Product Constructions For Critical Sets In Latin Squares

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Abstract

Let T be a partial latin square. If there exists two distinct latin squares M and N of the same order such that $M \cap N = T$, then $M \setminus T$ is said to be a latin trade. For a given latin square M it is possible to identify a subset of entries, termed a *critical set*, which intersects all latin trades in M and is minimal with respect to this property.

Stinson and van Rees have shown that under certain circumstances, critical sets in latin squares M and N can be used to identify critical sets in the direct product $M \times N$. This paper presents a refinement of Stinson and van Rees' results and applies this theory to prove the existence of two new families of critical sets.

1 Introduction

Let M be a latin square of order m, and T a partial latin square contained in M. If there exists a latin square N of order m, distinct from M and such that $M \cap N = T$, then $M \setminus T$ is said to be a latin trade. For a given latin square M it is possible to identify a subset of entries, termed a critical set, which intersects all latin trades in M and is minimal with respect to this property.

Critical sets have been studied since 1978, see for instance **[7]**. Recently there has been much interest in the possible sizes of critical

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sets, [**4, 10, 3, 2, 6**]. It has been shown ([**10, 3**]) that for a given m there exists critical sets of order m containing t entries where $\lfloor m^2/4 \rfloor \leq t \leq (m^2 - m)/2$ (see also [**6**]). There are examples of critical sets containing more than $(m^2 - m)/2$ entries, but to date very little is known about generating these critical sets. In 1982, Stinson and van Rees [**13**] proved that under certain circumstances, critical sets in latin squares M and N can be used to identify critical sets in the direct product $M \times N$. In this paper we will refine Stinson and van Rees' results and apply this theory to identify two new families of critical sets. These results are of interest as they indicate new techniques which may be useful in settling the question of the spectrum of critical sets.

2 Critical sets and latin trades

Let $X = \{1, \ldots, m\}$. A *partial latin square* P of order m is an $m \times m$ array containing symbols chosen from the set X in such a way that each element of X occurs at most once in each row and at most once in each column of the array. Thus P may contain a number of empty cells. Table 1 provides examples of partial latin squares of orders 2 and 4. For ease of exposition, a partial latin square P will be represented as a set of ordered triples $\{(i, j; k) \mid \text{element } k \in X \text{ occurs in cell } (i, j) \text{ of the array}\}$. Define $R(P) = \{i \mid (i, j; k) \in P\}$, $C(P) = \{j \mid (i, j; k) \in P\}$ and $E(P) = \{k \mid (i, j; k) \in P\}$. The *size* of the partial latin square P is $|P|$ and so represents the number of non-empty cells in P. The shape of a partial latin square P is the set of cells $S_P = \{(i, j) \mid (i, j; k) \in P\}$. If all the cells of the array are filled, then the partial latin square is termed a latin square. A *latin square* M, of order m, is an $m \times m$ array with entries chosen from the set X in such a way that each element of X occurs precisely once in each row and precisely once in each column of the array. Table 1 provides examples of latin squares of orders 2 and 4.

Throughout this paper the notation P_1, L_1, E_2 and L_2 will refer to the partial latin squares and latin squares given in Table 1.

Two partial latin squares P and Q of order m are *isotopic* if there exists three bijections θ, ϕ and ρ, respectively, mapping the rows, columns, and symbols of P to the rows, columns, and symbols of Q.

1	

P_1

1	2
2	1

L_1

1	2		
			3
	4		
		2	

E_2

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

L_2

Table 1: P_1 and E_2 are partial latin squares of orders 2 and 4 respectively, and L_1 and L_2 are latin squares of orders 2 and 4 respectively.

Formally, P and Q are isotopic if $Q = \{(\theta(i), \phi(j); \rho(k)) \mid (i, j; k) \in P\}$. Moreover, if $\theta = \phi = \rho$ then P and Q are said to be *isomorphic*.

Let P be a partial latin square of order m and $\{a, b, c\} = \{1, 2, 3\}$. Then the (a, b, c)-*conjugate* of P is denoted and defined by $P_{(a,b,c)} = \{(x_a, x_b; x_c) \mid (x_1, x_2; x_3) \in P\}$. For $\theta \in S_3$, the symmetric group on $\{1, 2, 3\}$, we define $\theta(x_1, x_2, x_3) = (x_{\theta(1)}, x_{\theta(2)}, x_{\theta(3)})$.

A partial latin square P of order m, is said to *complete to* the latin square M, if M is of order m and $P \subset M$. Note that if partial latin squares P and Q of order m are isotopic (or conjugates) then P completes to precisely r different latin squares of order m if and only if Q completes to precisely r different latin squares of order m. If M is the only latin square of order m which has symbol k in cell (i, j) for each $(i, j; k) \in P$, then P is termed a *uniquely completable* (UC) partial latin square in the latin square M and P is said to be UC to M. The literature contains a number of articles identifying general families of uniquely completable partial latin squares which are minimal with respect to this property, see for example [**7, 4, 10, 3, 2, 6**]. Such partial latin squares have been termed critical sets.

A *critical set* in a latin square M of order m is a partial latin square $\mathcal{P}$ contained in M, such that

1. $\mathcal{P}$ is a uniquely completable set in M, and

2. no proper subset of $\mathcal{P}$ satisfies 1.

The partial latin squares P_1 and E_2 given in Table 1 are examples of critical sets in the latin squares L_1 and L_2 respectively. Critical sets form the main focus of this paper.

Let P be a partial latin square of order m defined on an element set X. Then A_P is an *array of alternatives* for P if

1. A_P is an $m \times m$ array ;
2. whenever cell (i, j) of P is filled, cell (i, j) of A_P is empty;
3. whenever cell (i, j) of P is empty, cell (i, j) of A_P contains all elements of X not appearing in row i or column j of P.

Denote the set of symbols in cell (i, j) of A_P by $A_P(i, j)$.

The addition of a triple $(i, j; k)$ to P is said to be *forced* if either:

1. $A_P(i, j) = \{k\}$;
2. $\theta(i, j, k)$ satisfies 1 in $A_{P_{\theta(1,2,3)}}$ for some $\theta \in S_3$.

Further, let P and Q be two partial latin squares. The partial latin square P is *strong uniquely completetable* to Q if there exists a sequence of sets of triples $P = S_1 \subset S_2 \subset \ldots \subset S_f = Q$ such that each triple $t \in S_{v+1} \setminus S_v$ is forced in S_v. If Q is a latin square, then P is said to be strong UC to Q. A critical set $\mathcal{P}$ is a *strong critical set* in a latin square L, if $\mathcal{P}$ is strong UC to L and no proper subset of $\mathcal{P}$ satisfies this property. (For a more general definition of *semi-forced* and *near-strong* UC partial latin squares see [5].)

