,38_5)$	$(24_1,26_2,37_5,43_4)$	$(9_1,17_5,26_1,27_0)$
$(5_1,17_0,36_5,40_4)$	$(8_1,22_1,29_4,42_4)$	$(\infty_1,4_1,21_2,28_3)$
	$(\infty_1,31_4,32_0,39_5)$	

We construct a $(45,4,6;Z_6)$ GBRD on point set Z_{45}.

$(0_0,1_0,2_1,3_0)$	$(0_0,5_0,7_2,24_0)$	$(0_0,11_0,13_3,23_0)$
$(0_0,15_1,17_5,42_1)$	$(0_0,1_4,28_0,42_2)$	$(0_0,11_1,12_3,13_0)$
$(0_0,3_1,9_0,32_1)$	$(0_0,4_0,9_5,17_2)$	$(0_0,4_4,12_2,41_5)$
$(0_0,5_2,11_4,17_1)$	$(0_0,6_0,34_3,38_2)$	$(0_0,3_2,29_5,33_2)$
$(0_0,10_5,25_3,30_1)$	$(0_0,6_1,21_4,31_2)$	$(0_0,16_4,19_1,39_2)$
$(0_0,7_5,14_0,22_1)$	$(0_0,20_0,27_3,36_5)$	$(0_0,7_0,26_2,31_5)$
$(0_0,9_4,21_5,31_1)$	$(0_0,18_1,27_4,40_5)$	$(0_0,8_2,16_2,27_1)$
	$(0_0,10_0,21_2,31_3)$	

Lemma 9.6 *A $(4,6;Z_6)$ GBRGDD of type 2^n exists for $n\in\{4,5,6\}$.*

Proof: For $n=4$, apply Theorem 6.12 to the $(12,4,1)$ difference matrix of Table 2. For $n=5$, break the blocks of a $(5,4,3;Z_3)$ GBRD with a $(4,2;Z_2)$ GBRGDD of type 2^4 constructed in Theorem 6.14. For $n=6$, use a $(5,4,3;Z_3)$ GBRD to break the blocks of a $(5,2;Z_2)$ GBRGDD of type 2^6 constructed in Example 6.19. ■

Corollary 9.7 *A* $(37, 4, 6; Z_6)$ *GBRD exists.*

Proof: Take the $(4, 6; Z_6)$ GBRGDD of type 2^6 constructed in Lemma 9.6, and give all points a weight of 3, and using a TD(4, 3) as the ingredient design to get a $(4, 6; Z_6)$ GBRGDD of type 6^6; fill the groups of this design using an extra point in Theorem 5.4, to get a $(37, 4, 6; Z_6)$ GBRD. ■

Theorem 9.8 *A* $(4, 4, \lambda; Z_6)$ *GBRD exists iff* $\lambda \equiv 0 \pmod{12}$. *For* $v > 4$, *the necessary condition for a* $(v, 4, \lambda; Z_6)$ *GBRD is that* $\lambda \equiv 0 \pmod{6}$.

a. This condition is sufficient when $\lambda = 6$ *with the definite exception of a* $(5, 4, 6; Z_6)$ *GBRD and the possible exception of an* $(8, 4, 6; Z_6)$ *GBRDs.*

b. This condition is sufficient when $\lambda > 6$.

Proof: We look first at $\lambda = 6$. The non-existence for $v = 4$ is from Theorem 4.6, and was shown for $v = 5$ in [21].

Applying Theorem 6.12 to the $(n, 4, 1)$ difference matrices given in Theorem 4.3 for $n = 9$ and given in Table 2 for $n = 12$ or 15 gives $(4, 3; Z_3)$ GBRGDDs of types 3^4, 4^4 or 5^4; use these as the master design in Theorem 5.2 giving all points a weight of 2, and using a $(4, 2; Z_2)$ GBRGDD of type 2^4 as the ingredient design to get $(4, 6; Z_6)$ GBRGDDs of type 6^4, 8^4 or 10^4; fill these designs, possibly using an extra point in Theorem 5.4, to get a $(v, 4, 6; Z_6)$ GBRD for $v \in \{24, 25, 33, 40, 41\}$.

Similarly, starting with $(n, 5, 1)$ difference matrices for $n \in \{12, 20, 32\}$ noted in Table 2, we get $(5, 2; Z_2)$ GBRGDDs of type 6^5, 10^5 or 16^5; now break these designs with a $(5, 4, 3; Z_3)$ GBRD to get $(4, 6; Z_6)$ GBRGDDs of type 6^5, 10^5 or 16^5; fill these designs by Theorem 5.4 to get a $(v, 4, 6; Z_6)$ GBRD for $v \in \{30, 50, 80\}$.

