Properties of meanders

A. Panayotopoulos and P. Tsikouras
Department of Informatics
University of Piraeus
80 Karaoli & Dimitriou, 185 34 Piraeus
Greece

Abstract

In this paper we prove various properties of the meanders. We then use these properties in order to construct recursively the set of all meanders of any particular order.

1 Introduction

A road from west to east crosses r times a river flowing from south-west to east. We enumerate the bridges as they are located along the road (from west to east). The order of the bridges along the river determines a permutation μ on $[r] = \{1, 2, \ldots, r\}$. We call this permutation (and the corresponding geometrical representation) *a meander of order* r, [5].

All numbers are taken mod r. Obviously $\mu(i)$ is odd iff i is odd; also a meander of odd (resp. even) order finishes in a north-east (resp. south-east) direction; (see Fig.1).

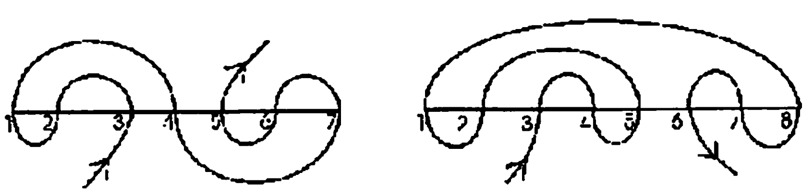

Fig.1. The meanders μ=3214765 and μ=34521876

JCMCC **46** (2003), pp. 181-190

We use the following notations:
$\mathcal{M}_r$: the set of all meanders of order r.
$\mathcal{M}_r(j) = \{\mu \in \mathcal{M}_r : \mu(1) = j\}$.
$\overline{\mathcal{M}}_r(k) = \{\mu \in \mathcal{M}_r : \mu(r) = k\}$.
$\mathcal{M}_r(j,k) = \mathcal{M}_r(j) \cap \overline{\mathcal{M}}_r(k)$, $j \neq k$.
U_μ: the set of pairs $\{\mu(i), \mu(i+1)\}$ of $\mu \in \mathcal{M}_r$ with i odd.
L_μ: the set of pairs $\{\mu(i), \mu(i+1)\}$ of $\mu \in \mathcal{M}_r$ with i even.

So, for example, for the meanders of Fig.1 we have:
μ=3214765, $U_\mu=\{\{1,4\},\{3,2\},\{7,6\}\}$, $L_\mu=\{\{2,1\},\{4,7\},\{6,5\}\}$
μ=34521876, $U_\mu=\{\{1,8\},\{3,4\},\{5,2\},\{7,6\}\}$, $L_\mu=\{\{2,1\},\{4,5\},\{8,7\}\}$

We note that the above sets of pairs U_μ, L_μ are nested, [7]. It is well known that the number of nested sets of pairs on $[2m]$ is given by the Catalan number

$$C_m = \frac{1}{m+1}\left(\begin{array}{c} \mathrm{m} \\ 2\mathrm{m} \end{array}\right).$$

In these sets, we say that a pair $\{a, b\}$ covers $c \in \{2, 3, \ldots, r-1\}$ if c lies between a and b. Finally each one of these nested sets covers c if at least one of its pairs covers c.

The wide range and the great importance of the applications of meanders in various areas, led to the intensification during the last decade of the effort to solve the problem of their enumeration [1],[2],[3],[4].

In this paper we give an equivalent definition of a meander, using permutations and nested sets of pairs; we relate a given meander μ to other meanders with particular properties and we use these properties in order not only to enumerate but also to construct recursively the set of all meanders of any particular order.

More specifically, in section 2 we present various relevant results, including some necessary conditions that enable us to construct $\mathcal{M}_{2n-1}(1)$ by checking the matching property of some nested sets. In section 3 we answer the corresponding problem for the set $\mathcal{M}_{2n}$, by using some outer pairs of the sets L_μ of the meanders $\mu \in \mathcal{M}_{2n}(1)$. In section 4, we complete the recursive construction of all meanders, since we present a way of constructing $\mathcal{M}_{2n}$ from $\mathcal{M}_{2n-1}$, and $\mathcal{M}_{2n+1}$ from $\mathcal{M}_{2n}$.

