Minimal Enclosings of Group Divisible Designs with Block Size 3 and Group Size 2

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Abstract: We obtain necessary conditions for the enclosing of a group divisible design with block size 3, *GDD(n, m; λ)*, into a group divisible design GDD(n, m+1; λ+x) with one extra group and minimal increase in λ. We prove that the necessary conditions are sufficient for the existence of all such enclosings for GDD's with group size 2 and $\lambda \le 6$, and for any λ when v is sufficiently large relative to λ.

Keywords: triple system, GDD, RGDD, complete graph, enclosing.

1. Introduction

In their authoritative work on triple systems [1], C.J. Colbourn and A. Rosa state that the existence of enclosings of partial triple systems, or even triple systems themselves, has yet to be studied. Motivated by this statement, Hurd, Munson, and Sarvate [5] solved the problem of existence of *minimal* enclosings for triple systems (or TS) into TS's with one extra point and a minimal increase in the index λ with $1 \le \lambda \le 6$. Since a TS can be considered a group divisible design with group size 1, in essence

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they added one new group of size 1 for their enclosings. We take a next logical step which is to find necessary conditions for enclosings, by adding one more group, of group divisible designs with group size two, and to prove that these necessary conditions are sufficient for minimal enclosings. In this paper we completely solve this problem for the case of minimal enclosings with group size two and index λ with $1 \leq \lambda \leq 6$. In fact we give minimal enclosings for all λ provided the minimal increase x is less or equal to 3.

Let V, a set of v points, be partitioned into m groups, each of size n. A *group divisible design with block size 3, GDD(n, m; λ)* is a collection of blocks or unordered triples of points of V, such that each pair of points from different groups occurs together in exactly λ blocks in the design and pairs of points from the same group never occur in the same block. It is clear that $m \geq 3$, since we are dealing with block size 3. A GDD in graph theoretic terminology can be considered as a graph decomposition into triangles of $\lambda(K_v\text{-}I)$ where I is a one-factor. In that sense, when n = 2, we want to enclose the graph decomposition of $\lambda(K_v\text{-}I)$ into that of $(\lambda+x)(K_{n+v}\text{-}I')$ where $I' \supseteq I$ is also a one-factor.

The problem we deal with is that of the *minimal standard enclosing* of $X = GDD(n, m; \lambda)$ into $Y = GDD(n, m+1; \lambda+x)$ for a minimal positive x. This means that, given X, we wish to construct a new *GDD*, say Y, on the points and groups of X along with one additional group of n points, and with the smallest possible increase in λ. Y contains, as a proper subdesign, all of the blocks from X.

The following necessary conditions for the existence of GDDs with block size 3 are well known, e.g. see Hanani [4] or Raghavarao [8].

Lemma 1.1. *There exists a GDD(n, m; λ) iff*

(a) 2 divides $\lambda n(m-1)$,

(b) 3 divides $\lambda n^2 m(m-1)$, and

(c) $m \geq 3$.

Lemma 1.1, along with a variety of sources including Fu, Rodger and Sarvate [2] and Lindner and Rodger [3], gives Table 1. Each cell (n, λ) in Table 1 represents these values for v mod 6, given λ mod 6 and n, and assuming n divides v. For example, to determine the condition on v for a GDD(3, m; 5) to exist, we must look at cell (3, 5), since n = 3 and $\lambda = 5$. This tells us that for this GDD to exist, it is necessary that $v \equiv 3 \pmod 6$. We note that $v \geq 3$ in all cases, since we are dealing with block size 3.

In Section 2 we derive new necessary conditions. Section 3 gives both a new general construction for enclosing of GDD's and a single construction for v = 6 and any index. In the succeeding sections we give examples and constructions for values of the index from 1 (mod 6) to 6 (mod 6). All the general constructions either use the new construction of Section 3 or a method used in [6] involving difference partitions which are described below.

A GDD is said to be *resolvable* if its blocks can be partitioned into classes such that each point occurs exactly once in each class. These classes are called *resolution classes.* We will refer to a resolvable GDD as an *RGDD(n, m; λ).* See, for example, Rees and Stinson [9] for the following necessary conditions and results for an RGDD(n, m; 1).

Lemma 1.2. *The following refer to the existence of an RGDD(n, m; 1):*

(a) *$n \equiv 1,5 \pmod 6$ implies $m \equiv 3 \pmod 6$.*
(b) *$n \equiv 3 \pmod 6$ implies $m \equiv 1 \pmod 2$.*
(c) *$n \equiv 2,4 \pmod 6$ implies $m \equiv 0 \pmod 3$*
(d) *$n \equiv 0$ mod 6 implies no congruential conditions.*
(e) *There is an RGDD(n, 3; 1) iff n is not 2 or 6.*
(f) *If $n \equiv 1,5 \pmod 6$ then there exists an RGDD(n, m; 1) iff m is odd.*
(g) *For $n \equiv 3 \pmod 6$, there is an RGDD(n, m; 1) iff m is odd.*
(h) *There is an RGDD(2, m; 1) iff $m \equiv 0 \pmod 3$, $m \geq 9$.*
(i) *There exists an RGDD(12, m; 1) if $m \geq 3$.*
(j) *If $m > 3$, there exists RGDD(6, m; 1) if m is not 11, 14.*