Property 2 of the definition of a critical set $\mathcal{P}$ ensures that for each element $(i, j; k) \in \mathcal{P}$, $\mathcal{P} \setminus \{(i, j; k)\}$ has at least two distinct completions. For example consider E_2, as given in Table 1, and remove element $(1, 2; 2)$. Then $E_2 \setminus \{(1, 2; 2)\}$ has the two completions, L_2 and L_2', given in Table 2. Table 2 also displays the differences between these two completions; that is, $L_2 \setminus L_2'$ and $L_2' \setminus L_2$.

The identification of partial latin squares with the properties exhibited by $L_2 \setminus L_2'$ and $L_2' \setminus L_2$ is crucial for the determination of critical sets and we give the following definition. Two partial latin squares I and I', of order m with $S_I = S_{I'}$, are said to be *row (column) balanced* if the symbols in each row (column) of I are the same as those in the corresponding row (column) of I'. If I and I' are row and column balanced then they are said to be *mutually balanced.* They are said to be *disjoint* if no cell in I contains the same symbol as the corresponding cell in I'. Given two disjoint, mutually balanced

1			
			3
	4		
		2	

$E_2 \setminus \{(1,2;2)\}$

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

L_2

1	3	4	2
4	2	1	3
2	4	3	1
3	1	2	4

L_2'

	2	3	4
2	1	4	
3		1	2
4	3		1

$L_2 \setminus L_2'$

	3	4	2
4	2	1	
2		3	1
3	1		4

$L_2' \setminus L_2$

Table 2: The possible completions of $E_2 \setminus \{(1,2;2)\}$ and their differences.

partial latin squares I and I' of order m, of the same shape, we say I is a *latin trade* and I' is the *disjoint mate* of I. (Note that in some earlier papers latin trades have been referred to as latin interchanges or critical partial latin squares.) A latin trade is said to be *minimal* if no proper subset of I contains a latin trade. An *intercalate* is a latin trade of size four. Let P be a UC set contained in a latin square M. Entry $(i,j;k) \in P$ is said to be *necessary for UC* if there exists a latin trade, $I \subset M$, such that $I \cap \mathcal{P} = \{(i,j;k)\}$. These ideas are summarized in the next lemma.

Lemma 1 *Let $\mathcal{P}$ be a critical set in a latin square M of order m. Then*

1. *for each latin trade $I \subset M$, $|\mathcal{P} \cap I| \geq 1$, and*
2. *each entry $(i,j;k) \in \mathcal{P}$ is necessary for UC.*

3 Products

Let M and N be two latin squares of orders m and n, with symbols chosen from the sets $X = \{1,2,\ldots,m\}$ and $Y = \{1,\ldots,n\}$. Suppose that P is a partial latin square contained in M and Q is a partial latin square contained in N. For $1 \leq r \leq m$, let $Y^r = \{(r-1)n+y \mid y \in Y\}$

and let Q^r be the array obtained from Q by adding $(r-1)n$ to each of the symbols in cells of Q. Consequently, Q^r is a partial latin square isomorphic to Q but based on symbols chosen from the set Y^r. Define the *completable product* of P and Q, with respect to M and N (written $P \otimes Q$) to be the partial latin square of order mn obtained by replacing each cell containing the symbol r of P with the array N^r and each cell containing the symbol s of $M \setminus P$ with the array Q^s. Let

$$\begin{aligned} PQ &= \{((u-1)n+x,(v-1)n+y;(w-1)n+z) \mid \\ &\quad (u,v;w) \in P \wedge (x,y;z) \in Q\}, \\ P\overline{Q} &= \{((u-1)n+x,(v-1)n+y;(w-1)n+z) \mid \\ &\quad (u,v;w) \in P \wedge (x,y;z) \in N \setminus Q\}, \\ \overline{P}Q &= \{((u-1)n+x,(v-1)n+y;(w-1)n+z) \mid \\ &\quad (u,v;w) \in M \setminus P \wedge (x,y;z) \in Q\}. \end{aligned}$$

Then the completable product of P and Q, with respect to M and N, is

$$P \otimes Q = PQ \cup P\overline{Q} \cup \overline{P}Q.$$

If P is UC to M and Q is UC to N then we will adopt the convention of referring to $P \otimes Q$ as the completable product of P with Q and omit the phrase "with respect to M and N". The completable product of two latin squares M and N is usually referred to as the *direct product* of M with N and is written $M \times N$. So

$$\begin{aligned} M \times N &= \{((a-1)n+d,(b-1)n+e;(c-1)n+f) \mid \\ &\quad (a,b;c) \in M \wedge (d,e;f) \in N\}. \end{aligned}$$

Table 3 provides an example of the completable product $P_1 \otimes E_2$ and the direct product $L_3 = L_1 \times L_2$.

For a given cell (i,j) of P, define the *block position* (i,j) of $P \otimes Q$ to be the cells of $P \otimes Q$ corresponding to the intersection of rows $(i-1)n+1$ to $(i-1)n+n$ with columns $(j-1)n+1$ to $(j-1)n+n$.

We observe that the latin square $M \times N$ is isomorphic to the latin square $N \times M$; the partial latin square $P \otimes Q$ is contained in $M \times N$; the partial latin square $P \otimes Q$ is isomorphic to the partial latin square $Q \otimes P$.

1	2	3	4	5	6		
2	1	4	3				7
3	4	1	2		8		
4	3	2	1			6	
5	6			1	2		
			7				3
	8				4		
		6				2	

$P_1 \otimes E_2$

1	2	3	4	5	6	7	8
2	1	4	3	6	5	8	7
3	4	1	2	7	8	5	6
4	3	2	1	8	7	6	5
5	6	7	8	1	2	3	4
6	5	8	7	2	1	4	3
7	8	5	6	3	4	1	2
8	7	6	5	4	3	2	1

$L_3 = L_1 \times L_2$

Table 3: Completable and Direct products

Gower [12], Bedford and Whitehouse [5], and Adams and Khodkar [1] have laid down conditions under which the completable product has UC.

Lemma 2 [5] *Let P be a partial latin square of order m that is UC to the latin square M and let Q be a partial latin square of order n that is UC to N. Let L be a latin square to which $P \otimes Q$ completes. Suppose that the addition of the triple $(i, j; k)$ is (semi)forced in P. Then L must contain a copy of N^k in block position (i, j). Further if P is (near)strong UC to the latin square M, then $P \otimes Q$ is (near)strong UC to $M \times N$.*

We will also require a generalization of a recently result by Donovan and Khodkar [11].