2 The set $\mathcal{M}_{2n-1}$

A permutation $\mu = \mu(1)\mu(2)\ldots\mu(2n-1)$ on $[2n-1]$ is a meander of order 2n-1 if:
1) $\mu(1)$ is odd.
2) The sets $U_\mu = \{\{\mu(i), \mu(i+1)\} : i = 1, 3, ..., 2n-3\}$ and
   $L_\mu = \{\{\mu(i), \mu(i+1)\} : i = 2, 4, ..., 2n-2\}$ are nested.
3) L_μ (resp. U_μ) does not cover $\mu(1)$ (resp. $\mu(2n-1)$).

The above definition of meanders, using nested sets, is obviously equivalent to the geometric definition of the introduction, considering nested arcs instead of nested pairs. In addition, we can easily deduce that every two consecutive values of μ cannot be both even or odd and hence $\mu(i)$ is odd iff i is odd; it is also clear that if $\mu \in \mathcal{M}_{2n-1}(j,k)$ with $j < k$, then $\mu(2) < k$.

The properties of the definition are complemented by the following proposition.

Proposition 2.1 *If $\mu \in \mathcal{M}_{2n-1}(j,k)$ with $j, k \neq 1, 2n-1$, then U_μ covers j and L_μ covers k.*

Proof: If $\mu(2n-1) > \mu(1)$, let h be the greatest odd number such that $\mu(h) < \mu(1)$; then $h \neq 2n-1$ and $\mu(h+2) > \mu(1)$. Since L_μ does not cover j, j does not lie between $\mu(h+1)$ and $\mu(h+2)$; so j must lie between $\mu(h)$ and $\mu(h+1)$, i.e. U_μ covers j.

If $\mu(2n-1) < \mu(1)$, let h be the greatest odd number such that $\mu(h) > \mu(1)$. Then $h \neq 2n-1$ and $\mu(h+2) < \mu(1)$. Since L_μ does not cover j, we get again that U_μ covers j.

We similarly prove that L_μ covers k. □

Lemma 2.2 *Let $\mu \in \mathcal{M}_{2n-1}$ and the permutations μ^* and $\bar{\mu}$ on $[2n-1]$ with $\mu^*(i) = \mu(2n-i)$ and $\bar{\mu}(i) = 2n - \mu(i)$. Then μ^*, $\bar{\mu} \in \mathcal{M}_{2n-1}$.*

Proof: By the definition of μ^*, $\mu^*(1)$ is odd iff $\mu(2n-1)$ is odd, which is true. Also U_{μ^*} and L_{μ^*} are nested, since $U_{\mu^*} = L_\mu$ and $L_{\mu^*} = U_\mu$. Finally, let some $\{\mu^*(i),\mu^*(i+1)\} \in L_{\mu^*}$ cover $\mu^*(1)$. Then $\{\mu(2n-i),\mu(2n-i-1)\}$ would cover $\mu(2n-1)$ with $2n-i-1$ odd (and hence $\{\mu(2n-i-1),\mu(2n-i)\} \in U_\mu$), contradicting condition 3 of the definition for μ. Similarly, for $\mu^*(2n-1)$.

By the definition of $\bar{\mu}$, $\bar{\mu}(1)$ is odd iff $2n-\mu(1)$ is odd, which is true. Also, if $\{\bar{\mu}(i),\bar{\mu}(i+1)\}$, $\{\bar{\mu}(h),\bar{\mu}(h+1)\}$ with $i, h \in \{1, 3, \ldots, 2n-3\}$ are two pairs of $U_{\bar{\mu}}$ with $\bar{\mu}(i) < \bar{\mu}(h) < \bar{\mu}(i+1) < \bar{\mu}(h+1)$, then $2n-\mu(i) < 2n-\mu(h) < 2n-\mu(i+1) < 2n-\mu(h+1)$, i.e $\mu(i) > \mu(h) > \mu(i+1) > \mu(h+1)$, contradicting the fact that U_μ is nested. Similarly we prove that $L_{\bar{\mu}}$ is nested. Finally the proof of condition 3 for μ is similar to the corresponding part of the proof for μ^*. □