Corollary 1.3. *If an RGDD(n, m; 1) exists, then an RGDD(n, m; λ) exists for all λ > 1.*

n \ λ	0	1	2	3	4	5
1	0,1,2,3,4,5	1,3	0,1,3,4	1,3,5	0,1,3,4	1,3
2	0,2,4	0,2	0,2	0,2,4	0,2	0,2
3	0,3	3	0,3	3	0,3	3
4	0,2,4	0,4	0,4	0,2,4	0,4	0,4
5	0,1,2,3,4,5	3,5	0,2,3,5	1,3,5	0,2,3,5	3,5
6	0	0	0	0	0	0

Table 1 - The spectrum of GDDs for parameters v, n, λ mod 6

One commonly used method of employing a GDD for enclosing purposes involves a technique known as *expanding a resolution class.* Suppose we have a GDD containing some arbitrary number of resolution classes. To expand a resolution class about some new point, y, we take each block {a, b, c} in the resolution class and replace it with three different blocks: {y, a, b}, {y, a, c}, {y, b, c}. This is helpful because it does not raise the index of the original elements, but it creates an index of 2 for the new point with all old points.

Another important technique we use in constructing these GDDs is that of graph factorization and partitioning. The complete graph, denoted K_j, is a set of j vertices and $j(j-1)/2$ edges such that every vertex is connected to every other vertex via an edge. A *one-factor* of K_j is a full set of parallel edges (i.e. no two edges in the set contain the same vertex, and each vertex appears once). When j is even say $j = 2t$, we define a *one-factorization* of K_{2t} to be a set of $2t-1$ one-factors such that the union of these $2t-1$ one factors gives the set of all edges of K_{2t}. When $j = 2t$, such a one-factorization always exists.

A *difference partition* of K_{2t} is a set of disjoint classes P_1, $P_2, \ldots, P_t$, where edge (i, j) is in P_k if and only if $i - j \equiv k \pmod{2t}$. Difference partitions and one-factorizations can be used in the construction of GDDs with group size 2 by carefully utilizing the following well-known results (see Stanton and Goulden **[10]** and Hurd and Sarvate **[6]**). For example, adding new point y to each edge in a one-factor (based on the points of X) turns each edge into a block for Y. This puts y in a block once with each point of X. If, further, each edge in a one-factorization is used to make new blocks, then the index for all old points is increased by 1. In the Lemma below, "triangles" is used in the sense of subgraph, but for our purposes, triangle equates to block of a GDD.

Lemma 1.4. *With respect to the complete graph K_{2t} we have:*

(a) The triangles $\{1 + i, 2 + i, 4 + i\}$ for $i = 1, 2, \ldots, 2t$ contain exactly the edges from P_1, P_2, P_3, and the graph K_{2t} may be factored into $2t-1$ one-factors such that six of the 1-factors can be combined into $2t$ triangles.

(b) The triangles $\{1 + i, 1 + x + i, 1 + x + y + i\}$ for $i = 1, 2, \ldots, 2t$ contain exactly the edges from P_x, P_y, P_{x+y}, where $(x + y) < t$.

(c) The pairs in P_t form a one-factor. The pairs in P_{2x+1} (for $2x + 1 < t$) may be divided into two one-factors. The pairs in P_{2x} may be divided into a two-factor (i.e. a set of n edges such that each vertex occurs in exactly 2 edges) if $2x < t$.

(d) If $2x + 1 < t$, then $P_{2x} \cup P_{2x+1}$ splits into four one-factors. If t is odd, the set $P_{t-1} \cup P_t$ can be split into 3 one-factors.

(e) For the complete graph K_{6s} the set $P_s \cup P_{2s}$ forms 4s distinct triangles and the set P_{2s} forms 2s distinct triangles.

2. Necessary Conditions

Two well known relationships between the parameters of a GDD(n, m; λ) with block size 3 are

$$\lambda(v\text{-}n) = 2r, \text{ and } vr = 3b.$$

These two relationships are used to obtain the first two conditions given in the Lemma below.

Lemma 2.1. *The following conditions are necessary for the enclosing of a* $X = GDD(n, m; \lambda)$ *into a* $Y = GDD(n, m+1; \lambda+x)$*:*

(a) $v(\lambda+x) \equiv 0 \bmod 2$

(b) $v(v+n)(\lambda+x) \equiv 0 \bmod 6$

(c) $b_y - b_x \geq nr_y$

where r_x *and* r_y *are the replication numbers for X and Y, respectively, and* b_x *and* b_y *are the number of blocks in X and Y, respectively.*

Proof: As

$$2r_y = (\lambda+x)[(v+n) - n] = v(\lambda+x)$$

condition (a) follows. Also

$$(v+n)r_y = v(v+n)(\lambda+x)/2 = 3b$$

implies condition (b). The third condition arises from the fact the number of new blocks that are needed to create *Y is* $b_y - b_x$ and that n is the number of new points in *Y*. Thus, the number of new blocks must be greater than the total number of times the new points (which do not appear together in any block) must appear in *Y*.

