On the Equivalences of the Wheel W_5, the Prism P and the Bipyramid B_5

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Abstract

The main results of this paper are the discovery of infinite families of flow equivalent pairs of B_5 and W_5 amallamorphs and infinite families of chromatically equivalent pairs of P and W_5^* homeomorphs, where B_5 is K_5 with one edge deleted, P is the Prism graph and W_5 is the join of K_1 and a cycle on 4 vertices. Six families of B_5 amallamorphs, with two families having 6 parameters, and 9 families of W_5 amallamorphs, with one family having 4 parameters, are discovered. Since B_5 and W_5 are both planar, all these results obtained can be stated in terms of chromatically equivalent pairs of B_5^* and W_5^* homeomorphs. Also three conjectures are made about non-existence of flow-equivalent amallamorphs or chromatically equivalent homeomorphs of certain graphs.

1 Introduction

Much information about the flow polynomial can be found in [3], [4] and [6]. Given a graph $G = (V, E)$ with vertex set V and edge set E, where multiple edges are allowed, let (D, f) be an ordered pair where D is an orientation of $E(G)$ and $f : E(G) \to \mathbf{Z}$ is an integer-valued function called a *flow*. An oriented edge of G is called an *arc*. For a vertex $v \in V(G)$, let $E^+(v)$ and $E^-(v)$ be the sets of arcs of D(G) with their tails at $v\}$ and with their heads at $v\}$, respectively.

Definition 1.1 *A* λ**-flow** *of a graph G is a flow f such that* $|f(e)| < \lambda$ *for* *every edge $e \in E(G)$ and for every vertex $v \in V(G)$*

$$\sum_{e \in E^+(v)} f(e) \equiv \sum_{e \in E^-(v)} f(e) \quad (mod\ \ \lambda).$$

Definition 1.2 *The* **support** *of f,* supp(f), *is the set of all edges of G with $f(e) \neq 0$. A λ-flow is* **nowhere-zero** *if $supp(f) = E(G)$.*

For a graph $G(V, E)$, the **cyclomatic number** of G, $\nu(G)$, is defined as $\nu(G) = |E(G)| - |V(G)| + \kappa(G)$ where $\kappa(G)$ denotes the number of components. In [6], Tutte defines the **flow polynomial**, $F(G, \lambda)$, of a graph G as a graph function and as a polynomial in an indeterminate λ with integer coefficients by

$$F(G, \lambda) = (-1)^{|E(G)|} \sum_{S \subseteq E(G)} (-1)^{|S|} \lambda^{\nu(G:S)}$$

where $(G : S)$ denotes the spanning subgraph of G with edge-set S. $F(G, \lambda)$ is a polynomial in λ which gives the number of nowhere-zero λ-flows in G independent of the chosen orientation. Tutte [6] also defines the **chromatic polynomial,** $P(G, \lambda)$, of a graph G by

$$P(G, \lambda) = \sum_{S \subseteq E(G)} (-1)^{|S|} \lambda^{\kappa(G:S)}.$$

When λ takes a positive integral value n, $P(G, n)$ is the number of "proper vertex" n-colorings of G. For more information on chromatic polynomials see [2]. It is often more convenient to work with the new variable $\omega = 1 - \lambda$. Tutte [6] states some important properties of the flow polynomial $F(G)$ of a graph G, where G may have multiple edges and/or loops, as follows:

Property 1.3 *$F(G, \omega)$ is a polynomial of degree $\nu = \nu(G)$. The coefficient of ω^ν is $(-1)^\nu$ and all terms in $F(G, \omega)$ have the same sign.*

Property 1.4 *If e is any edge of G, then $F(G, \lambda) = F(G'', \lambda) - F(G', \lambda)$, where G' and G'' are obtained from G by deleting and contracting the edge e, respectively.*

By a result of Jaeger [1], if G is planar, then $P(G^*, \lambda) = \lambda \cdot F(G, \lambda)$, where G^* is the planar dual of G. Two graphs are *homeomorphic* if both can be obtained from the same graph by inserting new vertices of degree 2 into its edges. Multigraphs with the same underlying simple graph were given the name *amallamorphs* by Read and Whitehead in [3]. Two graphs

G and H are said to be *chromatically equivalent* if $P(G, \lambda) = P(H, \lambda)$, while two graphs G and H are said to be *flow equivalent* if $F(G, \lambda) = F(H, \lambda)$.