Proposition 2.3 *The following relations hold:*

$$|\mathcal{M}_{2n-1}(j,k)| = |\mathcal{M}_{2n-1}(k,j)| \quad (2.3.1)$$
$$|\mathcal{M}_{2n-1}(j,k)| = |\mathcal{M}_{2n-1}(2n-k,2n-j)| \quad (2.3.2)$$
$$|\mathcal{M}_{2n-1}(j)| = |\mathcal{M}_{2n-1}(2n-j)| \quad (2.3.3)$$
$$|\mathcal{M}_{2n-1}(1,k)| = |\mathcal{M}_{2n-1}(1,2n-k)| \quad (2.3.4)$$

Proof: Relation 2.3.1 is a direct consequence of lemma 2.2 since for each $\mu \in \mathcal{M}_{2n-1}(j,k)$, $\mu^*(1) = \mu(2n-1) = k$ and $\mu^*(2n-1) = \mu(2n-2n+1) = \mu(1) = j$.

Relation 2.3.2 is obtained by combining relation 2.3.1 with lemma 2.2 from which we get that $|\mathcal{M}_{2n-1}(j,k)| = |\mathcal{M}_{2n-1}(2n-j,2n-k)|$. The last equality also proves 2.3.3 since
$\mathcal{M}_{2n-1}(j) = \bigcup_{k\in J} \mathcal{M}_{2n-1}(j,k)$ and $\mathcal{M}_{2n-1}(2n-j) = \bigcup_{k\in J} \mathcal{M}_{2n-1}(2n-j,2n-k)$, where $J = \{1,3,\ldots,2n-1\}$.

Finally, relation 2.3.4 is an immediate consequence of relation 4.2.2. □

The validity of the above formulae is displayed in Table 1.

Table 1: The values of $|\mathcal{M}_{13}(j,k)|$

$j \setminus k$	1	3	5	7	9	11	13	$\lvert\mathcal{M}_{13}(j)\rvert$
1		538	353	316	353	538	1828	3926
3	538		81	93	109	171	538	1530
5	353	81		42	65	109	353	1003
7	316	93	42		42	93	316	902
9	353	109	65	42		81	353	1003
11	538	171	109	93	81		538	1530
13	1828	538	353	316	353	538		3926
$\lvert\mathcal{M}_{13}(k)\rvert$	3926	1530	1003	902	1003	1530	3926	13820

Proposition 2.4 *If* $A = \{(j,k) \in J^2 : j+k = 2n, j < k\}$ *and* $B = \{(j,k) \in J^2 : j+k < 2n, j < k\}$ *then*

$$|\mathcal{M}_{2n-1}| = 2 \sum_{(j,k)\in A} |\mathcal{M}_{2n-1}(j,k)| + 4 \sum_{(j,k)\in B} |\mathcal{M}_{2n-1}(j,k)|.$$

Proof: Relation 2.3.1 justifies why it is enough to consider $j < k$ in both A and B, and hence multiply both sums with 2; furthermore, relation 2.3.2

justifies why it is enough to consider, in B, $j+k < 2n$ (and hence finally multiply the second sum with 4). □

Now for $q, k \in \{3, 5, \ldots, 2n-1\}$ let
$\mathcal{U}_{2n-1}(1,k) = \{U_\mu : \mu \in \mathcal{M}_{2n-1}(1,k)\}$.
$\mathcal{L}_{2n-1}(1,k) = \{L_\mu : \mu \in \mathcal{M}_{2n-1}(1,k)\}$.
$\mathcal{U}_{2n-1}(1,k;q)$:the subset of $\mathcal{U}_{2n-1}(1,k)$ consisting of its elements that cover q.
$\mathcal{L}_{2n-1}(1,k;q)$:the subset of $\mathcal{L}_{2n-1}(1,k)$ consisting of its elements that do not cover q.

In the rest of this section we will prove that in order to construct $\mathcal{M}_{2n-1}$, it is enough to work with a small number of pairs of nested sets, namely just with the nested sets $U \cup \{k, 2n\}$, $L \cup \{2n, j\}$ with $U \in \mathcal{U}_{2n-1}(1,k;j)$ and $L \in \mathcal{L}_{2n-1}(1,k;j)$, $1 < j < k$.