Lemma 3 *Let P and Q be (near)strong UC sets in latin squares M and N respectively. Let $I \subset M$ and, for $1 \leq i \leq r$, $J_i \subset N$ be partial latin squares such that $I \cap P = \{(a, b; c)\}$, $P \setminus \{(a, b; c)\}$ is (near)strong UC to $M \setminus I$, $J_i \cap Q = \{(d_i, e_i; f_i)\}$, and $Q \setminus \{(d_i, e_i; f_i) \mid 1 \leq i \leq r\}$ is (near)strong UC to $N \setminus (\cup_i J_i)$. Then the partial latin square $(P \otimes Q) \setminus \{\gamma_i \mid 1 \leq i \leq r\}$ is (near)strong UC to*

$$([(M \setminus I) \cup \{(a, b; c)\}] \otimes [(N \setminus (\cup_i J_i)) \cup \{(d_i, e_i; f_i) \mid 1 \leq i \leq r\}]) \setminus \{\gamma_i \mid 1 \leq i \leq r\}$$

where $\gamma_i = ((a-1)n + d_i, (b-1)n + e_i; (c-1)n + f_i)$.

The following lemma allows us to identify some of the entries of $P \otimes Q$ which are necessary for UC.

Lemma 4 *Let P and Q be a (near)strong critical sets contained, respectively, in the latin square M, of order m, and N of order n. Let $P\overline{Q}$ and $\overline{P}Q$ be as in the definition of completable product. Then the elements of $P\overline{Q}$ and $\overline{P}Q$ are necessary for the UC of $P \otimes Q$.*

Proof: Lemma 2 implies that $P \otimes Q$ is (near)strong UC to $M \times N$. Thus $P \otimes Q$ contains a critical set. It is necessary to show that for every entry $((u-1)n+x, (v-1)n+y; (w-1)n+z)$ of $(P\overline{Q} \cup \overline{P}Q)$ there exists a latin trade $\mathcal{I}$ in $M \times N$ such that $\mathcal{I} \cap (P \otimes Q) = \{((u-1)n+x, (v-1)n+y; (w-1)n+z)\}$.

The entries in $\overline{P}Q$ correspond to the entries $(u, v; w) \in M \setminus P$. All such block positions (u, v) contain isomorphic copies of Q. Since Q is a critical set in N for each entry $(x, y; z)$ in Q there exists a latin trade $I_{(x,y;z)}$ in N such that $Q \cap I_{(x,y;z)} = \{(x, y; z)\}$. Hence for each $((u-1)n+x, (v-1)n+y; (w-1)n+z) \in \overline{P}Q$ there exists a latin trade, namely $\{(u-1)n+x'; (v-1)n+y'; (w-1)n+z') \mid (x', y'; z') \in I_{(x,y;z)}\}$, which meets $P \otimes Q$ in $((u-1)n+x, (v-1)n+y; (w-1)n+z)$ alone.

The entries of $P\overline{Q}$ correspond to the entries $(u, v; w) \in P$. Since P is a critical set in M, for all entries $(u, v; w) \in P$ there exists a latin trade $I_{(u,v;w)} \in M$ such that $I_{(u,v;w)} \cap P = \{(u, v; w)\}$. Thus for each entry $((u-1)n+x, (v-1)n+y; (w-1)n+z) \in P\overline{Q}$ there exists a latin trade, namely $\{((u'-1)n+x, (v'-1)n+y; (w'-1)n+z) \mid (u', v'; w') \in I_{(u,v;w)}\}$, which meets $P \otimes Q$ in the entry $((u-1)n+x, (v-1)n+y; (w-1)n+z)$ alone.

The above result suggests that the structure of the latin trades in M and N is important in the identification of elements in $P \otimes Q$ which are necessary for unique completion. The purpose of this paper is to advance our knowledge of the structure of latin trades in $M \times N$, and to shed new light on the necessity for UC of entries in the set $\{((a-1)n+d, (b-1)n+e; (c-1)n+f) \mid (a, b; c) \in P \wedge (d, e; f) \in Q\}$. Two further results which will be useful in our discussion are given below.

Let $\mathcal{I} \subseteq (M \times N)$ be a partial latin square and define functions $f_M(\mathcal{I})$ and $f_N(\mathcal{I})$ and sets $proj_M(\mathcal{I})$ and $proj_N(\mathcal{I})$ as follows:

$f_M : \mathcal{I} \longrightarrow M$, where

$$f_M(((u-1)n+x,(v-1)n+y;(w-1)n+z)) = (u,v;w),$$
$$f_N : \mathcal{I} \longrightarrow M, \text{ where}$$
$$f_N(((u-1)n+x,(v-1)n+y;(w-1)n+z)) = (x,y;z), \text{ and}$$
$$proj_M(\mathcal{I}) = \{(u,v;w) \mid \exists \alpha \in \mathcal{I}, f_M(\alpha) = (u,v;w)\},$$
$$proj_N(\mathcal{I}) = \{(x,y;z) \mid \exists \alpha \in \mathcal{I}, f_N(\alpha) = (x,y;z)\}.$$

If $\mathcal{I}$ is a latin trade that meets each block position of $M \times N$ in at most one entry, then it will be shown that the image of f_M is a latin trade in M, as is the image of f_N in N.

Lemma 5 *Let M and N be latin squares of order m and n respectively. Let $\mathcal{I}$ be a latin trade in $M \times N$ such that for each $i, j \in \{1, \ldots, m\}$ block position (i,j) of $M \times N$ meets $\mathcal{I}$ in at most one entry. Let f_M and $proj_M(\mathcal{I})$ be as defined above. Then $proj_M(\mathcal{I})$ is a latin trade in M. In addition, if $\mathcal{I}$ is a minimal latin trade, then up to isotopism $\mathcal{I}$ and $proj_M(\mathcal{I})$ have the same shape.*

Proof: Let $\mathcal{I}'$ denote the disjoint mate of $\mathcal{I}$. The partial latin square $\mathcal{I}'$ is contained in the latin square $M' = ((M \times N) \setminus \mathcal{I}) \cup \mathcal{I}'$. Define $f_{M'}$ and $proj_{M'}$ as follows:

$$f_{M'} : \mathcal{I}' \to M'$$
$$f_{M'}(((u-1)n+x,(v-1)n+y;(w-1)n+z)) = (u,v;w),$$
$$proj_{M'}(\mathcal{I}') = \{(u,v;w) \mid \exists \alpha \in \mathcal{I}', f_{M'}(\alpha) = (u,v;w)\}.$$