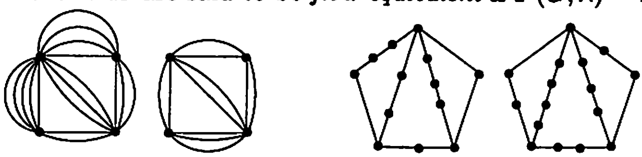


Figure 1: Amallamorphic graphs and homeomorphic graphs

Given a multigraph M, whose underlying graph is G, consider a bundle of multiplicity n and let K be the graph obtained by contracting this bundle in G to a vertex and H that obtained by deleting this bundle. By using Property 1.4 of flow polynomials repeatedly, Read and Whitehead [3] arrive at the "SRF", or the **Sheaf Removal Formula:**

$$F(M, \omega) = (-1)^n \left[\frac{\omega^n - 1}{1 - \omega} F(K, \omega) + F(H, \omega) \right].$$

2 Equivalences of B_5 and P

In this section we study the flow equivalence of the planar graph B_5 shown in Figure 2. The underlying simple graph of B_5 is the bipyramid on 5 vertices which is obtained by adjoining a vertex adjacent to 3 of the vertices of K_4. The planar dual of B_5 is B_5^*, also known as the Prism graph P. In Figure 2 a letter labeling an edge of B_5 indicates the edge multiplicity, while a letter labeling an edge of P indicates the number of edges in the subdivision of that edge.

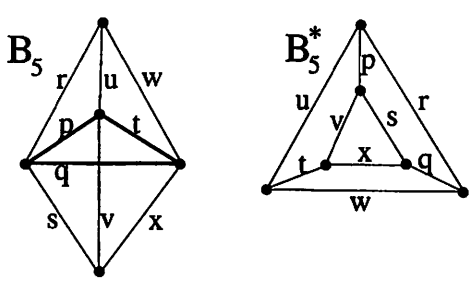


Figure 2: The graph B_5 and its planar dual $B_5^* \cong P$

To compute the flow polynomial of a B_5 amallamorph we need to apply the SRF a few times. Since these computations are tedious and lengthy, we omit them and just state the result:

$$
\begin{aligned}
F(B_5,\omega) = & \frac{(-1)^{p+q+r+s+t+u+v+w+x}}{(1-\omega)^4}\Bigg\{ 4\omega + 8\omega^2 + 5\omega^3 + \omega^4 \\
- & (2\omega + 3\omega^2 + \omega^3)\left[\omega^u + \omega^r + \omega^w + \omega^v + \omega^s + \omega^x\right] \\
- & (\omega + 2\omega^2 + \omega^3)\left[\omega^p + \omega^q + \omega^t\right] \\
+ & (\omega + \omega^2)\Big[\omega^{u+v} + \omega^{r+v} + \omega^{w+v} + \omega^{u+s} + \omega^{r+s} + \omega^{w+s} + \omega^{u+x} \\
+ & \omega^{r+x} + \omega^{w+x} + \omega^{w+p} + \omega^{x+p} + \omega^{u+q} + \omega^{v+q} + \omega^{r+t} + \omega^{s+t}\Big] \\
+ & (\omega + \omega^2)\Big[\omega^{u+r+p} + \omega^{v+s+p} + \omega^{r+w+q} + \omega^{s+x+q} + \omega^{u+w+t} + \\
+ & \omega^{v+x+t}\Big] - \omega\left[\omega^{u+r+v+s+p} + \omega^{r+w+s+x+q} + \omega^{u+w+v+x+t}\right] \\
- & \omega\Big[\omega^{w+v+s+p} + \omega^{u+r+x+p} + \omega^{r+w+v+q} + \omega^{u+s+x+q} + \omega^{u+w+s+t} \\
+ & \omega^{r+v+x+t}\Big] \\
- & \omega\left(\omega^{w+x+p} + \omega^{u+v+q} + \omega^{r+s+t}\right) - \omega^2\left[\omega^p + \omega^q + \omega^t\right] \qquad (2.1) \\
- & \omega\left[\omega^{u+r+w+p+q+t} + \omega^{v+s+x+p+q+t}\right] + \omega^{p+q+r+s+t+u+v+w+x}\Bigg\}
\end{aligned}
$$

By FORTRAN programs, we obtained a list of all nonisomorphic B_5 amallamorphs with 10 through 30 edges and their flow polynomials. These cases are found in a table in Theorem 2.2. In order to complete the proof of Theorem 2.2, we first need to establish the following lemma.