Lemma 2.5 *α) For every nested set U on $[2n-1]\backslash\{k\}, k \in \{3,5,...,2n-1\}$ that does not cover k, there exists $\mu \in \mathcal{M}_{2n-1}(1,k)$ with $U_\mu = U$.*
β) For every nested set L on $[2n-1]\backslash\{1\}$ there exists $\mu \in \mathcal{M}_{2n-1}(1,2n-1)$ with $L_\mu = L$.
γ) For every nested set L on $[2n-1]\backslash\{1\}$ that covers $k \in \{3,5,\ldots,2n-3\}$, there exists $\mu \in \mathcal{M}_{2n-1}(1,k)$ with $L_\mu = L$.

Proof: α) We will construct recursively such a meander μ, with $\mu(1) = 1$, $\mu(k) = 2n-1$, $\mu(2n-1) = k$, $\mu(i) \in \{2,3,...,k-1\}$, $\forall\, i \in \{2,3,...,k-1\}$ and $\mu(i) \in \{k+1, k+2, ..., 2n-2\}$, $\forall\, i \in \{k+1, k+2, ..., 2n-2\}$.

Suppose that $i \neq 1, k, 2n-1$ and that $\mu(1)$,$\mu(2)$,...,$\mu(i-1)$ have been defined. We define $\mu(i) \in A = \{2,3,...,k-1\} \bigcup \{k+1, k+2, ..., 2n-2\}$ as follows:

If i is even, let $\mu(i)$ be the unique element of $[2n-1] \setminus \{k\}$ such that $\{\mu(i-1), \mu(i)\} \in U$. If on the other hand i is odd let

$$\mu(i) = \begin{cases} \mu(i-1) - 1, & \text{if } \mu(i-1) - 1 \in I_{i-1}, \\ \min(A \setminus I_{i-1}), & \text{otherwise.} \end{cases}$$

where I_{i-1} is the image of $[i-1]$ under μ.

The proofs of β and γ are similar. □

From lemma 2.5, we have the following result, for $q < k$.

Proposition 2.6 *The following relations hold:*

$|\mathcal{U}_{2n-1}(1,k)| = C_\alpha C_\beta$, where $\alpha = \frac{k-1}{2}$, $\beta = \frac{2n-1-k}{2}$.
$|\mathcal{L}_{2n-1}(1,k)| = C_{n-1} - C_\alpha C_\beta$.
$|\mathcal{U}_{2n-1}(1,k;q)| = C_\beta(C_\alpha - C_\gamma C_\delta)$, where $\gamma = \frac{q-1}{2}$, $\delta = \frac{k-1}{2}$.
$|\mathcal{L}_{2n-1}(1,k;q)| = C_\gamma(C_\varepsilon - C_\beta C_\delta)$, where $\varepsilon = \frac{2n-1-q}{2}$.

The following necessary conditions will help us construct the set $\mathcal{M}_{2n-1}$ from the set $\mathcal{M}_{n-1}(1)$.

Proposition 2.7 *If $\mu \in \mathcal{M}_{2n-1}(j,k)$, $1 < j < k$, then $U_\mu \in \mathcal{U}_{2n-1}(1,k;j)$ and $L_\mu \in \tilde{\mathcal{L}}_{2n-1}(1,k;j)$, where $\tilde{\mathcal{L}}_{2n-1}(1,k;j)$ is the set that we obtain, if we replace j with 1 in every element of $\mathcal{L}_{2n-1}(1,k;j)$.*

Proof: Applying lemma 2.5α for $U = U_\mu$, we obtain $\mu_1 \in \mathcal{M}_{2n-1}(1,k)$ with $U_{\mu_1} = U_\mu$. Furthermore, by proposition 2.1 follows that U_μ (and hence U_{μ_1}) covers j, so that $U_\mu \in \mathcal{U}_{2n-1}(1,k;j)$.