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Part (c) of Lemma 2.1, for our purposes, is better expressed as in Corollary 2.2. We use this condition extensively.

Corollary 2.2. *For any enclosing of GDD(n, m; λ) into GDD(n, m+1; λ+x), it is necessary that* $x(m\text{-}2) \geq \lambda$.

Proof: We substitute the value of b_y - b_x and r_y to obtain

$$[(v+n)(\lambda+x)v - (v\text{-}n)v\lambda]/6 \geq nv(\lambda+x)/2$$

and then simplify to yield

$$vx \geq n(\lambda+2x).$$

Since $v = mn$, solving for the index gives

$$x(m-2) \geq \lambda.$$

■

n \ λ	1	2	3	4	5	6
1	(1,5) (3,1)	(0,1) (1,4) (3,2) (4,1)	(1,3) (3,1) (5,1)	(0,1) (1,2) (3,2) (4,2)	(1,1) (3,1)	(0,1) (1,6) (2,1) (3,2) (4,3) (5,2)
2	(0,1) (2,2)	(0,1) (2,1)	(0,1) (2,3) (4,1)	(0,1) (2,2)	(0,1) (2,1)	(0,1) (2,3) (4,1)
3	(3,1)	(0,1) (3,2)	(3,1)	(0,1) (3,2)	(3,1)	(0,1) (3,2)
4	(0,1) (4,2)	(0,1) (4,1)	(0,1) (2,1) (4,3)	(0,1) (4,2)	(0,1) (4,1)	(0,1) (2,1) (4,3)
5	(3,1) (5,5)	(0,1) (2,1) (3,2) (5,4)	(1,1) (3,1) (5,3)	(0,1) (2,2) (3,2) (5,2)	(3,1) (5,1)	(0,1) (1,2) (2,3) (3,2) (4,3) (5,6)
6	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)

Table 2 – Minimal x for GDDs given parameters λ and n for sufficiently large v.

Using Table 1 and Lemma 2.1, we have created Table 2 which shows the minimal possible x required for enclosing in each case for large enough v. Each ordered pair (v, x) in the cell (n, λ) represents these values for the given v *and* λ mod 6, i.e. a GDD(n, m; λ) can be minimally enclosed in a GDD(n, m+1; λ+x). For example, if we wish to find the minimal enclosing for a GDD(3, m; 5), we would look at cell (3, 5). The (3, 1) entry means that v = 6t + 3 and x = 1. Therefore, one should try to enclose a GDD(3, 2t+1; 6s+5) into a GDD(3, 2t+2; 6s+6) for all possible values of m which satisfy Corollary 2.2. The aim of this paper is to show the existence of all these possible enclosings for each entry in line 2 of Table 2.

3. New General Results

In Sections 4 to 9 we consider GDD with $\lambda = 6s+1$ to $\lambda = 6s+6$ for $s \geq 0$, respectively. However, in this section we first establish a new general construction for arbitrary group size which we apply later several times. Then we give an application of Corollary 2.2 for $v = 6$ and any index λ. As will be seen, the enclosings possible for $v = 6t$ will tend to vary for $t = 1$, $t = 2$, and $t > 2$, and it is convenient to handle the special case in this section.

Theorem 3.1. *Let $X = GDD(n, m; \lambda)$. Suppose that there exists $Z = RGDD(n, m; x)$, that $\lambda + x$ is even and that Corollary 2.2 is satisfied. Then X may be enclosed by $Y = GDD(n, m+1; \lambda+x)$.*

Proof: The design Y consists first of the blocks of X and the blocks of Z. This raises the index of points of X with each other to $\lambda+x$. Now, the replication number r_z for Z is also the number of resolution classes and, for block size 3, is given by $r_z = nx(m-1)/2$. To finish the enclosing, we must expand each of the n new points with $(\lambda+x)/2$ resolution classes from Z. We need $n(\lambda+x)/2$ resolution classes. However, Corollary 2.2 implies

$$\lambda \leq x(m-2),$$
$$\lambda + x \leq xm - x = x(m-1), \text{ so}$$
$$n(\lambda + x)/2 \leq nx(m-1)/2.$$

But the left hand side is the number of resolution classes needed and the right hand side is the number available. It is easy to see that the index is $\lambda+x$ for all points not in the same group. So X is enclosed by Y, a $GDD(n, m+1; \lambda+x)$.