Combining these with the direct constructions of de Launey and Seberry for $v = 7$, 9–16, 18 and 20 and with our $v = 6$, 17, 21, 22, 26, 28, 29, 32, 34, 35, 38, 44, 45 and prime power $q \equiv 3 \pmod{4}$ from Examples 9.1, 9.2, 9.5 9.3 and 9.4 covers $v \leq 35$ plus 38, 44, 45, 47, 50, 59 and 80.

We next turn to PBDs with nice block sizes which we may break with $(k, 4, 6; Z_6)$ GBRDs. We may truncate one group of a TD$(7, q)$, then fill the groups using w extra points.

q	w	v	q	w	v
7	0	42, 43, 48, 49	8	1	57
9	0,1	54, 55, 60, 61, 63, 64	11	0,1	66, 67, 73, 75–78
13	0,1	79, 87–92	16	0	96, 97

We may spike one block of a TD$(6, q)$ to size 9 through $q + 1$, then fill the groups.

q	v	q	v	q	v
11	69–72	13	81–86	16	99–106

We next aim to truncate one group of a TD$(5,n)$ or two groups of a TD$(6,n)$ to produce a $(\{4,5,6\},1)$ GDD with a nice group type on a total of $(v-w)/2$ points, use this GDD as the master design in Theorem 5.2 giving all points a weight of 2, and use the GBRGDDs from Lemma 9.6 as the ingredient designs to get $(4,6;Z_6)$ GBRGDD; we fill these designs, using w extra points in Theorem 5.4, to get a $(v,4,6;Z_6)$ GBRD.

v	w	GDD group type	v	w	GDD group type
39	1	$4^4 3^1$	52/53	0/1	$5^4 3^2$
62	0	$7^4 3^1$	65	1	$7^4 4^1$
68	0	$7^4 6^1$	74	0	$8^4 5^1$
93	1	$9^4 6^1 4^1$	94/95	0/1	$9^4 6^1 5^1$

An application of Theorem 5.6 gives $v \in \{36,46,51,56\}$, since $v = n(6-1)+1$, and $v = 98 = 14 \cdot 7$, and we note that $v = 37$ is given in Corollary 9.7.

Applying Theorem 6.12 to the $(24,5,1)$ difference matrix given in Table 2 gives a $(5,6;Z_6)$ GBRGDD of type 4^5, and thence a $(\{4,5\},6;Z_6)$ GBRGDD of type $4^4 3^1$; use this as the master design in Theorem 5.2 giving all points a weight of 3, and using $(4,1)$ GDDs of type 3^4 and 3^5 (obtained from a TD$(4,3)$ or punctured AG$(2,4)$) as the ingredient designs to get a $(4,6;Z_6)$ GBRGDD of type $12^4 9^1$; fill this design using an extra point in Theorem 5.4, to get a $(58,4,6;Z_6)$ GBRD.

To deal with $v \geq 105$, we may truncate $k-9$ groups of a TD(k,q), where $k = \min(16, q+1)$, then fill the groups; the truncated group sizes used are 0, 1, 6, 7, 9–16, 18–20, 23–25, remembering a truncated group size cannot exceed q. This construction covers the ranges given below.

q	v	q	v
11	[105–132]	13	[123–182]
16	[150–256]	25	[231–400]

Finally, for $v \geq 387$, we may truncate one group of a TD$(7,n)$ to a size in the range 9–14, then fill the groups. Here the condition $n \geq 63$ ensures that the TD exists.

We look now at $\lambda > 6$. de Launey and Seberry showed that designs exist for all v when $\lambda = 12$; when $\lambda > 6$ de Launey and Seberry's problem was with nine particular $(v,4,18;Z_6)$ GBRDs (for $v = 8$ and eight larger v's). We only need concern ourselves with $v = 8$ since we have just shown that a $(v,4,6;Z_6)$ GBRD exists for those other eight problematical v's. Now we have an $(8,4,18;EA(18))$ GBRD (given below in Theorem 9.10); collapse points over a normal subgroup of order 3, using Theorem 6.4, for an $(8,4,18;Z_6)$ GBRD. ■

9.2 The Group of Order 18

Example 9.9 We construct a $(5, 4, 18; Z_3 \times Z_6)$ GBRD on point set Z_5.

$$
\begin{array}{ccc}
(0_{00}, 1_{00}, 2_{01}, 3_{04}) & (0_{00}, 1_{02}, 2_{10}, 3_{14}) & (0_{00}, 1_{15}, 2_{05}, 3_{23}) \\
(0_{00}, 1_{10}, 2_{03}, 3_{25}) & (0_{00}, 1_{11}, 2_{23}, 3_{00}) & (0_{00}, 1_{25}, 2_{24}, 3_{15})
\end{array}
$$

We construct a $(6, 4, 18; Z_3 \times Z_6)$ GBRD on point set $Z_6 \cup \{\infty\}$.