In order now to prove that $L_\mu \in \tilde{\mathcal{L}}_{2n-1}(1,k;j)$ let L be the set obtained from L_μ, by replacing 1 with j. Applying lemma 2.5β or γ for L, we obtain $\mu_2 \in \mathcal{M}_{2n-1}(1,k)$ with $L_{\mu_2} = L$. Furthermore, by definition, L_μ does not cover j and so L_{μ_2} does not cover j either. Thus $L \in \mathcal{L}_{2n-1}(1,k;j)$ so that $L_\mu \in \tilde{\mathcal{L}}_{2n-1}(1,k;j)$. □

We now note the following: If we know $\mathcal{M}_{2n-1}(1)$ and hence $\mathcal{M}_{2n-1}(1,k)$ for every $k \in \{3,5,\ldots,2n-1\}$, we get the sets $\mathcal{U}_{2n-1}(1,k)$, $\mathcal{L}_{2n-1}(1,k)$ and from them we get the sets $\mathcal{U}_{2n-1}(1,k;j)$, $\mathcal{L}_{2n-1}(1,k;j)$ for $1 < j < k$; furthermore from $\mathcal{L}_{2n-1}(1,k;j)$ we get $\tilde{\mathcal{L}}_{2n-1}(1,k;j)$.

Now, following the procedure presented in [6], for each $U \in \mathcal{U}_{2n-1}(1,k;j)$ and for each $L \in \tilde{\mathcal{L}}_{2n-1}(1,k;j)$ such that $U \cup \{k,2n\}$, $L \cup \{2n,j\}$ are matching, we get a meander $\mu \in \mathcal{M}_{2n-1}(j,k)$ with $U_\mu = U$, $L_\mu = L$. According to lemma 2.5 and proposition 2.7, we thus construct from the set $\mathcal{M}_{2n-1}(1)$ all meanders of $\mathcal{M}_{2n-1}(j,k)$ for $j < k$; by lemma 2.2, we get $\mathcal{M}_{2n-1}(j,k)$ for $j > k$, too. So, we construct $\mathcal{M}_{2n-1}(j,k)$ for every $j \neq k$, i.e we construct the required set $\mathcal{M}_{2n-1}$.

3 The set $\mathcal{M}_{2n}$

A permutation $\mu = \mu(1)\mu(2)\ldots\mu(2n)$ on $[2n]$ is a meander of order 2n if:
1) $\mu(1)$ is odd.
2) The sets $U_\mu = \{\{\mu(i),\mu(i+1)\} : i = 1,3,\ldots,2n-1\}$ and
$L_\mu = \{\{\mu(i),\mu(i+1)\} : i = 2,4,\ldots,2n-2\}$ are nested.
3) L_μ does not cover either $\mu(1)$ or $\mu(2n)$.

Obviously, for the meanders of even order, the above definition is again equivalent to the geometric definition of the introduction and we also have that $\mu(i)$ is odd iff i is odd. Furthermore, $\mu(1) < \mu(2n)$, since otherwise we would have an odd number of elements in the set $\{i+1, i+2, \dots, 2n\}$, which is a contradiction, since i is odd.

Lemma 3.1 *α) Let $\mu \in \mathcal{M}_{2n}$ and the permutation $\tilde{\mu}$ on $[2n]$ with $\tilde{\mu}(i) = 2n+1-\mu(2n+1-i)$. Then $\tilde{\mu} \in \mathcal{M}_{2n}$.*

β) Let $\mu \in \mathcal{M}_{2n}(1)$ and the permutation $\hat{\mu}$ on $[2n]$ with $\hat{\mu}(1) = 1$ and $\hat{\mu}(i) = 2n+2-\mu(i)$, $i = 2, 3, \dots, 2n$. Then $\hat{\mu} \in \mathcal{M}_{2n}(1)$.

Proof: α) $\tilde{\mu}(1)$ is odd iff $2n+1-\mu(2n)$ is odd, which is true, since $\mu(2n)$ is even.