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Theorem 3.2 *Any $GDD(2, 3; \lambda)$ can be minimally enclosed into $Y = GDD(2, 4; 2\lambda)$.*

Proof: First we use λ-copies of a difference partition $\{P_1, P_2, P_3\}$ of K_6. The pairs in P_3 correspond to the groups of X and are not used to make blocks. Use λ-copies of P_1 to make blocks with new point 7 and λ-copies of P_2 to make blocks with new point 8. Since the index is 2λ for all points, we have enclosed X into Y. By Corollary 2.2 with $v = 6$ and $m = 3$, we get $x \geq \lambda$. Thus the enclosing is minimal.

■

4. $\lambda \equiv 1 \pmod 6$

Note that row 1 of Table 2, where n = 1, has been dealt with in [5]. For the rest of the paper, the group size n is 2. Noting the cells of Table 2, we can see that to enclose a GDD(2, m; 1) into a GDD(2, m+1; 1+x), it is necessary that v = 6t or v = 6t + 2.

For v = 6t, x = 1 is minimal by Table 2. For the case of t = 1 see Theorem 3.2, and for t = 2 we show existence through a specific construction. For $t \geq 3$, we will apply Theorem 3.1.

Example 4.1 We consider the case where t = 2. We will be enclosing X = GDD(2, 6; 1) into Y = GDD(2, 7; 2). We know that X has 20 blocks, and the replication number is 5, and that Y has 56 blocks and a replication number of 12. So it is necessary to add 36 new blocks to X to create Y. Consider a difference partition of K_{12}. We can consider P_6 as the groups of X (by relabeling if necessary) since it is a partition of 12 vertices into pairs (a one-factor). We then create a set of 12 new blocks from P_4 by adding 13 to each edge. Similarly, we create another set of 12 new blocks from P_5 by adding 14 to each edge. Thus, we have created 24 new blocks, and the index of 13 and 14 with each number is 2, as required. To get the remaining 12 blocks, we use part (a) of Lemma 1.4, which allows us to form blocks from the remaining partition elements P_1, P_2, and P_3. Adding these 36 new blocks with the original 20 blocks of X, gives us Y.

Theorem 4.2 *Let X = GDD(2, 3t; 6s+1) and $t \geq 1$. Then X can be minimally enclosed into a Y = GDD(2, 3t+1; 6s+2) provided Corollary 2.2 is satisfied.*

Proof: In view of the examples, we assume t >2. We consider Z, an *RGDD(2, 3t; 1)*. Z exists by Lemma 1.2 part (h), since $m \equiv 0$ mod 3 and $m \geq 9$. To use resolution classes to get the new blocks with the new points, 6s+2 resolution classes are needed (3s+1 for each of the two new points to be added to X to form Y). This is exactly guaranteed by the hypothesis that Corollary 2.2 is satisfied, and the theorem follows by Theorem 3.1.

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For v = 6t+2, we can see that x = 2 is minimal. We will enclose a *GDD(2, 3t+1; 1)* into a *GDD(2, 3t+2; 3)*.

Example 4.3 We consider the case where t = 1. To enclose X = GDD(2, 4; 1) into Y = GDD(2, 5; 3), we will use a difference partition P_1, P_2, P_3, P_4 of K_8 to increase λ by 1 for points of X, and then follow by taking a copy of X to increase λ again. We identify P_4 with the groups of X, and so P_4 is not used to make new blocks. For new point 9, make blocks with the pairs from the two-factor P_2 and one of the one-factors comprising P_1. For new point 10, use P_3 and the remaining pairs from P_1 for new blocks.

Theorem 4.4 *Any X = GDD(2, 3t+1; 6s+1) can be minimally enclosed into a Y = GDD(2, 3t+2; 6s+3) for all t ≥ 1 when Corollary 2.2 is satisfied.*

Proof: We first suppose the index is 1 (s = 0) and modify the proof for the general case. The case for t = 1 is in Example 4.3. Now suppose t > 1. Then Y will consist of the blocks of two copies of X and additional blocks formed from (the equivalent of) 3 one-factors for each of the two new points and other one-factors formed into triangles (new blocks without new points). We will use K_{6t+2} to increase the index of each old pair by 1 to get the required index 3. We partition K_{6t+2} into its difference partition, which has 3t+1 classes of edges. We now apply a well-known partition of indices technique (see, e.g., Hurd and Sarvate [6]) to construct the necessary blocks. We consider the cases of t odd and t even separately.

Case: t even – Since t is even, we have 3t+1 = 6j+1 sets in our difference partition for some j. We identify P_{6j+1} (a one-factor) with the set of groups. We will use part (b) of Lemma 1.4 to construct the triangles from the partition of indices, so we need to take the remaining indices, i = 1, 2, ..., 6j, of the sets of the difference partition and partition them into triples {a, b, a+b} where $a+b < 3t+1$, except for 3 indices that we will use to create the blocks with the new points. The triples are

{1, 3j, 3j+1}, {3, 3j-1, 3j+2}, ..., {2j-3, 2j+2, 4j-1}, {2j-1, 2j+1, 4j}, and

{2, 5j, 5j+2}, {4, 5j-1, 5j+3}, ..., {2j-2, 4j+2, 6j}.