$$
\begin{array}{ccc}
(0_{00}, 1_{00}, 2_{01}, 3_{04}) & (0_{00}, 1_{02}, 2_{00}, 3_{15}) & (0_{00}, 1_{10}, 2_{03}, 3_{25}) \\
(\infty_{00}, 0_{00}, 1_{11}, 2_{23}) & (\infty_{00}, 0_{01}, 1_{21}, 2_{15}) & (\infty_{00}, 0_{12}, 1_{03}, 2_{02}) \\
(\infty_{00}, 0_{10}, 1_{05}, 3_{24}) & (\infty_{00}, 0_{13}, 1_{20}, 3_{25}) & (\infty_{00}, 0_{14}, 1_{22}, 3_{04})
\end{array}
$$

We construct an $(8, 4, 18; Z_3 \times Z_6)$ GBRD on point set $Z_7 \cup \{\infty\}$.

$$
\begin{array}{ccc}
(0_{00}, 1_{00}, 2_{01}, 3_{00}) & (0_{00}, 1_{02}, 2_{15}, 3_{01}) & (0_{00}, 1_{15}, 2_{03}, 3_{23}) \\
(0_{00}, 1_{03}, 2_{11}, 4_{01}) & (0_{00}, 1_{11}, 2_{04}, 4_{20}) & (0_{00}, 1_{25}, 2_{23}, 4_{11}) \\
(\infty_{00}, 0_{14}, 1_{24}, 4_{12}) & (\infty_{00}, 0_{05}, 1_{20}, 4_{23}) & (\infty_{00}, 0_{00}, 2_{02}, 4_{15}) \\
(\infty_{00}, 0_{01}, 2_{11}, 5_{25}) & (\infty_{00}, 0_{03}, 2_{22}, 4_{13}) & (\infty_{00}, 0_{10}, 3_{21}, 6_{04})
\end{array}
$$

We construct a $(9, 4, 18; Z_3 \times Z_6)$ GBRD on point set Z_9.

$$
\begin{array}{ccc}
(0_{00}, 2_{00}, 4_{01}, 6_{00}) & (0_{00}, 1_{00}, 4_{02}, 6_{15}) & (0_{00}, 2_{02}, 4_{10}, 6_{01}) \\
(0_{00}, 1_{01}, 2_{03}, 3_{10}) & (0_{00}, 1_{03}, 2_{11}, 4_{23}) & (0_{00}, 1_{11}, 4_{04}, 6_{22}) \\
(0_{00}, 3_{03}, 5_{25}, 6_{12}) & (0_{00}, 4_{22}, 5_{12}, 8_{01}) & (0_{00}, 1_{10}, 2_{25}, 6_{02}) \\
(0_{00}, 1_{12}, 2_{04}, 3_{22}) & (0_{00}, 1_{04}, 2_{23}, 5_{24}) & (0_{00}, 1_{21}, 3_{11}, 6_{24})
\end{array}
$$

Theorem 9.10 *A* $(4, 4, \lambda; EA(18))$ *GBRD exists iff* $\lambda \equiv 0 \pmod{36}$. *For* $v > 4$, *the necessary condition for a* $(v, 4, \lambda; EA(18))$ *GBRD is that* $\lambda \equiv 0 \pmod{18}$.

a. *This condition is sufficient when* $\lambda = 18$ *with the possible exception of* $v = 10, 11, 12, 18, 20$ *and* 23.

b. *This condition is sufficient when* $\lambda > 18$.

Proof: We look first at $\lambda = 18$. The non-existence for $v = 4$ is from Theorem 4.6.

We note that a $(k, 4, 9; EA(9))$ GBRD exists for $k = 4$. We may use this GBRD as the ingredient to break the blocks of any $(v, 4, 2; Z_2)$ GBRD, in particular, for $v = 7$, 13, 16, 19 and 22, where the design is given by Theorem 7.1. We may also use this GBRD as the ingredient to break the blocks of any $(4, 2; Z_2)$ GBRGDD, in particular, for GBRGDDs of types 4^4 and 8^4, where the design is given by Theorem 6.14.

Bennett et al. [7] determined the PBD closure of $K = \{5, 6, 7, 8, 9\}$, and determined that 10–20, 22–24, 27–29, 32–34 were the definite exceptions.

Since we constructed $v = 7$ above, and gave $v = 5$, 6, 8 and 9 in Example 9.9, we may break the blocks of the $(v, K, 1)$ PBD to get the GBRD, so we only have to deal with this PBD exception set.

Start with Baker's $(15, 7, 3; Z_3)$ GBRD or from Mathon's $(45, 12, 3; Z_3)$ GBRD and remove some points. If we avoid getting blocks of size 4, 5 or 8, we may then break the resulting design using $(k, 4, 6; Z_6)$ GBRDs; in particular, we obtain $(v, 4, 18; EA(18))$ GBRDs for $v = 14$ or 15, and $v = 24$, 27–29, 33, 34.