Let now $\{\tilde{\mu}(i), \tilde{\mu}(i+1)\}, \{\tilde{\mu}(h), \tilde{\mu}(h+1)\} \in U_{\tilde{\mu}}$ such that $\tilde{\mu}(i) < \tilde{\mu}(h) < \tilde{\mu}(i+1) < \tilde{\mu}(h+1)$; then $2n+1-\mu(2n+1-i) < 2n+1-\mu(2n+1-h) < 2n+1-\mu(2n-i) < 2n+1-\mu(2n-h)$, i.e. $\mu(2n+1-i) > \mu(2n+1-h) > \mu(2n-i) > \mu(2n-h)$ with $\{\mu(2n-h), \mu(2n+1-h)\}$, $\{\mu(2n-i), \mu(2n+1-i)\} \in U_\mu$ contradicting the fact that U_μ is nested.

If now $\{\tilde{\mu}(i), \tilde{\mu}(i+1)\} \in L_{\tilde{\mu}}$ covers $\tilde{\mu}(1)$, then $\{2n+1-\mu(2n+1-i), 2n+1-\mu(2n-i)\}$ would cover $2n+1-\mu(2n)$ i.e. $\{\mu(2n-i), \mu(2n-i+1)\}$ would cover $\mu(2n)$ which is a contradiction. Similarly, if $\{\tilde{\mu}(i), \tilde{\mu}(i+1)\}$ covers $\tilde{\mu}(2n)$.

β) The proof is similar and it is omitted. □

We can similarly prove the following lemma.

Lemma 3.2 *α) Let $\mu \in \mathcal{M}_{2n}(2n)$ and the permutation $\bar{\mu}$ on $[2n]$ with $\bar{\mu}(i) = 2n - \mu(i)$, Then $\bar{\mu} \in \mathcal{M}_{2n}(2n)$.*

β) Let $\mu \in \mathcal{M}_{2n}(j)$ and the permutation $\dot{\mu}$ on $[2n]$, with $\dot{\mu}(i) = \mu(i+h)$, where $h = \mu^{-1}(j-1)$. Then $\dot{\mu} \in \overline{\mathcal{M}}_{2n}(j-1)$.

γ) Let $\mu \in \mathcal{M}_{2n}(j)$ and the permutation $\ddot{\mu}$ on $[2n]$, with $\ddot{\mu}(i) = 2n+1-\mu(2n+1-i)$. Then $\ddot{\mu} \in \overline{\mathcal{M}}_{2n}(2n+1-j)$.

The following proposition is now clear:

Proposition 3.3 *The following relations hold:*

$$|\mathcal{M}_{2n}(j,k)| = |\mathcal{M}_{2n}(2n+1-k, 2n+1-j)| \quad (3.3.1)$$
$$|\mathcal{M}_{2n}(j,2n)| = |\mathcal{M}_{2n}(2n-j, 2n)| \quad (3.3.2)$$
$$|\mathcal{M}_{2n}(j)| = |\mathcal{M}_{2n}(2n+2-j)| \quad (3.3.3)$$
$$|\mathcal{M}_{2n}(1,k)| = |\mathcal{M}_{2n}(1, 2n+2-k)| \quad (3.3.4)$$

The validity of the above formulae is displayed in Table 2.

Table 2: The values of $|\mathcal{M}_{12}(j,k)|$

$j \setminus k$	2	4	6	8	10	12	$\lvert\mathcal{M}_{12}(j)\rvert$
1	538	221	155	155	221	538	1828
3		132	66	52	67	221	538
5			95	51	52	155	353
7				95	66	155	316
9					132	221	353
11						538	538
$\lvert\mathcal{M}_{12}(k)\rvert$	538	353	316	353	538	1828	3296

Proposition 3.4 *If* $J = \{1, 3, ..., 2n-1\}, K = \{2, 4, ..., 2n\}, A = \{(j,k) \in J \times K : j+k = 2n+1, j < k\}$ *and* $B = \{(j,k) \in J \times K : j+k < 2n, j < k\}$ *then*

$$|\mathcal{M}_{2n}| = \sum_{(j,k)\in A} |\mathcal{M}_{2n}(j,k)| + 2 \sum_{(j,k)\in B} |\mathcal{M}_{2n}(j,k)|.$$

The proof is similar to that of proposition 2.4.