The remaining sets are P_{2j}, P_{4j+1}, and P_{5j+1}. We now divide P_{5j+1} into two one-factors as described in part (c) of Lemma 1.4. We add one new point to each edge in the first one-factor and the other new point to each edge in the other one-factor. Then we add the first new point to the edges in P_{2j} and the other new point to the edges in P_{4j+1}.

Case: t odd – Since t is odd, we have $3t+1 = 6n+4$ sets in our difference partition for some n. We identify P_{6n+4} with the set of groups. The needed triples in the case of an odd t are:

{1, 3n+2, 3n+3}, {3, 3n+1, 3n+4}, ..., {2n-1, 2n+3, 4n+2}, {2n+1, 2n+2, 4n+3} and

{2, 5n+3, 5n+5}, {4, 5n+2, 5n+6}, ..., {2n-2, 4n+5, 6n+3}

The remaining sets are P_{2n}, P_{4n+4}, and P_{5n+4}. Since it is necessary for one of the sets to have an odd index so we can apply part (c) Lemma 1.4, we replace the triple {2n-3, 2n+4, 4n+1} with {2n, 2n+4, 4n+4} in the set of triples above. Part (b) of Lemma 1.4 is then applied to construct triangles from the triples. We are left with the classes P_{2n-3}, P_{4n+1}, and P_{5n+4}. Now we, divide P_{4n+1} into two one-factors as described in part (c) of Lemma 1.4. We add one new point to each edge in the first one-factor and the other new point to each edge in the other one-factor. Then we add the first new point to the edges in P_{2n-3} and the other new point to the edges in P_{5n+4}.

Now suppose the index is $\lambda = 6s+1$ for $s > 0$. Rather than a second copy of X and one decomposition of K_{6t+2}, we use two copies of K_{6t+2} to make new blocks. Assume t is even (the odd case is similar). We partition the copies as in the even case above. There are six indices "left over," two each of P_{2j}, P_{4j+1}, and P_{5j+1}. One copy of each is used as described in the case above for $\lambda = 1$. The other three indices (that is, the corresponding P's) comprise six one-factors. We use these six one-factors to make blocks with one of the new points. Now one of the two new points requires s-triples to create new blocks and one requires (s-1)-triples. Thus, 2s-1 triples are needed. We note that the hypothesis from Corollary 2.2 becomes

$$s \leq 2j-1.$$

We note that there are 2j-1 triples in each partition of K_{6t+2}. In other words,

$$2s - 1 < 2(2j-1).$$

That is, the left hand side is the number of necessary triples of indices to make new blocks and the right hand side is the number available. Thus, there are ample one-factors to make all needed blocks with the two new points. The one-factors corresponding to any other triples of indices are used to make blocks without new points as indicated in Lemma 1.4.

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It is worth noting that the separation of cases in the proof for index 1 and for index 6s+1 is necessary. Two copies of K_{6t+2}, necessary for the general case, can not be used for the index 1 case since there would be no way to apply Lemma 1.4 to the six indices not in triples. As will be seen, a similar comment is appropriate for the proofs of Theorems 6.3, 7.2, and 9.3.

5. $\lambda \equiv 2 \pmod 6$

We consider the problem of enclosing a GDD(2, m; 6s+2) into a GDD(2, m+1; 6s+2+x). Noting the cells of Table 2, we can see that it is necessary that v = 6t or v = 6t + 2. For v = 6t, we can see that 1 is the minimal possible x by Table 2. Checking the necessary condition by Corollary 2.2, we can see that this x does not work for t = 1 but works for t ≥ 2, the general case here.

When using a difference partition for the proof of Theorem 4.4, we used Lemma 1.4(b) which required three related P_i to make triangles, i.e, blocks for Y without new points. In the next theorem we will use Lemma 1.4(e) so that the two P_i corresponding to indices t and 2t can be used together to make triangles. For the general case at hand, if the enclosing is to be accomplished via a difference partition, this use of P_t and P_{2t} (equivalent to 4 one-factors) is forced. This is because, for K_{6t}, there are 6t-1 one-factors in all, the one-factor P_{3t} is not used (it corresponds to the groups), and because six one-factors are to be used to make blocks with the 2 new points. This leaves 6t-1-1-6 = 6(t-2)+4 one-factors left to make blocks without new points. Since Lemma 1.4(b) "consumes" six one-factors at a time, there must be four one-factors used in some way to make blocks. Interestingly, the previous partitions can not be used in this case to allow use of P_t and P_{2t}. They and P_{3t} occur in different triples and there are too many parts "left over." However, two new partitions are constructed in the proof of the next theorem which overcome this difficulty, and exactly the right number of indices are not in triples of the sort (x, y, x+y). In the remaining sections, when we use difference partitions in a proof, we will prove only the odd case or the even case, not both, or we will just refer to Theorem 4.4 or 5.1 to indicate the partition to be used when the difficulties of balancing indices are straightforward.