For $v = 17$ and 32, we may take the $(4, 18; EA(18))$ GBRGDDs of types 4^4 and 4^8 constructed above, and fill the groups, possibly using an extra point in Theorem 5.4.

This establishes the $\lambda = 18$ case.

We now look at $\lambda > 18$. Here de Launey and Seberry had 25 open parameter sets, but we will establish our result for $\lambda > 18$ directly.

For $\lambda = 36$, if a $(v, 4, 6; Z_6)$ GBRD exists, we may break the blocks with a $(4, 4, 6; Z_3)$ GBRD for a $(v, 4, 36; EA(18))$ GBRD, so we only need consider the exceptions in Theorem 9.8, i.e., just $v = 4$, since we have a $(v, 4, 18; EA(18))$ GBRD for $v = 5$ and 8. For $v = 4$, we have a $(4, 4, 9; EA(9))$ GBRD; we may break the blocks with a $(4, 4, 4; Z_2)$ GBRD for a $(4, 4, 36; EA(18))$ GBRD.

For $\lambda = 54$, we note that a $(v, 4, 6; Z_2)$ GBRD exists for all $v > 4$; break the blocks with a $(4, 4, 9; EA(9))$ GBRD for a $(v, 4, 54; EA(18))$ GBRD for all $v > 4$. ■

9.3 Some Groups of Order 54 or More

In this subsection we consider elementary Abelian groups of order $n = 2 \cdot 3^s$ with $n \geq 54$.

Theorem 9.11 *Let $n = 2 \cdot 3^s \geq 54$. A $(4, 4, \lambda; EA(n))$ GBRD exists iff $\lambda \equiv 0 \pmod{2n}$. For $v > 4$, the necessary condition for a $(v, 4, \lambda; EA(n))$ GBRD is that $\lambda \equiv 0 \pmod{n}$.*

a. If $n = 54$, this condition is sufficient when $\lambda = n$ with the possible exception of $v = 8$.

b. If $n > 54$, this condition is sufficient when $\lambda = n$.

c. For $n \geq 54$, this condition is sufficient when $\lambda > n$.

Proof: For most of these results, we can take the $(v, 4, \lambda; Z_6)$ GBRD given by Theorem 9.8, and break its blocks with a $(4, 4, n/6; EA(n/6))$ GBRD, noting that $n/6 \geq 9$, so our only problems arise from the exceptions and possible exceptions in Theorem 9.8.

Again, we have non-existence for $v = 4$ from Theorem 4.6 if λ is an odd multiple of n.

So we only need consider $\lambda = n$. For $v = 5$, we can take the $(8, 7, 6; Z_6)$ GBRD given by Corollary 6.18, remove 3 points and break the blocks of the resulting $(5, \{4, 5\}, 6; Z_6)$ GBRD with $(k, 4, n/6; EA(n/6))$ GBRDs with $k = 4$ and 5 to get a $(5, 4, n; EA(n))$ GBRD.

For $v = 8$, we can take the $(8, 4, 18; EA(18))$ GBRD given by Theorem 9.10 and break its blocks with a $(4, 4, n/18; EA(n/18))$ GBRD provided $n/18 \geq 9$. ■

10 Some 4 mod 8 cases

Here we consider EA(n) groups of order $n = 4 \cdot 3^s \geq 12$. There is no general restriction on any $v \geq 4$.

10.1 The Group of Order 12

For the group EA(12) de Launey and Seberry solved the existence problem for $\lambda > 12$, but gave no examples of $(v, 4, 12; EA(12))$ GBRDs for $v \equiv 2, 3 \pmod 4$.

We begin with some GBRDs with block size 5.

Example 10.1 We construct a $(31, 5, 4; EA(4))$ GBRD on the point set Z_{31}.

$$B_0 = (0_e, 1_e, 3_e, 4_a, 12_b) \quad C_0 = (0_e, 2_a, 9_a, 5_b, 28_b).$$

Multiply B_0 and C_0 by 1, 26 and 25 to generate six base blocks.

Theorem 10.2 *If $q \equiv 1 \pmod{10}$ is a prime (sic) and $41 \leq q \leq 1151$, then a $(q, 5, 4; EA(4))$ GBRD exists.*

Proof: In [3], these designs were given as frame resolvable $(5, 1)$ GDDs of type 4^q constructed over $GF(4) \times Z_q$. Their conversion to GBRDs follows from Theorem 6.2. No frame resolvable solution of with the structure considered in [3] was found for q=31, and no $(45, 5, 1)$ BIBD has an automorphism of order 11 [32, pp. 16]. ■

Example 10.3 Our $(6, 4, 12; Z_2 \times Z_6))$ GBRD is actually formed on the points $I_2 \times Z_3$, but we present it on the points Z_6 where the development is with the order 3 automorphism $x \mapsto x + 2$.