Similarly to proposition 2.6 (and using corresponding notation for meanders of even order) we have the following results.
Proposition 3.5 *The following relations hold:*

$|\mathcal{U}_{2n}(1,k)| = C_n - C_\alpha C_\beta$, *where* $\alpha = \frac{k}{2}$, $\beta = \frac{2n-k}{2}$.
$|\mathcal{L}_{2n}(1,k)| = C_\beta C_{\gamma-1}$.
$|\mathcal{U}_{2n}(1,k;q)| = C_\beta(C_\alpha - C_\gamma C_\delta)$, *where* $\gamma = \frac{q-1}{2}$, $\delta = \frac{k+1-q}{2}$.
$|\mathcal{L}_{2n}(1,k;q)| = C_\gamma(C_\varepsilon - C_\beta C_\delta)$, *where* $\varepsilon = \frac{2n+1-q}{2}$.

A pair $\{\alpha, b\}$ of a nested set S is called outer pair if there is no pair $\{c,d\} \in S$ such that $c < \alpha < b < d$. We can easily find the set OPS of outer pairs of S, since

$$OPS = \{\{\alpha_i + 1, \alpha_{i+1}\} \in S : i \in \{0, 1, ..., h\}, \alpha_0 = 0, \alpha_{h+1} = 2n\}.$$

For $\mu \in \mathcal{M}_{2n}$, let $\overline{OPL_\mu} = \{\{\mu(\rho), \mu(\rho+1)\} \in OPL_\mu : \mu(2n) < \mu(\rho)\}$.

Proposition 3.6 *The following relation holds:*

$$|\mathcal{M}_{2n}| = |\mathcal{M}_{2n}(1)| + \sum_{\mu\in\mathcal{M}_{2n}(1)} |\overline{OPL_\mu}|,$$

Proof: It is enough to prove that $\sum_{\mu\in\mathcal{M}_{2n}(1)} |\overline{OPL}_\mu| = |\mathcal{M}_{2n}\backslash\mathcal{M}_{2n}(1)|$; for this, we prove that there exists a 1-1 correspondence between the sets $\bigcup_{\mu\in\mathcal{M}_{2n}(1)} \overline{OPL}_\mu$ and $\mathcal{M}_{2n}\backslash\mathcal{M}_{2n}(1)$. Let $\mu \in \mathcal{M}_{2n}(1)$, $\{\mu(\rho), \mu(\rho+1)\} \in \overline{OPL}_\mu$ and the permutation μ_ρ on $[2n]$ with $\mu_\rho(i) = \mu(\rho + i)$. Then $\mu_\rho \in \mathcal{M}_{2n}\backslash\mathcal{M}_{2n}(1)$, since otherwise we would have $\mu(\rho+1) = \mu_\rho(1) = 1 = \mu(1) < \mu(2n) < \mu(\rho)$, i.e. the pair $\{\mu(\rho), \mu(\rho+1)\} \in L_\mu$ would cover $\mu(1)$, contradicting condition 3 of the definition of $\mu \in \mathcal{M}_{2n}$. Conversely, every $\nu \in \mathcal{M}_{2n}\backslash\mathcal{M}_{2n}(1)$ corresponds to $\mu \in \mathcal{M}_{2n}(1)$ with $\mu(i) = \nu(q+i-1)$, $i = 2, 3, \ldots, 2n$ and the pair $\{\mu(2n-h+1), \mu(2n-h+2)\} \in \overline{OPL}_\mu$, where $h = \nu^{-1}(1)$. □

The proof of the above proposition gives a method to construct the set $\mathcal{M}_{2n}$ from the set $\mathcal{M}_{2n}(1)$ since, in order to construct $\mathcal{M}_{2n}(1)\backslash\mathcal{M}_{2n}(1)$ it is enough, for each $\mu \in \mathcal{M}_{2n}(1)$ and for each $\{\mu(\rho), \mu(\rho+1)\} \in \overline{OPL}_\mu$ to form a new permutation $\mu' \in \mathcal{M}_{2n}(1)\backslash\mathcal{M}_{2n}(1)$, by defining $U_{\mu'} = U_\mu$ and $L_{\mu'} = (L_\mu\backslash\{\mu(\rho), \mu(\rho+1)\}) \cup \{1, \mu(2n)\}\}$.