Theorem 5.1 *An X = GDD(2, 3t; 6s+2) can be minimally enclosed into a Y = GDD(2, 3t+1; 6s+3) for all* $t \geq 2$ *provided Corollary 2.2 is satisfied.*

Proof: We use difference partitions as in Theorem 4.4. This time, however, we must use a different set of triples representing the indices of the difference classes. The even and odd cases are considered separately. We first suppose $\lambda = 2$.

Case: t even – Since t is even, t = 2j for some j. Therefore we have 6j difference classes. We identify P_{6j} as the set of groups and so we remove it. Next we partition the difference classes with the following triples:

(1, 3j-1, 3j), (3, 3j-2, 3j+1), ..., (2j-1, 2j, 4j-1), and

(2, 5j, 5j+2), (4, 5j-1, 5j+3), ..., (2j-2, 4j+1, 6j-1)

It can be verified that the P_{5j+1} and P_{4j} are not included in any triple. We remove the triple (2j-1, 2j, 4j-1) as well. Next we use part (e) of Lemma 1.5 to combine P_{4j} and P_{2j} into triangles (note that 2j = t and 4j = 2t). Next we divide P_{5j+1} into two one factors, and add the first new point to one one-factor and the second new point to the other. Finally, we add the first new point to the edges in P_{2j-1} and the second new point to the edges in P_{4j-1}. It can be verified that these new blocks along with the blocks of X form a GDD(6t+2, 2, 3).

Case: t odd – t is odd, therefore t = 2j+1 for some j. So 6t = 12j+6 and we have 6j+3 difference classes. We identify P_{6j+3} with the groups of X. An appropriate partition of the indices is

(1, 3j+1, 3j+2), (3, 3j, 3j+3), ..., (2j-1, 2j+2, 4j+1), and

(2, 5j+2, 5j+4), (4, 5j+1, 5j+5), ..., (2j-2, 4j+4, 6j+2).

This time, the remaining indices are 2j, 2j+1, 4j+2, 4j+3, 5j+3, and 6j+3. This time we use part (e) of Lemma 1.5 on P_{2j+1} and P_{4j+2} (note that 2j+1 = t and 4j+2 = 2t), and divide P_{5j+3} into two one factors. The rest follows the even case exactly.

Now suppose the index $\lambda = 6s + 2$. In addition to the blocks created as above, we need s triples for each of the two new points. It is easy to see that there are 2j-1 triples of indices. Thus, we want

$$2s \leq 2j - 1$$

or equivalently, $s < j$.

The condition from Corollary 2.2 ($\lambda \leq m - 2$) in this case reduces to

$$6s+2 \leq 3t - 2.$$

Since t = 2j, this becomes

$$s \leq j - 2/3.$$

That is, $s < j$.

■

From Table 2, for the case of v = 6t+2, we see that x = 1 is minimal, and Corollary 2.2 tells us that these enclosings may exist for all $t \geq 1$.

Theorem 5.2 *An X = GDD(2, 3t+1; 6s+2) can be minimally enclosed into a Y = GDD(2, 3t+2; 6s+3) for all $t \geq 1$ provided Corollary 2.2 is satisfied.*

Proof: When t = 1 and $\lambda = 2$, use a one-factorization of K_8, which has 7 one-factors. One one-factor corresponds to the set of groups and three each are used for each of the two new points. The general enclosing uses the partition from Theorem 4.4, and the argument for the sufficiency of the number of triples of indices is the same as at the end of the proof of Theorem 5.1.

■

6. $\lambda \equiv 3 \pmod 6$

We consider the problem of enclosing a GDD(2, m; 6s+3) into a GDD(2, m+1; 6s+3+x). Noting the cells of Table 2, we can see that it is necessary that v = 6t, v = 6t+2 or v = 6t+4. In the case of v = 6t, x = 1 is minimal, and Corollary 2.2 tells us that this enclosing may exist for all $t \geq 2$. For t = 1, we know x = 3 by Theorem 3.2.

Example 6.1 We consider the case where t = 2. We will be enclosing X = GDD(2, 6; 3) into Y = GDD(2, 7; 4). We use the difference partition $P_1, P_2, ..., P_6$ for K_{12}. Use P_4 by itself to make triangles, by Lemma 1.4(e). Identify P_6 with the blocks of X. Use two of the partitions left to make blocks with the first new point and use the other two partitions to make blocks with the second new point.

Theorem 6.2 *Let X = GDD(2, 3t; 6s+3) and $t \geq 2$. Then X can be minimally enclosed into a Y = GDD(2, 3t+1; 6s+4) provided Corollary 2.2 is satisfied.*

Proof: In view of Example 6.1, we may assume $t \geq 3$. We consider Z, an *RGDD(2, 3t; 1)*. Z exists by Lemma 1.2 part (h), since $m \equiv 0$ mod 3 and $m \geq 9$. We also know that $r_z = m-1 = 3t-1$. It is necessary that $r_z \geq n(\lambda + x) / 2 = 6s+4$, but this is exactly the condition guaranteed by Corollary 2.2. We can then apply Theorem 3.1 to construct the new blocks of Y.