It will be shown that $proj_M(\mathcal{I})$ and $proj_{M'}(\mathcal{I}')$ form a latin trade and its disjoint mate. The definition of a latin trade implies that $((u-1)+x,(v-1)n+y) \in \mathcal{S}_{\mathcal{I}}$ if and only if $((u-1)+x,(v-1)n+y) \in \mathcal{S}_{\mathcal{I}'}$. Hence $(u,v) \in \mathcal{S}_{proj_M(\mathcal{I})}$ if and only if $(u,v) \in \mathcal{S}_{proj_{M'}(\mathcal{I}')}$. Thus $proj_M(\mathcal{I})$ and $proj_{M'}(\mathcal{I}')$ have the same shape. Assume that they are not disjoint. Then there exists an entry $(u,v;w) \in proj_M(\mathcal{I})$ such that $(u,v;w) \in proj_{M'}(\mathcal{I}')$. But this implies there exists $x, x', y, y', z, z' \in Y$ such that $((u-1)n+x,(v-1)n+y;(w-1)n+z) \in \mathcal{I}$ and $((u-1)n+x',(v-1)n+y';(w-1)n+z') \in \mathcal{I}'$. But recall that $\mathcal{I}$ and $\mathcal{I}'$ have the same shape and $\mathcal{I}$ intersects each block position of $M \times N$ in at most one entry. Thus $x' = x$ and $y' = y$. In addition, $z' = z$. Therefore $proj_M(\mathcal{I})$ and $proj_{M'}(\mathcal{I}')$ are disjoint. Finally assume that row u of $proj_M(\mathcal{I})$ contains symbol w but row u of $proj_{M'}(\mathcal{I}')$ does not. It follows that for some $x \in Y$ there exists $z \in Y$ such that symbol $(w-1)n+z$ occurs in row $(u-1)n+x$ of $\mathcal{I}$, however for all $z' \in Y$, symbol $(w-1)n+z'$ does not occur in row $(u-1)n+x$ of $\mathcal{I}'$. But this is a contradiction as $\mathcal{I}$ and $\mathcal{I}'$ are mutually balanced. So $proj_M(\mathcal{I})$ and $proj_{M'}(\mathcal{I}')$ are row balanced. A similar argument shows that $proj_M(\mathcal{I})$ and $proj_{M'}(\mathcal{I}')$ are also column balanced. Hence $proj_M(\mathcal{I})$ is a latin trade in M with disjoint mate $proj_{M'}(\mathcal{I}')$. It is immediate that up to isotopism $\mathcal{I}$ and $proj_M(\mathcal{I})$ have the same shape.

Corollary 6 *Let M and N be latin squares of order m and n respectively. Let $\mathcal{I}$ be a latin trade in $M \times N$. Then $\mathcal{J} = \{(x-1)m+u, (y-1)m+v; (z-1)m+w) \mid ((u-1)n+x, (v-1)n+y; (w-1)n+z) \in \mathcal{I}\}$ is a latin trade in $N \times M$. Assume that $\mathcal{J}$ is such that for each $i, j \in \{1, \ldots, n\}$ block position (i,j) of $N \times M$ meets $\mathcal{J}$ in at most one entry. Let $f_N(\mathcal{J})$ and $proj_N(\mathcal{J})$ be as defined above. Then $proj_N(\mathcal{J})$ is a latin trade in N and up to isotopism $proj_N(\mathcal{J})$ and $\mathcal{J}$ have the same shape.*

The next result will be important in proving the main result of this paper.

Lemma 7 *Let M and N be latin squares of order m and n respectively and χ be a latin square of order mn distinct from $M \times N$. Let $(i, j; k)$ be an arbitrary element of M. Let $\chi_{(i,j)}$ denote the set of cells corresponding to the intersection of rows $(i-1)n+1, \ldots, (i-1)n+n$ with columns $(j-1)n+1, \ldots, (j-1)n+n$ of χ. Let $Q \subset N$ be a critical set. Assume that in $\chi_{(i,j)}$, $M \times N$ and χ agree in Q^k. In addition, assume that for each symbol $z \in Y^k \setminus \{(k-1)n+1\}$, $\chi_{(i,j)}$ contains n cells occupied by symbol z, $n-1$ cells occupied by symbol $(k-1)n+1$ and precisely one cell occupied by the symbol $(w-1)n+u$, for $w \in X \setminus \{k\}$ and $u \in Y$. Then in the cells corresponding to $\chi_{(i,j)}$, $M \times N$ and χ agree in precisely n^2-1 cells.*

Proof: By assumption there exists $r, s \in Y$ such that cell $((i-1)n+r, (j-1)n+s)$ of χ contains symbol $(w-1)n+u$, $w \in X \setminus \{k\}$ and $u \in Y$. Each of the symbols $(k-1)n+2, \ldots, (k-1)n+n$ occurs n times in $\chi_{(i,j)}$. Hence each of these symbols occurs in each row and each column of $\chi_{(i,j)}$. Symbol $(k-1)n+1$ occurs exactly $n-1$ times in $\chi_{(i,j)}$. Consequently symbol $(k-1)n+1$ does not occur in any cell of row $(i-1)n+r$ or column $(j-1)n+s$ of $\chi_{(i,j)}$.

Focus on $\chi_{(i,j)}$. If symbol $(w-1)n+u$ is replaced by symbol $(k-1)n+1$, the result is an $n \times n$ subarray which contains a latin square on the symbols $(k-1)n+1, \ldots, (k-1)n+n$. Assume that this $n \times n$ latin square, denoted A, is distinct from N^k. Then it follows that N^k and A differ in a latin trade. However, N^k and A agree in all cells corresponding to Q^k, and Q is near-strong UC to N. Thus Q^k intersects all latin trades in N^k, leading to a contradiction. Therefore, in the cells corresponding to $\chi_{(i,j)}$, χ and $M \times N$ differ in precisely one cell.

4 Doubling construction

In [13], Stinson and van Rees used the completable product to construct critical sets in latin squares of even order. Their result is as follows.

Lemma 8 [13] *Assume that $\mathcal{Q}$ is a critical set in a latin square N of order n and that for each $(i,j;k) \in \mathcal{Q}$, there exists an intercalate $I_{(i,j;k)} \subset N$ with the property that $I_{(i,j;k)} \cap \mathcal{Q} = \{(i,j;k)\}$. Then $P_1 \otimes \mathcal{Q}$ is a critical set in the latin square $L_1 \times N$.*

Proof: Lemma 2 implies that $P \otimes \mathcal{Q}$ is UC. Lemma 4 implies that each of the entries in block positions $(1,2),(2,1),(2,2)$ is necessary for UC. Lemma 4 also implies that in block position $(1,1)$ the entries corresponding to $N \setminus \mathcal{Q}$ are necessary for UC. For the remaining entries of block position $(1,1)$ it is noted that for each $(i,j;k) \in \mathcal{Q}$ there exists an intercalate $I_{(i,j;k)} = \{(i,j;k),(i,j';k'),(i',j;k'),(i',j';k)\}$ in N with the property that $I_{(i,j;k)} \cap \mathcal{Q} = \{(i,j;k)\}$. Hence for each such $(i,j;k)$ in block position $(1,1)$ of $P \otimes \mathcal{Q}$ there exists an intercalate $\mathcal{I} = \{(i,j;k),(i,j'+n;k'+n),(i'+n,j;k'+n),(i'+n,j'+n;k)\}$ such that $\mathcal{I} \cap (P_1 \otimes \mathcal{Q}) = \{(i,j;k)\}$.

We now prove that if there exists a latin trade which meets $P_1 \otimes \mathcal{Q}$ in block position $(1,1)$ in only one entry then the latin trade is an intercalate.