$$\begin{array}{lll}
(0_{00}, 1_{00}, 2_{00}, 3_{01}) & (0_{00}, 1_{02}, 2_{15}, 3_{00}) & (0_{00}, 1_{03}, 2_{12}, 4_{15}) \\
(0_{00}, 1_{10}, 2_{14}, 4_{04}) & (0_{00}, 1_{15}, 3_{03}, 4_{13}) & (0_{00}, 1_{05}, 2_{01}, 5_{05}) \\
(0_{00}, 1_{13}, 2_{04}, 5_{03}) & (0_{00}, 1_{04}, 3_{13}, 5_{12}) & (0_{00}, 1_{14}, 3_{10}, 5_{01}) \\
 & (0_{00}, 2_{05}, 3_{11}, 5_{14}) &
\end{array}$$

We construct a $(7, 4, 12; Z_2 \times Z_6)$ GBRD on point set Z_7.

$$\begin{array}{lll}
(0_{00}, 1_{00}, 2_{01}, 3_{00}) & (0_{00}, 1_{02}, 2_{15}, 3_{01}) & (0_{00}, 1_{03}, 3_{05}, 4_{13}) \\
(0_{00}, 1_{15}, 3_{12}, 4_{03}) & (0_{00}, 1_{04}, 3_{15}, 5_{13}) & (0_{00}, 1_{10}, 3_{02}, 5_{12})
\end{array}$$

We construct a $(10, 4, 12; Z_2 \times Z_6)$ GBRD on point set $Z_9 \cup \{\infty\}$.

$$\begin{array}{lll}
(0_{00}, 1_{00}, 2_{01}, 3_{00}) & (0_{00}, 1_{02}, 2_{15}, 3_{04}) & (0_{00}, 1_{03}, 2_{11}, 5_{02}) \\
(0_{00}, 1_{12}, 4_{00}, 7_{13}) & (0_{00}, 2_{03}, 5_{15}, 7_{14}) & (0_{00}, 1_{11}, 3_{15}, 5_{05}) \\
(\infty_{00}, 0_{04}, 1_{14}, 5_{11}) & (\infty_{00}, 0_{00}, 3_{01}, 6_{03}) & (\infty_{00}, 0_{15}, 3_{05}, 5_{13}) \\
 & (\infty_{00}, 0_{02}, 4_{12}, 5_{10}) &
\end{array}$$

We construct a $(11, 4, 12; Z_2 \times Z_6)$ GBRD on point set Z_{11}.

$$\begin{array}{lll}
(0_{00}, 1_{00}, 2_{01}, 8_{00}) & (0_{00}, 1_{02}, 3_{01}, 7_{02}) & (0_{00}, 2_{00}, 4_{02}, 7_{15}) \\
(0_{00}, 2_{03}, 5_{12}, 8_{15}) & (0_{00}, 1_{15}, 5_{03}, 6_{13}) & (0_{00}, 2_{11}, 5_{05}, 7_{13}) \\
(0_{00}, 1_{12}, 2_{15}, 3_{02}) & (0_{00}, 1_{04}, 2_{12}, 4_{10}) & (0_{00}, 1_{11}, 3_{04}, 7_{01}) \\
 & (0_{00}, 1_{05}, 4_{15}, 7_{14}) &
\end{array}$$

Theorem 10.4 *For $v \geq 4$, the necessary condition for a $(v, 4, \lambda; EA(12))$ GBRD is that $\lambda \equiv 0 \pmod{12}$.*

a. This condition is sufficient when $\lambda = 12$ with the possible exceptions of $v = 14$, 15, 18 and 23.

b. This condition is sufficient when $\lambda > 12$.

Proof: First we consider $\lambda = 12$.

If $v = 5$, 8, 9 or 12, then we may take the $(v, 4, 3; Z_3)$ GBRD given by Theorem 8.2, and break its blocks with a $(4, 4, 4; EA(4))$ GBRD to produce the desired $(v, 4, 12; EA(12))$ GBRD. A $(4, 4, 12; EA(12))$ GBRD is given by the difference matrix in Table 2, so we have the ingredient designs to deal with the $v \equiv 0, 1 \pmod 4$ case, using Lemma 8.4.c. By Example 10.3, we have the remaining GBRDs with $v \leq 12$, and since the PBD closure of $K = \{4\text{–}12\}$ is 14, 15, 18, 19, 23 (see [8]), we only need deal with $v = 19$ for our $\lambda = 12$ result.

Application of Theorem 5.6 gives a $(19, 4, 12; EA(12))$ GBRD, since $19 = 6 \cdot (4 - 1) + 1$.

For $\lambda > 12$, de Launey and Seberry established the sufficiency. ■

10.2 The Group of Order 36

For the group EA(36) de Launey and Seberry had two open cases for $\lambda > 36$, namely $(v, 4, 108; EA(36))$ GBRDs for $v = 15$ and 23. However, they gave no examples of $(v, 4, 36; EA(36))$ GBRDs for $v \equiv 2, 3 \pmod 4$.