■

According to Table 2, for v = 6t+2, x = 3 is minimal.

Theorem 6.3 *Let X = GDD(2, 3t+1; 6s+3). Then X can be minimally enclosed into a Y = GDD(2, 3t+2; 6s+ 6) for all $t \geq 1$ provided Corollary 2.2 is satisfied.*

Proof: First, let $\lambda = 3$ and let Z denote a GDD(2, 3t+1; 1). We add two copies of the blocks of Z to those of X. Now increase the index for points of X to 6 by using a difference partition exactly as in the proof of Theorem 4.4. For t odd or even, there are 3 indices outside the partition, corresponding to six one-factors for one of the new points. Use the partitions corresponding to any other triple to get six one-factors to make blocks with the second new point. For the general case, $\lambda = 6s+3$, and we need 3 copies of K_{6t+2} rather than using Z at all. In addition to 18 one-factors outside the partitions (Theorem 4.4), we will need 2s-1 triples of indices to make blocks with the new points. Since $x = 3$, Corollary 2.2 guarantees this.

■

Table 2 shows us that in the case of $v = 6t+4$, $x = 2$ is minimal, and Corollary 2.2 tells us that this enclosing may exist for all $t \geq 1$.

Theorem 6.4 *Let $X = GDD(2, 3t+2; 6s+3)$. Then X can be minimally enclosed into a $Y = GDD(2, 3t+3; 6s+5)$ for all $t \geq 1$ provided Corollary 2.2 is satisfied.*

Proof: First suppose $\lambda = 3$. We will use two copies of K_{6t+4} to increase the index of points of X by 2. We utilize the technique introduced in Theorem 4.4 We consider only the case for t odd, the other case being similar. Since t is odd, we have $3t+2 = 6j+5$ for some j. The sets in our difference partition are $P_1, P_2, \ldots, P_{6j+5}$. We identify P_{6j+5} with the set of groups in both copies of K_{6t+4}. The partition, from Theorem 4.4, which we use is the following:

$\{1, 3n+2, 3n+3\}, \{3, 3n+1, 3n+4\}, \ldots, \{2n+1, 2n+2, 4n+3\}$
and
$\{2, 5n+3, 5n+5\}, \{4, 5n+2, 5n+6\}, \ldots, \{2n-2, 4n+5, 6n+3\}, \{2n, 4n+4, 6n+4\}$.

The set P_{5n+4} is left over in both copies. In the first copy of K_{6t+4}, we turn all of the triples into triangles. In the other copy, we remove one triple, the one containing, say, the sets P_{2n+1}, P_{2n+2}, and P_{4n+3}. We convert the remaining triples into triangles. We now have five P_i's remaining which correspond to the 10 one-factors we need to make blocks with the two new points. The general case for $\lambda = 6s+3$ is completed as before.

■

7. $\lambda \equiv 4 \pmod 6$

We have two cases to consider: v = 6t, and v = 6t + 2. Table 2 shows us that in the case of v = 6t, x = 1 is minimal, and Corollary 2.2 tells us that this enclosing may exist for all $t \geq 2$.

Theorem 7.1 *Let X = GDD(2, 3t; 6s+4) and $t \geq 2$. Then X can be minimally enclosed into a Y = GDD(2, 3t+1; 6s+5) provided Corollary 2.2 is satisfied.*

Proof: Use the partition from Theorem 5.1.

■

Table 2 shows us that in the case of v = 6t+2, x = 2 is minimal, and Corollary 2.2 tells us that this enclosing may exist for all $t \geq 1$.

Theorem 7.2 *An X = GDD(2, 3t+1; 6s+4) can be minimally enclosed into a Y = GDD(2, 3t+2; 6s+ 6) for all $t \geq 1$ provided Corollary 2.2 is satisfied.*

Proof: Use 2 difference partitions of K_{6t+2} and this construction is similar to the construction in Theorem 4.4.

■

8. $\lambda \equiv 5 \pmod 6$

We have two cases to consider: v = 6t, and v = 6t + 2. We begin with the case of v = 6t. Table 2 shows us that in the case of v = 6t, x = 1 is minimal, and Corollary 2.2 tells us that this enclosing may exist for all $t \geq 3$.

Example 8.1 Let t = 2. Using Corollary 2.2, it is clear that x = 2 will work, so we will enclose X = GDD(2, 6; 5) into Y = GDD(2, 7; 7) by using two copies of a one-factorization of K_{12} and the two corresponding difference partitions. The details are straightforward.

Theorem 8.2 *Let X = GDD(2, 3t; 6s+5) and $t \geq 3$. Then X can be minimally enclosed into a Y = GDD(2, 3t+1; 6s+6) provided Corollary 2.2 is satisfied.*

Proof: We consider Z, an *RGDD(2, 3t; 3)*. Z exists by Lemma 1.2 part (h), since m ≡ 0 mod 3 and m ≥ 9. We can then apply Theorem 3.1 to construct the new blocks of Y.