Lemma 2.1 *The graphs G and H in Figure 3 are flow equivalent.*

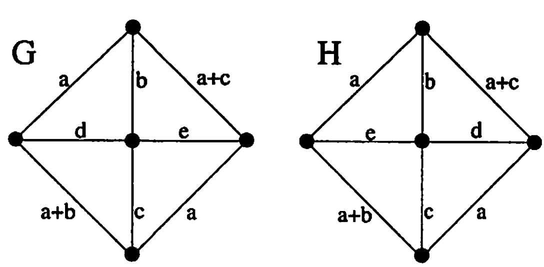


Figure 3: Flow equivalent graphs

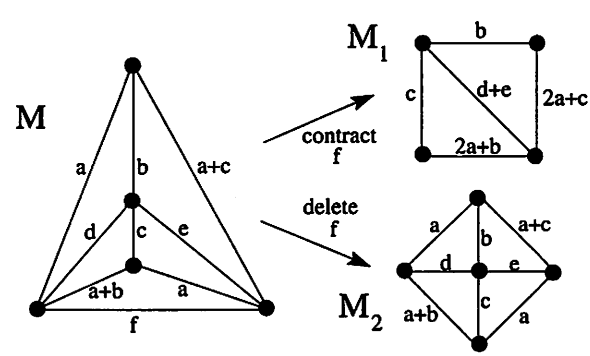
Proof: Apply SRF to the bundles d and e of G. This gives four graphs whose polynomials sum to that of G. Now apply SRF to the bundles e and d of H. The same four graphs are produced, showing that G and H have the same flow polynomial. ∎

Theorem 2.2 *The following 6 pairs of B_5 amallamorphs are flow equivalent for every choice of positive integers a, b, c, d, e and f:*

	p	q	r	s	t	u	v	w	x
1	e	f	a	a+b	d	b	c	a+c	a
	d	f	a	a+b	e	b	c	a+c	a
2	a	a	1	e	f	d	a+b	c	a+b+1
	a+b	a+b	1	e	f	d	a	c	a+1
3	a	a+1	1	c	a	b	1	d	b
	a	a	1	b	a+1	b	c	d	1
4	a	a+2	1	c	a+b	1	1	1	b
	a+b+1	a	1	c	a+1	1	1	1	b
5	a	a+1	b	2	a+2	1	c	2	1
	a	a+1	2	c	a+2	b	1	1	2
6	a	a+1	1	2	a+3	1	b	1	1
	a	a+2	1	1	a+2	1	b	1	2

Proof: For the sake of brevity, we shall only prove the flow equivalence of pair #1. Let M and N be the multigraphs whose edge multiplicitis are described by pair #1. As shown in Figure 4, we proceed by applying the SRF to the bundle f of M and N. Doing so results in the following:

$$F(M,\omega) = (-1)^f \left[\frac{\omega^f - 1}{1-\omega} F(M_1,\omega) + F(M_2,\omega)\right]$$



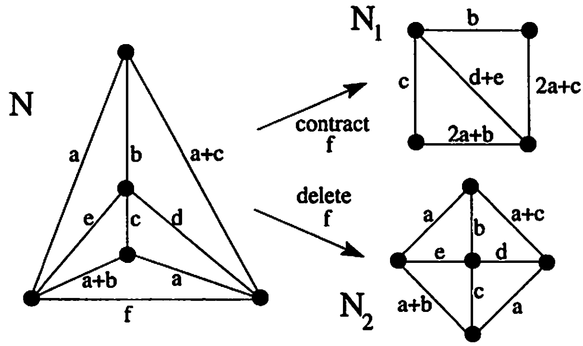


Figure 4: SRF applied to bundle f of M and to bundle f of N

$$F(N,\omega) = (-1)^f \left[\frac{\omega^f - 1}{1-\omega} F(N_1,\omega) + F(N_2,\omega)\right]$$

First we notice that $M_1 \cong N_1$. Then by the Lemma 2.1, $F(M_2,\omega) = F(N_2,\omega)$. Hence $F(M,\omega) = F(N,\omega)$. Flow equivalence of all the other pairs in the table can be established in a similar manner. ∎