We now need a PBD result of Lenz [41], improved by Mullin et al. [46]; note that [46, Result 3.5a] inadvertently omits the value 10.

Lemma 10.5 *If* $v \notin \{6, 10, 11, 12, 14, 15, 18, 19, 23, 26, 27, 30, 51\}$ *and* $v \geq 4$, *then a* $(v, \{4, 5, 7, 8, 9\}, 1)$ *PBD exists.*

Theorem 10.6 *For* $v \geq 4$, *the necessary condition for a* $(v, 4, \lambda; EA(36))$ *GBRD is that* $\lambda \equiv 0 \pmod{36}$.

a. This condition is sufficient when $\lambda = 36$ *with the possible exception of* $v = 6$, 10, 11 *and* 18.

b. This condition is sufficient when $\lambda > 36$.

Proof: First we look at $\lambda = 36$.

If $v = 5$, 8, 9 or 12, then we may take the $(v, 4, 3; Z_3)$ GBRD given by Theorem 8.2, and break its blocks with a $(4, 4, 12; EA(12))$ GBRD to produce the desired $(v, 4, 12; EA(12))$ GBRD. A $(4, 4, 36; EA(36))$ GBRD is given by Theorem 4.13, so we have the ingredient designs to deal with the $v \equiv 0, 1 \pmod 4$ case, using Lemma 8.4.c. If $v = 7$, we may take the $(8, 7, 6; Z_6)$ GBRD given by Corollary 6.18, remove a point, then break its blocks with $(k, 4, 6; Z_6)$ GBRDs. If $v = 19$, we may take the $(19, 4, 4; EA(4))$ GBRD given in [23], and break its blocks with a $(4, 4, 9; EA(9))$ GBRD. If $v = 30$, we may take the $(31, 5, 4; EA(4))$ GBRD given in Example 10.1, remove a point and then break its blocks with $(k, 4, 9; EA(9))$ GBRDs. From Abel and Ling's designs (see Remark 7.4) we may get a $(23, \{4, 5\}, 4; EA(4))$ GBRD and, adding 4 infinite points and possibly deleting a finite point, we may get a $(5, 4; EA(4))$ GBRGDD of type $1^{23}4^1$ and a $(\{4, 5\}, 4; EA(4))$ GBRGDD of type $1^{22}4^1$. Now we may break the blocks of these designs with $(k, 4, 9; EA(9))$ GBRDs for $k = 4$ and 5, and then fill in the group if necessary with a $(4, 4, 36; EA(36))$ GBRD to get a $(v, 4, 36; EA(36))$ GBRD for $v = 23$, 26 and 27. For $v = 14$ or 15, we may take Baker's $(15, 7, 3; Z_3)$ GBRD, possibly remove a point, then break its blocks with $(k, 4, 12; EA(12))$ GBRDs.

Now we may use Lemma 10.5 to get a PBD whose blocks we can break. We have already dealt with 12, 19, 23, 26, 27 and 30 in the possible exception set of Lemma 10.5. We may also construct a $(51, \{4, 7, 12\}, 1)$ PBD (by truncating one group of a TD(5, 11) to size 6, then filling the groups using an extra point) and so deal with another possible exception.

We now consider $\lambda > 36$. For $\lambda = 72$, we may break the blocks of a $(v, 4, 6; Z_3)$ GBRD with a $(4, 4, 12; EA(12))$ GBRD for our result. For $\lambda = 108$, we may break the blocks of a $(v, 4, 12; EA(4))$ GBRD with a $(4, 4, 9; EA(9))$ GBRD for our result. ■

10.3 Some Groups of Order 108 or More

For the group EA(108) de Launey and Seberry had no open cases for $\lambda > 108$; however, they gave no examples of $(v, 4, 108; EA(108))$ GBRDs for $v \equiv 2, 3 \pmod 4$.

Theorem 10.7 *Let $n = 4 \cdot 3^s \geq 108$. For $v \geq 4$, the necessary condition for a $(v, 4, \lambda; EA(n))$ GBRD to exist is that $\lambda \equiv 0 \pmod n$.*

a. If $n = 108$, this condition is sufficient when $\lambda = n$ with the possible exception of $v = 14$ and 18.

b. If $n > 108$, this condition is sufficient when $\lambda = n$ with the possible exception of $v = 18$.

c. For $n \geq 108$, this condition is sufficient when $\lambda > n$.

Proof: For most of these GBRDs, we can take the $(v, 4, 36t; EA(36))$ GBRD given by Theorem 10.6, and break its blocks with a $(4, 4, n/36; EA(n/36))$ GBRD, or else take the $(v, 4, 12t; EA(12))$ GBRD given by Theorem 10.4, and break its blocks with a $(4, 4, n/12; EA(n/12))$ GBRD to produce the desired $(v, 4, nt; EA(n))$ GBRD. So, for $n/36 \geq 9$, our only possible exception is the intersection of the possible exceptions in Theorems 10.4 and 10.6. However, for $n = 108$, the former construction is not available, so we must deal with the possible exceptions in Theorem 10.4.