■

Table 2 shows us that in the case of v = 6t+2, x = 1 is minimal, and Corollary 2.2 tells us that this enclosing may exist for all t ≥ 2. We start by finding the x for the case of t = 1.

Example 8.3 For the case of t = 1 and X = GDD(2, 4; 5), Corollary 2.2 shows that x = 3 is minimal, but a GDD(2, 5; 8) does not exist by Table 1, so we must let x = 4. We may enclose X into Y = GDD(2, 5; 9) - just use 4 copies of a difference partition for K_8. Identify P_4 with the groups. Make triangles with one copy of P_1, P_2 and P_3. Use the other 3 copies of those partitions to get the 18 one-factors needed to make blocks with the 2 new points.

Theorem 8.4 *Let X = GDD(2, 3t+1; 6s+5). Then X can be minimally enclosed into a Y = GDD(2, 3t+2; 6s+6) for all t ≥ 2 provided Corollary 2.2 is satisfied.*
Proof: The construction of this enclosing is identical to that of Theorem 4.4.

■

9. λ ≡ 0 (mod 6)

Finally, we consider the problem of enclosing a GDD(2, m; 6s+6) into a GDD(2, m+1; 6s+6+x). Noting the cells of Table 2, we can see that it is necessary that v = 6t, v = 6t + 2, or v = 6t + 4. Table 2 shows us that in the case of v = 6t, x = 1 is minimal, and Corollary 2.2 tells us that this enclosing may exist for all t ≥ 3.

Example 9.1 When t = 1, using Theorem 3.2, we can see that x = 6 is minimal. When t = 2, x = 2 suffices. To see this, we enclose X = GDD(2, 6; 6) into Y = GDD(2, 7; 8). Use two copies of K_{12} and the method of difference partitions.

Theorem 9.2 *Let $X = GDD(2, 3t; 6s+6)$ and $t \geq 3$. Then X can be minimally enclosed into a $Y = GDD(2, 3t+1; 6s+7)$ provided Corollary 2.2 is satisfied.*

Proof: Use the method of difference partitions (Theorem 4.4). In each case, odd t or even t, one needs 14 one-factors for the new blocks. But this is straightforward.

■

Table 2 shows us that in the case of $v = 6t+2$, $x = 3$ is minimal, and Corollary 2.2 tells us that this enclosing may exist for all $t \geq 1$.

Theorem 9.3 *Let $X = GDD(2, 3t+1; 6s+6)$. Then X can be minimally enclosed into a $Y = GDD(2, 3t+2; 6s+9)$ for all $t \geq 1$ provided Corollary 2.2 is satisfied.*

Proof: Use the method from Theorem 4.4, and use 3 copies of K_{6t+2}.

■

Table 2 shows us that in the case of $v = 6t+4$, $x = 1$ is minimal, and Corollary 2.2 tells us that this enclosing may exist for all $t \geq 2$.

Example 9.4 Using Corollary 2.2, we can see that $x = 2$ is minimal for $X = GDD(2, 5; 6)$. Therefore it is necessary to enclose X into $Y = GDD(2, 6; 8)$. We take two copies of K_{10} and partition them into 18 one-factors. Two of these correspond to the groups. For the other 16 one-factors, use 8 with each of the two new points to make blocks.

Theorem 9.5 *Let $X = GDD(2, 3t+2; 6s+6)$. Then X can be minimally enclosed into a $Y = GDD(2, 3t+3; 6s+7)$ for all $t \geq 2$ provided Corollary 2.2 is satisfied.*

Proof: This is straightforward using the method of Theorem 4.4.

■

10. An Alternate Approach

A possible approach to the problems considered here using graph decompositions could be undertaken, and we outline this attack as helpfully indicated by an anonymous referee. Each of the n new vertices has in one sense no choice in what it requires: it needs to be allocated the edges in a $(\lambda+x)$-regular graph of multiplicity at most x defined on the old vertices, but never using the edges in a one-factor F (F forms the groups). So the problem here is really to show that, whenever the necessary conditions are satisfied, there exists an edge-disjoint decomposition of the multigraph $x(K_{mn} - F)$ into n $(\lambda+x)$-regular graphs and a lot of triangles. There is a result in the literature that does almost precisely this and has a short proof. It finds the required triangles in such a way that the complement in xK_{mn} has a one-factorization if mn is even and a 2-factorization when mn is odd. In the case of this paper, mn is even since $n = 2$; so the only thing it does not do is guarantee that one one-factor is repeated x times in the complement - these would be the edges never allowed to be used as they define the groups. However, this should be essentially already done except in a few extreme cases. In any case, this approach should be shorter, and the case for $n = 3$ and one extra group added could similarly be settled using the result mentioned as one can ensure that the triangles contain x copies one parallel class to form the groups.

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