Lemma 9 *Assume that Q is a critical set in a latin square N of order n. If there exists a latin trade $\mathcal{I} \in (L_1 \times N)$ such that $|\mathcal{I} \cap (P_1 \otimes Q)| = 1$ and $\mathcal{I}$ contains precisely one entry from block position $(1,1)$, then $\mathcal{I}$ is an intercalate.*

Proof: Assume $\mathcal{I}$ meets block position $(1,1)$ in the entry $(d,e;f)$. Let $\mathcal{I}'$ denote the disjoint mate of $\mathcal{I}$. Without loss of generality assume that block position $(1,2)$ of $((L_1 \times N) \setminus \mathcal{I}) \cup \mathcal{I}'$ contains one occurrence of the symbol f, $n-1$ occurrences of the symbol $n+1$ and n occurrences of each of the symbols $n+2, \dots, 2n$. Then by Lemma 7, $((L_1 \times N) \setminus \mathcal{I}) \cup \mathcal{I}'$ and $L_1 \times N$ agree in $n^2 - 1$ cells of block position $(1,2)$ and similarly for block position $(2,1)$. Now assume that in block position $(2,2)$ the latin squares $((L_1 \times N) \setminus \mathcal{I}) \cup \mathcal{I}'$ and $L_1 \times N$ are disjoint in at least two cells. Then for $1 \le i < i' \le n$ and $1 \le j < j' \le n$ block position $(2,2)$ of $((L_1 \times N) \setminus \mathcal{I}) \cup \mathcal{I}'$ contains the entries $(n+i, n+j; n+1)$ and $(n+i', n+j'; n+1)$. But this implies that in block position $(1,2)$ of $((L_1 \times N) \setminus \mathcal{I}) \cup \mathcal{I}'$ symbol $n+1$ does not occur in column $n+j$ or column $n+j'$. This is a contradiction. Thus block position $(2,2)$ of $((L_1 \times N) \setminus \mathcal{I}) \cup \mathcal{I}'$ and $L_1 \times N$ agree in $n^2 - 1$ cells and $\mathcal{I}$ must contain four symbols. Hence $\mathcal{I}$ is an intercalate.

Corollary 10 *Let $(i,j;k) \in Q$ and let $I \subset N$ be a latin trade such that $I \cap Q = \{(i,j;k)\}$. Assume that for all such I, $|I| > 4$. Then for all latin trades $\mathcal{I} \subset (L_1 \times N)$ such that $(i,j;k) \in \mathcal{I}$, $|\mathcal{I} \cap (P_1 \otimes Q)| > 1$.*

Proof: Assume that there exists a latin trade $\mathcal{I}$ with $(i,j;k) \in \mathcal{I}$. Since $P_1 \otimes Q$ is UC it follows that $|\mathcal{I} \cap (P_1 \otimes Q)| \ge 1$. If $|\mathcal{I} \cap (P_1 \otimes Q)| = 1$, then $\mathcal{I}$ meets block position $(1,1)$ in precisely one entry and Lemma 9 implies that $\mathcal{I}$ is an intercalate. So $|proj_N(\mathcal{I})| = 1$ or 4. If $|proj_N(\mathcal{I})| = 1$, then $proj_N(\mathcal{I}) = \{(i,j;k)\}$ and $\mathcal{I} = \{(i,j;k), (i, n+j; n+k), (n+i, j; n+k), (n+i, n+j; k)\}$. But $(i,j;k) \in Q$ and $(i,j;k), (i, n+j; n+k), (n+i, j; n+k), (n+i, n+j; k) \in (P_1 \otimes Q)$, and we have a contradiction. Thus $|proj_N(\mathcal{I})| \ne 1$. Consequently, $|proj_N(\mathcal{I})| = 4$ and Corollary 6 implies that $proj_N(\mathcal{I})$ is an intercalate in N. But $(i,j;k) \in proj_N(\mathcal{I})$ and by assumption $|proj_N(\mathcal{I})| > 4$ which gives a contradiction.

Corollary 11 *Assume $\mathcal{Q}$ is a critical set in a latin square N, of order n. Fix $(d,e;f) \in \mathcal{Q}$ and assume that for each $(i,j;k) \in \mathcal{Q} \setminus \{(d,e;f)\}$, there exists an intercalate $I_{(i,j;k)} \subset N$ with the property that $I_{(i,j;k)} \cap \mathcal{Q} = \{(i,j;k)\}$. In addition, assume that all latin trades $I \subset N$ such that $(d,e;f) \in I$, $|I| > 4$. Then $L_1 \times N$ contains the critical set*

$$(P_1 \otimes \mathcal{Q}) \setminus \{(d,e;f)\}.$$

Corollary 12 *Let $\mathcal{Q}$ be a critical set in a latin square N of order n. For r a positive integer and $1 \leq i \leq r$, fix $(d_i, e_i; f_i) \in \mathcal{Q}$ such that the following conditions are satisfied:*

1. *for all $(u,v;w) \in \mathcal{Q} \setminus \{(d_i,e_i;f_i) \mid 1 \leq i \leq r\}$, there exists an intercalate $I_{(u,v;w)} \in N$ such that $\mathcal{Q} \cap I_{(u,v;w)} = \{(u,v;w)\}$;*

2. *for $1 \leq i < j \leq r$ there exists partial latin square $G_i \subset N$ such that $(d_i,e_i;f_i) \in G_i$, $(\cup_i G_i) \cap \mathcal{Q} = \{(d_i,e_i;f_i) \mid 1 \leq i \leq r\}$, $\mathcal{Q} \setminus \{(d_i,e_i;f_i) \mid 1 \leq i \leq r\}$ is (near) strong UC to $N \setminus (\cup_i G_i)$, and $R(G_i) \cap R(G_j) = \emptyset$, $C(G_i) \cap C(G_j) = \emptyset$;*

3. *for $1 \leq i \leq r$ and all latin trades J_i such that $\mathcal{Q} \cap J_i = \{(d_i,e_i;f_i)\}$, we have $|J_i| > 4$.*

Then $L_1 \times N$ contains the critical set

$$(P_1 \otimes \mathcal{Q}) \setminus \{(d_i,e_i;f_i) \mid 1 \leq i \leq r\}.$$

Proof: Assume that for some $\ell \in \{1,\ldots,r\}$, the entry $(d_\ell, e_\ell; f_\ell)$ is necessary for UC in $P_1 \otimes \mathcal{Q}$. If J_i is a latin trade with the property that $J_i \cap \mathcal{Q} = \{(d_i,e_i;f_i)\}$, then $|J_i| > 4$. Thus Corollary 10 implies that, for $1 \leq i \leq r$, if there exists a latin trade $\mathcal{J}_i$ in $L_1 \times N$ such that $(d_i,e_i;f_i) \in \mathcal{J}_i$, then $|\mathcal{J}_i \cap (P_1 \otimes \mathcal{Q})| > 1$. Lemma 3 implies that $\forall i$, $J_i \subset (\cup_i G_i)$ and so if $(d_\ell,e_\ell;f_\ell)$ is to be necessary for UC then there exists $j \neq \ell$ such that $(d_j,e_j;f_j) \in \mathcal{J}_\ell$. However, for all $1 \leq i < j \leq r$, $R(G_i) \cap R(G_j) = \emptyset$, and $C(G_i) \cap C(G_j) = \emptyset$, and so

$$\begin{aligned} \mathcal{J}_\ell \subseteq \; & \{(d_\ell,e_\ell;f_\ell), (x, n+y; n+z), (n+x, y; n+z), \\ & (n+x, n+y; z) \mid (x,y;z) \in G_\ell \setminus \{(d_\ell,e_\ell;f_\ell)\}\}. \end{aligned}$$

Thus we have a contradiction and so for $1 \le i \le r$ the entries $(d_i, e_i; f_i)$ are not necessary for UC. The result follows.