Corollary 2.3 *The 6 pairs of B_5 amallamorphs of Theorem 2.2 are also chromatically equivalent P homeomorphs for every choice of positive integers a, b, c, d, e and f.*

3 Equivalences of W_5

In this section we study the flow equivalence of the planar graph W_5, which is join of K_1 and a cycle on 4 vertices as shown in Figure 5. Since all wheels are self-dual, we also show the planar dual of W_5. In Figure 5 a letter labeling an edge of W_5 indicates the edge multiplicity, while a letter labeling an edge of W_5^* indicates the number of edges in subdivision of that edge.

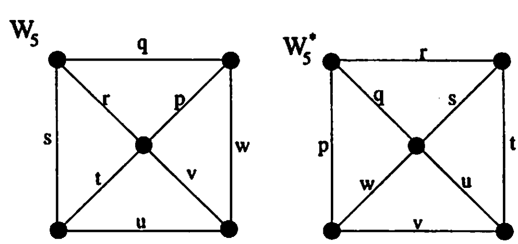


Figure 5: The wheel W_5 and its planar dual

To compute the flow polynomial of a W_5 amallamorph we need to apply the SRF a few times. Since these computations are tedious and lengthy, we omit them and just state the result:

$$
\begin{aligned}
F(W_5,\omega) = \; & \frac{(-1)^{p+q+r+s+t+u+v+w}}{(1-\omega)^4}\Bigg\{ 3\omega + 6\omega^2 + 4\omega^3 + \omega^4 \\
- \; & (\omega + 2\omega^2 + \omega^3)\left[\omega^p + \omega^r + \omega^t + \omega^v\right] \\
- \; & (2\omega + 3\omega^2 + \omega^3)\left[\omega^q + \omega^s + \omega^u + \omega^w\right] \qquad (3.1) \\
+ \; & (\omega + \omega^2)\left[\omega^{p+q+r} + \omega^{r+s+t} + \omega^{t+u+v} + \omega^{p+v+w}\right] + (\omega+\omega^2)\cdot \\
& \Big[\omega^{p+s} + \omega^{q+s} + \omega^{q+t} + \omega^{p+u} + \omega^{q+u} + \omega^{r+u} + \omega^{s+u} + \omega^{q+v} \\
+ \; & \omega^{s+v} + \omega^{q+w} + \omega^{r+w} + \omega^{s+w} + \omega^{t+w} + \omega^{u+w}\Big] + \omega^2\left[\omega^{p+t} + \omega^{r+v}\right] \\
- \; & \omega\left[\omega^{p+q+r+s+t} + \omega^{r+s+t+u+v} + \omega^{p+q+r+v+w} + \omega^{p+t+u+v+w}\right] \\
- \; & \omega\left[\omega^{p+q+r+u} + \omega^{q+t+u+v} + \omega^{r+s+t+w} + \omega^{q+s+u+w} + \omega^{p+s+v+w}\right] \\
- \; & \omega\left(\omega^{p+s+u} + \omega^{q+s+v} + \omega^{q+t+w} + \omega^{r+u+w}\right) + \omega^{p+q+r+s+t+u+v+w}\Bigg\}
\end{aligned}
$$

By FORTRAN programs, all nonisomorphic W_5 amallamorphs with 9 through 30 edges and their flow polynomials were obtained. Below we list all the cases found. Since flow equivalence of all these pairs can be established in a similar manner to that of Theorem 2.2, we omit their proof.

Theorem 3.1 *The following 9 pairs of W_5 amallamorphs are flow equivalent for every choice of positive integers a, b, c and d:*

	p	q	r	s	t	u	v	w
1	b	1	c	a	b	d	a+1	c
	a	1	c	b	a	d	b+1	c
2	a+1	1	a	b	a+1	1	b+1	a
	a+2	1	a	a+1	b	1	b	a
3	a+1	1	b	1	a+3	3	a	2
	a+2	1	b	1	a	3	a+2	2
4	a+1	1	a+1	a	a+1	a	a	a+1
	a	1	a	a+2	a+1	a	a	a+1

	p	q	r	s	t	u	v	w
5	a+2	a+2	2	1	a+3	1	a	2
	a	a+4	2	1	a+1	1	a+2	2
6	3	1	2	a+4	a	2	a	1
	3	1	2	a	a+2	2	a+2	1
7	3	1	2	2	a+2	a	a+2	1
	3	1	2	2	a	a+4	a	1
8	a	4	2	1	a+2	2	4	3
	a	2	2	a+3	4	1	3	3
9	4	3	2	1	a	1	4	2
	3	2	2	3	5	1	a	1