For $v = 15$, we may take Baker's $(15, 7, 3; Z_3)$ GBRD, then break its blocks with a $(4, 4, 36; EA(36))$ GBRD. We have a $(\{4, 5\}, 4; EA(4))$ GBRGDD of type $1^{23}u^1$ with $u = 0$, from Abel and Ling's construction (see Remark 7.4); break the blocks with a $(k, 4, 27; EA(27))$ for $k = 4$ or 5 to get a $(23, 4, 108; EA(108))$ GBRD ■

11 The 0 mod 24 cases

Here we consider EA(n) groups of order $n = 24 \cdot h$.

We first need a PBD result primarily due to Lenz [41]. We give the improved version of Ling et al. [42, Theorem 1.1].

Lemma 11.1 *If $v \geq 4$, then a $(v, \{4, 5, 6\}, 1)$ PBD exists if and only if $v \notin \{7, 8, 9, 10, 11, 12, 14, 15, 18, 19, 23\}$.*

Theorem 11.2 *Let $n = 24h$. For $v \geq 4$, the necessary and sufficient condition for a $(v, 4, \lambda; EA(n))$ GBRD is that $\lambda \equiv 0 \pmod{n}$.*

Proof: It suffices to establish the result for $\lambda = n$.

Note that, by Theorem 4.13, a $(4, 4, g; EA(g))$ GBRD exists for $g = n$, $g = n/3$ and $g = n/6$. For $v = 5$ or 8, we may break a $(v, 4, 3; Z_3)$ GBRD with a $(4, 4, n/3; EA(n/3))$ GBRD. If we have a $(v, 4, 6; Z_6)$ GBRD, we may break it with a $(4, 4, n/6; EA(n/6))$ GBRD; In particular, we have GBRDs for $v = 6$ and 7, and the exception set of Lemma 11.1. We also have $v = 4$, so we have the GBRDs to break the blocks of the PBDs given by Lemma 11.1, which with its exception set covers all $v \geq 4$. ■

12 Other Prime Factors

Here we consider EA(n) groups of order $n = 2^r3^sh$ with $\gcd(6, h) = 1$ and $h > 1$. Up to now we have mostly just considered $h = 1$. The case $h > 1$ doesn't alter the basic residue classes relating v and λ. Apart from the necessary condition that $\lambda \equiv 0 \pmod{n}$, this case is largely like the corresponding case with $h = 1$, and any $(v, 4, \lambda/h; EA(n/h))$ GBRD can be broken with a $(4, 4, h; EA(h))$ GBRD to yield a $(v, 4, \lambda; EA(n))$ GBRD.

However, there are cases where we do not have the $(v, 4, \lambda/h; EA(n/h))$ GBRD available to break; most of these cases are due to our failure to construct them, but there are some cases where it is known that they do not exist.

One such class is $v = 4$ with $n/h \equiv \lambda/h \equiv 2 \pmod 4$: here, since h is odd, we also have $n \equiv \lambda \equiv 2 \pmod 4$, and Theorem 4.6 gives us non-existence for our larger group EA(n).

We are not able to say much about about the three sporadic series resulting from de Launey and Sarvate's work [21] i.e., the $(10, 4, 2h; EA(2h))$ GBRDs, $(7, 4, 4h; EA(4h))$ GBRDs and $(5, 4, 6h; EA(6h))$ GBRDs.

The other non-existence result we have is for a $(4, 4, 3; Z_3)$ GBRD. Here, by Evans's result [27] (given in Theorem 4.9) we know $(4, 4, 3h; EA(3h))$ GBRDs exist for all odd $h > 1$ (see Theorem 4.13).

13 Summary of Open Cases

We summarize our results for $G = EA(n)$ when n is of the form $n = 2^r3^s$; See Section 12 for comments on other n.

For $\lambda > |G|$, apart from the integrality conditions on the BIBD and the non-existence result for $v = 4$ given in Theorem 4.6, it is probable that there are no other non-existing designs. However, we still leave the case $(18, 4, 18; EA(9))$ GBRD open.