We provide an application of these results in Section 6. In Section 5 we generalize the result to the completable product of strong critical sets P and Q.

5 Generalization for strong critical sets

The Stinson and van Rees result has been generalized by Donovan, Gower and Khodkar, [**8**]. Their result (together with the Bedford and Whitehouse results [**5**]) essentially shows that given (near)strong critical sets $P \subset M$ and $Q \subset N$, with the property that for each pair of entries $\eta \in P$ and $\nu \in Q$ there exists isotopic latin trades $IM \subset M$ and $IN \subset N$, such that $IM \cap P = \{\eta\}$ and $IN \cap Q = \{\nu\}$, then $P \otimes Q$ is a critical set in $M \times N$. However, the conditions placed on P and Q are too restrictive for this result to be of great use. In this section we seek to weaken these conditions and identify subsets of $P \otimes Q$ which form critical sets.

Lemma 13 *Let M and N be latin squares defined on the sets $X = \{1, \dots, m\}$ and $Y = \{1, \dots, n\}$, respectively. Let P and Q be partial latin squares which are, respectively, strong UC to M and N. Fix $(a, b; c) \in P$ and $(d, e; f) \in Q$, and let $I \subset M$ and $J \subset N$ such that $P \cap I = \{(a, b; c)\}$ and $Q \cap J = \{(d, e; f)\}$ with $P \setminus \{(a, b; c)\}$ strong UC to $M \setminus I$ and $Q \setminus \{(d, e; f)\}$ strong UC to $N \setminus J$. Let $\gamma = ((a-1)n + d, (b-1)n + e; (c-1)n + f)$. If there exists a latin trade $\mathcal{I} \subset (M \times N)$ such that*

$$(P \otimes Q) \cap \mathcal{I} = \{\gamma\},$$

then, for each $i, j \in \{1, \dots m\}$, $\mathcal{I}$ intersects block position (i, j) of $M \times N$ in at most one entry.

Proof: Assume that there exists a latin trade $\mathcal{I}$ such that

$$(P \otimes Q) \cap \mathcal{I} = \{\gamma\}.$$

By Lemma 3

$$\mathcal{I} \subseteq \{((i-1)n+g,(j-1)n+h;(k-1)n+l) \mid (i,j;k) \in I \wedge (g,h;l) \in J\}.$$

Hence if $\mathcal{I}$ intersects block position (i,j), then there exists $k \in X$ such that $(i,j;k) \in I$.

Let χ be a completion of $([(M \setminus I) \cup \{(a,b;c)\}] \otimes [(N \setminus J) \cup \{(d,e;f)\}]) \setminus \{\gamma\}$ distinct from $M \times N$. Denote the array of alternatives for $([(M \setminus I) \cup \{(a,b;c)\}] \otimes [(N \setminus J) \cup \{(d,e;f)\}]) \setminus \{\gamma\}$ by

$$\mathcal{A}_\chi = A_{([(M\setminus I)\cup\{(a,b;c)\}]\otimes[(N\setminus J)\cup\{(d,e;f)\}])\setminus\{\gamma\}}$$

and let $\mathcal{A}_\chi(g,h)$ denote the set of symbols in cell (g,h) of $\mathcal{A}_\chi$.

The partial latin square P is strong UC to M, and since $P \subseteq (M \setminus I) \cup \{(a,b;c)\}$ it follows that $(M \setminus I) \cup \{(a,b;c)\}$ is strong UC to M. Consider the array of alternative $A_{(M\setminus I)\cup\{(a,b;c)\}}$. It may be assumed that without loss of generality that there exists a cell (i,j) such that $A_{(M\setminus I)\cup\{(a,b;c)\}}(i,j) = \{q\}$ for some $q \in X$. If not consider the appropriate conjugate of $(M \setminus I) \cup \{(a,b;c)\}$. If $A_{(M\setminus I)}(i,j) = \{q\}$ then by Lemma 2 block position (i,j) of χ must contain symbols chosen from the set Y^q and so χ and $M \times N$ agree in block position (i,j). Otherwise $\{q,q'\} \subseteq A_{(M\setminus I)}(i,j)$ where $q' \in X \setminus \{q\}$. The partial latin squares $M \setminus I$ and $(M \setminus I) \cup \{(a,b;c)\}$ differ in the occurrence of symbol c in cell (a,b). Hence the only difference in the array of alternatives $A_{(M\setminus I)\cup\{(a,b;c)\}}$ and $A_{(M\setminus I)}$ is the addition of symbol c to some cells of $A_{(M\setminus I)}$. Thus $q' = c$ and $i = a$ or $j = b$. Without loss of generality assume $j = b$. Then $\forall y \in Y$ and $\forall z \in Y \setminus \{e\}$ if $h \in \mathcal{A}_\chi((i-1)n+y,(b-1)n+z)$ (that is, in the array of alternatives, symbol h occurs in any column corresponding to block position (i,b) except $(b-1)n+e$), then $h \in Y^q$. In addition, $\forall y \in Y$ if $h \in \mathcal{A}_\chi((i-1)n+y,(b-1)n+e)$ (that is, in the array of alternatives, symbol h occurs in column $(b-1)n+e$ of block position (i,b)), then $h \in Y^q \cup \{(c-1)n+f\}$. But note that symbol $(c-1)n+f$ can be placed in at most one cell in column $(b-1)n+e$ of block position (i,b) and all remaining cells in column $(b-1)n+e$ of block position (i,b) must be chosen from the set Y^q. Thus Lemma 7 may be used to prove that block position (i,j) of χ and of $M \times N$ intersect in at least n^2-1 entries.

Since P is a strong UC set, it may be assumed without loss of generality, that there exists a cell (i', j') such that

$$A_{(M\setminus I)\cup\{(a,b;c),(i,j;q)\}}(i', j') = \{p\}$$

for some $p \in X$ and we repeat the above argument and prove that block position (i', j') of χ and of $M \times N$ intersect in at least $n^2 - 1$ entries. This argument is repeated until all block positions have been dealt with.