Corollary 3.2 *The 9 pairs of* W_5 *amallamorphs of Theorem 3.1 are also chromatically equivalent* W_5^* *homeomorphs for every choice of positive integers a, b, c and d.*

4 Some Conjectures on Graph Equivalences

In[4], we studied some graphs, each of which admitted many infinite families of flow-equivalent amallamorphs. Furthermore, many infinite families of flow-equivalent amallamorphs of the Petersen graph was discovered in [5].

However, we then examined a class of graphs none of which showed any trace of flow equivalence. The number of nonisomorphic amallamorphs of a graph can grow very rapidly as the size grows. With the available computing capabilities and within the time allowed, we searched among all the possible amallamorphs of reasonable order of each of these 13 graphs in Figure 6. None of the graphs in Figure 6 exhibited any flow-equivalence among its amallamorphs. This led us to make the following conjecture.

Conjecture 4.1 *The graphs in Figure 6 do not have any flow-equivalent amallamorphs.*

Since all the 13 graphs in Figure 6 are planar, they possess a planar dual. As we have already seen, an infinite family of flow-equivalent amallamorphs of a planar graph G also signals the presence of an infinite family of chromatically equivalent homeomorphs of G^*. For the sake of completeness, we also make the following conjecture.

Conjecture 4.2 *The planar duals of graphs in Figure 6 do not have any chromatically equivalent homeomorphs.*

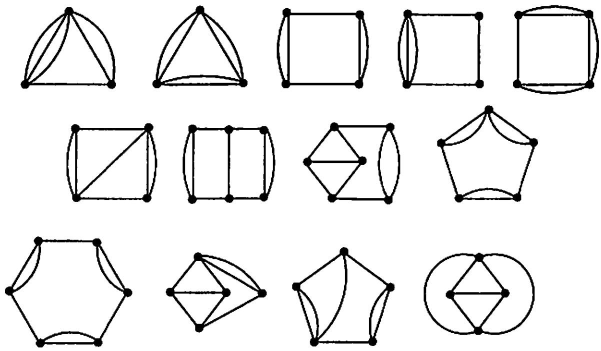


Figure 6: Some graphs admitting no equivalence

We now have seen some graphs which allow some sort of equivalence, flow or chromatic, among their amallamorphs or homeomorphs. Some self-dual graphs like the wheels W_n, possess both flow-equivalent amallamorphs and chromatically equivalent homeomorphs. It is desirable to find a set of properties for a graph to have in order to exhibit any form of equivalence. Equally interesting is the set properties that guarantee the non-existence of any equivalence.

Conjecture 4.3 *There exists a set of properties for a graph G so that G admits no flow-equivalent amallamorphs or chromatically equivalent homeomorphs.*

It is worth searching for necessary conditions or sufficient conditions which point to the existence or non-existence of flow-equivalent amallamorphs or chromatically equivalent homeomorphs of a graph.

References

[1] Jaeger, F., "Even subgraph expansion for the flow polynomial of cubic plane maps", *J. Combinat. Theory, Ser. B* **52**, 1991, p. 259–273.

[2] Read, R. C. & Tutte, W. T., "Chromatic polynomials", *in Selected Topics in Graph Theory, Academic Press*, 1988, p. 15–42.

[3] Read, R. C. & Whitehead, E. G. Jr, "Chromatic polynomials of homeomorphism classes of graphs", Discrete Math., **204**(1999), p. 337–356.

[4] Shahmohamad, H., "On nowhere-zero flows, chromatic equivalence and flow equivalence of graphs", PhD Thesis, University of Pittsburgh, 2000.

[5] Shahmohamad, H. & Whitehead, E. G. Jr, "Homeomorphs and Amallamorphs of the Petersen Graph", To appear in *Discrete Mathematics*

[6] Tutte, W. T. , Graph Theory, *Addison-Wesley, Reading, Mass.*, 1984.