Table 3: The Open Cases

The cases with $\lambda = \|G\|$			
$\|G\|$	v restriction	Non-exist.	Unknown
$\|G\| \equiv 1 \pmod 2$			
1	1, 4 (mod 12)		None
3	0, 1 (mod 4)	4	None
9	0, 1 (mod 4)		None
≥ 27	0, 1 (mod 4)		None
$\|G\| \equiv 2 \pmod 4$			
2	1 (mod 3)	4, 10	None
6	$v \geq 4$	4, 5	8
18	$v \geq 4$	4	10, 11, 12, 18, 20, 23
54	$v \geq 4$	4	8
≥ 162	$v \geq 4$	4	None
$\|G\| \equiv 4 \pmod 8$			
4	1 (mod 3)	7	10, 22, 34, 46
12	$v \geq 4$		14, 15, 18, 23
36	$v \geq 4$		6, 10, 11, 18
108	$v \geq 4$		14, 18
≥ 324	$v \geq 4$		18
$\|G\| \equiv 0 \pmod 8$			
$3 \nmid n$	1 (mod 3)		10
$3 \mid n$	$v \geq 4$		None

The cases with $\lambda > \|G\|$		
$\|G\|$	v	Unknown λ
9	18	18

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Appendix

Since we rely so heavily on de Launey and Seberry's earlier work [23], we take the opportunity here to correct some minor errors in that article (all references in this appendix will be to [23]). Only the first two errors we mention below had any impact on our present article.

Their Theorem 2.2 is confusingly stated, and contradicts an statement of theirs on page 248 regarding $n = \lambda = 3h$ odd. There is also a minor lacuna in their proof for the case $n = \lambda = 24h$. We correct (and improve) their Theorem 2.2 in our corresponding Theorem 4.13.

The non-existence of a $(15, 5, 2)$ BIBD was overlooked during the proof of their Theorem 5.2.2 and Lemma 5.6.1; in the former case, a $(15, 4, 6; Z_6)$ GBRD was also constructed directly on page 290, but in the later case they provide no alternative construction for $(v, 4, 18; EA(9))$ GBRDs for $v = 14$ nor $v = 15$. (These values were consequently also omitted from their Theorem 10.1 summarizing the possible exceptions.)

There are arithmetic errors in Appendix Tables 1 and 2, invalidating some PBD constructions given in Theorems 1.2.9 and 1.2.11. These can be remedied by constructing a $(96, \{8, 9, 12\}, 1)$ PBD by truncating one group of a TD$(9, 11)$ to size 7, then using an extra point when filling in the groups, or constructing $(v, \{4, 19\}, 1)$ PBDs by appealing to our Theorem 7.5 for $v = 91$ and $v = 139$.

The construction of a $(27, \{4, 5, 6\}, 1)$ PBD (by removing 4 oval points from PG$(2, 5)$) was omitted from the table on page 269, and the value 27 should not be in the exception set on line 6.

The construction of a $(19, 4, 4; Z_2)$ GBRD given in the proof of Theorem 4.2.1 is flawed (but one can take two copies of the $(19, 4, 2; Z_2)$ GBRD given in Theorem 4.1.1).

In some difference families, the signing of $\{\infty\}$ (with the identity element) was omitted.

Page 229, Abstract: $v \geq 40$ should be $v \geq 48$; de Launey has later quoted this result (correctly) as $v > 50$, see [20, Theorem III.4.8].

Page 256, Example 5.1.2: the last base block of the second difference family should be $\{2x + 1_0, 2_1, x + 2_0, 1_1\}$.

Page 262, $v = 9$: the fourth and fifth base blocks should be:

$$(0_1, 1_{a^2w}, 3_{aw}, 7_{aw^2}) \text{ and } (0_1, 1_{a^2w^2}, 4_a, 6_{w^2}).$$

Page 268, $v = 9$: the fourth base block should be $(0_e, 1_{ab}, 4_a, 6_e)$.

Page 274, Lemma 7.3.1 should read "There exists a GBRD$(v, 4, 18; Z_6)$ for $v = 5, 6$."

Page 290, $v = 12$: the first base block should be $(\infty_0, 0_0, 1_1, 2_2)$.

Page 292, $v = 10$: the first base block should be $(\infty_0, 0_0, 1_1, 2_3)$.

Page 292, $v = 14$: the first base block should be $(\infty_0, 0_0, 2_0, 7_2)$.

Page 293, $v = 11$: the fourth, seventh and ninth base blocks should be: $(\infty_{0e}, 0_{0u}, 3_{1w}, 1_{2uw})$, $(0_{0e}, 1_{0e}, 3_{1e}, 6_{1u})$ and $(0_{0e}, 2_{0e}, 4_{0u}, 6_{1e})$.
Page 293, $v = 11$: the fourth base block should be $(0_{0e}, 1_{2u}, 5_{1w}, 7_{0uw})$.
Page 293, $v = 14$: the twelfth base block should be $(0_{0e}, 2_{2u}, 6_{0w}, 9_{1uw})$.
Page 293, $v = 15$: the fourth and fifth base blocks should be: $(0_{0e}, 1_{1u}, 2_{0w}, 11_{0uw})$ and $(0_{0e}, 1_{2u}, 6_{2w}, 8_{0e})$.
Page 294, $v = 18$: the tenth base block should be $(0_{0e}, 1_{0u}, 2_{0w}, 5_{2uw})$.

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