Corollary 14 *Let I, J, P, Q, M and N be as in Lemma 13. Let $\mathcal{I} \subset (M \times N)$ be a latin trade such that*

$$(P \otimes Q) \cap \mathcal{I} = \{\gamma\},$$

and let θ denote an isomorphism from $P \otimes Q$ to $Q \otimes P$, such that $\theta((u-1)n + x, (v-1)n + y; (w-1)n + z)) = ((x-1)m + u, (y-1)m + v; (z-1)m + w)$. Let $\delta = \theta(\gamma)$. Then there exists a latin trade $\mathcal{J}$ such that $\theta(\mathcal{I}) = \mathcal{J}$ and $(Q \otimes P) \cap \mathcal{J} = \{\delta\}$. Furthermore, for each $i, j \in \{1, \ldots n\}$, $\mathcal{J}$ intersects block position (i, j) of $(N \times M)$ in at most one cell.

Proof: It is easy to see that

$$(P \otimes Q) \cap \mathcal{I} = \{\gamma\} \quad \Rightarrow \quad (Q \otimes P) \cap \mathcal{J} = \{\delta\},$$

where $\mathcal{J}$ is the appropriate isomorphic image of $\mathcal{I}$. If $\mathcal{J}$ intersects block position (g, h) then there exists an $l \in Y$ such that $(g, h; l) \in J$. Using the fact that $(N \setminus J) \cup \{(d, e; f)\}$ is strong UC to N we may proceed as in the proof of Lemma 13 to verify that $\mathcal{J}$ intersects each of the block positions (g, h), where $(g, h; l) \in J$ for some $l \in Y$, in at most one cell.

It is now possible to show that if I and J are minimal latin trades, then the latin trades I, J and $\mathcal{I}$ all have the same shape.

Corollary 15 *Let I, J, P, Q, M and N be as in Lemma 13 with I and J minimal latin trades. If there exists a latin trade $\mathcal{I} \subset (M \times N)$ such that*

$$(P \otimes Q) \cap \mathcal{I} = \{\gamma\},$$

then up to isotopism the latin trades $\mathcal{I}$, I and J all take the same shape.

Proof: Lemma 3 indicates that $\mathcal{I} \subseteq \{((u-1)n+x, (v-1)n+y; (w-1)n+z) \mid (u,v,w) \in I \wedge (x,y;z) \in J\}$. Thus $proj_M(\mathcal{I}) \subseteq I$ and $proj_N(\mathcal{I}) \subseteq J$. Lemma 5 and Corollary 6 indicate that $proj_M(\mathcal{I})$ and $proj_N(\mathcal{I})$ are latin trades which, up to isotopism, take the same shape as $\mathcal{I}$. But I and J are minimal, therefore $I = proj_M(\mathcal{I})$ and $J = proj_N(\mathcal{I})$. The result now follows.

For completeness we prove the following proposition.

Proposition 16 *Let M and N be latin squares defined on the sets $X = \{1,\ldots,m\}$ and $Y = \{1,\ldots,n\}$, respectively. Let P and Q be partial latin squares which, respectively, are (near) strong UC to M and N. Fix $(a,b;c) \in P$ and $(d,e;f) \in Q$, and let $I \subset M$ and $J \subset N$ be latin trades such that $P \cap I = \{(a,b;c)\}$ and $Q \cap J = \{(d,e;f)\}$. If S_I and S_J, then there exists a latin trade $\mathcal{I}$ in $M \times N$ with $S_{\mathcal{I}} = S_I = S_J$ such that $(P \otimes Q) \cap \mathcal{I} = \{\gamma\}$, where $\gamma = ((a-1)n+d, (b-1)n+e; (c-1)n+f)$. Hence, γ is necessary in $P \otimes Q$ for the UC to $M \times N$.*

Proof: Define

$$\begin{aligned}\mathcal{I} &= \{((i-1)n+i, (j-1)n+j; (k-1)n+\ell) \mid \\ &\quad (i,j;k) \in I \text{ and } (i,j;\ell) \in J\}.\end{aligned}$$

(Note that since I and J have the same shape $\mathcal{I}$ is well-defined.) Then $\mathcal{I}$ is a latin trade in $M \times N$ with disjoint mate

$$\begin{aligned}\mathcal{I}' &= \{((i-1)n+i, (j-1)n+j; (k'-1)n+\ell') \mid \\ &\quad (i,j;k') \in I' \text{ and } (i,j;\ell') \in J'\},\end{aligned}$$

where I' and J' are disjoint mates of I and J, respectively. Obviously, $\mathcal{I}$, I and J have the same shape. Moreover, $(P \otimes Q) \cap \mathcal{I} = \{\gamma\}$.

Lemma 17 *Let P and Q be strong critical sets in the latin squares M and N respectively. Let $P\overline{Q}$ and $\overline{P}Q$ be as in the definition of the completable product. For r a positive integer, let $(a,b;c) \in P$ and $\{(d_i,e_i;f_i) \mid 1 \le i \le r\} \subseteq Q$ such that the following conditions are satisfied:*

1. *there exists a minimal latin trade $I \subset M$ such that $P \cap I = \{(a,b;c)\}$, and $P \setminus \{(a,b;c)\}$ is strong UC to $M \setminus I$;*

2. *for $1 \le i < j \le r$, there exists minimal latin trades J_i such that $Q \cap J_i = \{(d_i, e_i; f_i)\}$, $R(J_i) \cap R(J_j) = \emptyset$, $C(J_i) \cap C(J_j) = \emptyset$, and $Q \setminus \{(d_i, e_i; f_i) \mid 1 \le i \le r\}$ is strong UC to $N \setminus (\cup_i J_i)$;*

3. *the sets S_I and S_{J_i}, $1 \le i \le r$, are not isotopic.*

Then there exists a critical set C such that

$$(P\overline{Q} \cup \overline{P}Q) \subseteq C \subseteq (P \otimes Q) \setminus \{((a-1)n + d_i, (b-1)n + e_i; (c-1)n + f_i) \mid 1 \le i \le r\}.$$

Proof: Lemma 2 indicates that $P \otimes Q$ is UC to $M \times N$ and hence contains a critical set. Lemma 4 implies that the entries of $(P\overline{Q} \cup \overline{P}Q)$ are necessary for UC and therefore $(P\overline{Q} \cup \overline{P}Q)$ is a subset of a critical set. For $1 \le i \le r$, assume that there exists latin trades $\mathcal{J}_i$ such that $\mathcal{J}_i \cap (P \otimes Q) = \{((a-1)n + d_i, (b-1)n + e_i; (c-1)n + f_i)\}$. Since for all $1 \le i < j \le r$, $R(J_i) \cap R(J_j) = \emptyset$, and $C(J_i) \cap C(J_j) = \emptyset$, Lemma 3 implies that

$$\mathcal{J}_i \subseteq \{((u-1)n + x, (v-1)n + y; (w-1)n + z) \mid (u, v; w) \in I \wedge (x, y; z) \in J_i\},$$

and so $\mathcal{J}_i \cap \mathcal{J}_j = \emptyset$. In addition, each such latin trade meets each block position of $M \times N$ in at most one entry. For i, $1 \le i \le r$, we know I and J_i are minimal, so Lemma 15 implies that up to isotopism the latin trades I and $\mathcal{J}_i$ take the same shape. But this is a contradiction. Hence no such latin trades exist and the entries $((a-1)n + d_i, (b-1)n + e_i, (c-1)n + f_i)$, $1 \le i \le r$, are not necessary for UC. The result now follows.