4 The general case

For the set $\mathcal{M}_r$ we have the following results.

Proposition 4.1 *From the set $\mathcal{M}_r$ we can construct the set $\mathcal{M}_{r+1}(1)$.*

Proof: For each $\mu \in \mathcal{M}_r$ define the permutation μ^+ on $[r+1]$ such that $\mu^+(1) = 1$ and $\mu^+(i+1) = \mu(i) + 1, i \in [r]$. Then $\mathcal{M}_{r+1}(1) = \{\mu^+ : \mu \in \mathcal{M}_r\}$. □

Notice that inversely, from the set $\mathcal{M}_{r+1}(1)$ we can construct the set $\mathcal{M}_r$. Indeed, for each $\mu \in \mathcal{M}_{r+1}(1)$ define the permutation μ^- on [r] such that $\mu^-(i) = \mu(i+1) - 1, i \in [r]$.

Proposition 4.2 *The following relations hold:*

$$|\mathcal{M}_{2n-1}(1)| = |\mathcal{M}_{2n-2}| \quad (4.2.1)$$
$$|\mathcal{M}_{2n-1}(1,k)| = |\mathcal{M}_{2n-2}(2n-k)| \quad (4.2.2)$$
$$|\mathcal{M}_{2n}(1)| = |\mathcal{M}_{2n-1}| \quad (4.2.3)$$
$$|\mathcal{M}_{2n}(1,k)| = |\mathcal{M}_{2n-1}(k-1)| \quad (4.2.4)$$

Proof: Relations 4.2.1 and 4.2.3 are direct consequences of the proof of proposition 4.1 and of the subsequent remark.

For the proof of 4.2.2 it is enough to realize that there is a 1-1 correspondence between each $\mu \in \mathcal{M}_{2n-1}(1,k)$ and a permutation $\mathring{\mu} \in \mathcal{M}_{2n-2}(2n-k)$ defined by $\mathring{\mu}(i) = 2n - \mu(2n-i)$.

Similarly, for 4.2.4 consider for each $\mu \in \mathcal{M}_{2n}(1,k)$ the permutation $\overset{\circ\circ}{\mu} \in \mathcal{M}_{2n-1}(1,k)$ with $\overset{\circ\circ}{\mu}(i) = \mu(2n+1-i)-1$. $\square$

The validity of the relations 4.2.1 and 4.2.2 becomes clear by the existence of the corresponding common values in Tables 1 and 2.

Proposition 4.3 *From the set $\mathcal{M}_r$ we can construct every $\mathcal{M}_s$, $s < r$.*
The proof of this proposition is an immediate consequence of the remark after proposition 4.1.

Corollary 4.4 *If $\mathcal{M}_r[\eta] = \{\mu \in \mathcal{M}_r : \mu(i) = i, i \in [\eta]\}, \eta \leq r$, then $|\mathcal{M}_r[\eta]| = |\mathcal{M}_{r-\eta}|$.*

References

[1] P. Di Francesco, O. Golinelli and E. Guitter, Meanders: A Direct Enumeration Approach. *Nucl. Phys.* **482** (1996), 497-535.

[2] M.G. Harris, A Diagrammatic Approach to the Meander Problem. *http://arxiv.org/abs/hep-th/9807193.*

[3] I. Jensen, Enumeration of plane meanders. Preprint, *http://arxiv.org/abs/cond-mat/9910313.*

[4] I. Jensen and A. J. Guttmann, Critical exponents of plane meanders. *J. Phys. A: Math. Gen.* **33** (2000), L187-L192.

[5] S.K. Lando and A.K. Zvonkin, Plane and projective meanders. *Theoretical Computer Science* **117** (1993), 227-241.

[6] A. Panayotopoulos, A. Sapounakis and P. Tsikouras, The matching property of nested sets. *Australasian Journal of Combinatorics* **15** (1997), 17-24.

[7] A. Sapounakis and P. Tsikouras, Nested sets of pairs, *Journal of Information and Optimization Sciences* **16** (1995), 549-555.