6 Applications

Let P_1, E_2, L_1 and L_2 be, respectively, the partial latin squares and the latin squares given in Table 1. For $n \ge 2$, let $L_n = L_1 \times L_{n-1}$ and $P_n = P_1 \otimes P_{n-1}$. By Lemma 8 P_n is a critical set in L_n, with $|P_n| = 4^n - 3^n$.

Lemma 18 *The set $(P_1 \otimes E_2) \setminus \{(1, 2; 2)\}$ is a critical set of order 8.*

Proof: Lemma 2 implies that $P_1 \otimes E_2$ has a UC to $L_1 \times L_2$. For each of the entries $(i,j;k) \in E_2 \setminus \{(1,2;2)\}$ there exists an intercalate I such that $I \cap E_2 = \{(i,j;k)\}$. Using Table 2 it can be seen that if $I \subset E_2$ is a latin trade such that $I \cap E_2 = \{(1,2;2)\}$, then $|I| = 12$. Thus Corollary 10 can be applied and the result follows.

Lemma 19 *For all $n \geq 3$ define $\mathcal{E}_n$ to be*

$$\begin{aligned}&(P_n \setminus \{(2^n-2, 2^n-2; 1), (2^n-1, 2^n-1; 1), (2^n-1, 2^n-3; 3),\\ &(2+4r, 1+4r; 2), (1+4r, 3+4r; 3), (3+8s, 2+8s; 4) \mid\\ &0 \leq t \leq 2^{n-2}-1, 0 \leq s \leq 2^{n-3}-1\}) \cup\\ &\{(2^n-2, 2^n; 3), (2^n-1, 2^n-2; 4), (2^n, 2^n-1; 2)\}.\end{aligned}$$

Then $(P_1 \otimes \mathcal{E}_n) \setminus \{(2^n-3, 2^n-2; 2)\}$, is a critical set of order 2^{n+1} and size $4^{n+1} - 3^{n+1} - 3.2^{n-1} - 3.2^{n-3} - 1$.

Proof: In the Appendix we show that $\mathcal{E}_n$ is a critical set, for all $n \geq 3$. In addition, it is shown that for each entry $(i,j;k) \in \mathcal{E}_n \setminus \{(2^n-3, 2^n-2; 2)\}$ there exists an intercalate I such that $I \cap \mathcal{E}_n = \{(i,j;k)\}$. If $I \subset L_n$ is a latin trade such that $I \cap \mathcal{E}_n = \{(2^n-3, 2^n-2; 2)\}$, then $|I| > 4$. Hence by Corollary 11, $(2^n-3, 2^n-2; 2)$ is not necessary for unique completion and so $(P_1 \otimes \mathcal{E}_n) \setminus \{(2^n-3, 2^n-2; 2)\}$ is a critical set. It is shown in the Appendix that $|\mathcal{E}_n| = 4^n - 3^n - 2^{n-1} - 2^{n-3}$. Hence $|(P_1 \otimes \mathcal{E}_n) \setminus \{(2^n-3, 2^n-2; 2)\}| = 2^{2n} - 1 + 3(4^n - 3^n - 2^{n-1} - 2^{n-3}) = 4^{n+1} + 3^{n+1} - 3.2^{n-1} - 3.2^{n-3} - 1$ as required.

To present this construction we start with the back circulant latin square of even order. For ease of exposition the symbols occurring in this latin square will be chosen from the set $Y = \{0, 1 \ldots, n-1\}$ and the latin square will take the from $B_n = \{(i, j; i + j (\text{mod } n)) \mid 0 \leq i, j \leq n-1\}$. In 1978, [7], Curran and van Rees showed that for even n the back circulant latin square B_n contains the critical set C_n of order n and size $n^2/4$, where

$$\begin{aligned} C_n &= \{(i, j; i + j (\text{mod } n) \mid 0 \leq i \leq n/2 - 1,\\ &\quad 0 \leq j \leq n/2 - 1 - i\} \cup\\ &\quad \{(i, j; i + j (\text{mod } n) \mid n/2 + 1 \leq i \leq n - 1,\\ &\quad 3n/2 - i \leq j \leq n - 1\}.\end{aligned}$$

In addition, they showed that for each entry $(i,j;k) \in C_n$ there exists an intercalate $I \subset B_n$ such that $I \cap C_n = \{(i,j;k)\}$. Later Donovan and Howse, [**9**], adapted C_n to construct a critical set F_n, of size $n^2/4+2$. They showed that, for even n, the set

$$F_n = (C_n \setminus \{(0,0;0)\}) \cup \{(0,n/2;0),(n/2,0;0),(n/2,n/2;n/2)\}$$

is a critical set in

$$\begin{aligned} A_n &= (B_n \setminus \{(0,0;0),(0,n/2;n/2),(n/2,0;n/2),(n/2,n/2;0)\}) \\ &\quad \cup\{(0,0;n/2),(0,n/2;0),(n/2,0;0),(n/2,n/2;n/2)\}. \end{aligned}$$

If $n \equiv 0 (\text{mod } 4)$, then for each entry $\nu \in F_n$ there exists an intercalate $I \subset A_n$ such that I intersects F_n in ν alone. However, if $n \equiv 2 (\text{mod } 4)$, this is only true for entries in $F_n \setminus \{(0,n/2;0),(n/2,0;0),(n/2,n/2;n/2)\}$. It will be shown that in the completable product $P_1 \otimes F_n$ the entries $\{(0,n/2;0),(n/2,0;0)\}$ are not necessary for UC. The result will be a critical set of order $2n$ and size $n^2+3(\frac{n^2}{4}+2)-2 = \frac{7n^2}{4}+4$.

Lemma 20 *For all $n \equiv 2 (\text{mod } 4)$, $n > 2$, the partial latin square $(P_1 \otimes F_n) \setminus \{(0,n/2;0),(n/2,0;0)\}$ is a critical set of order $2n$ and size $\frac{7n^2}{4}+4$.*

Proof: The partial latin square $P_1 \otimes F_n$ is a UC set of order $2n$ and size $n^2+3(\frac{n^2}{4}+2) = \frac{7n^2}{4}+6$. The proof of Lemma 8 implies that each of the entries $(P_1 \otimes F_n) \setminus \{(0,n/2;0),\ (n/2,0;0),(n/2,n/2;n/2)\}$ is necessary for UC. Lemma 23 in the Appendix indicates that $(n/2,n/2;n/2)$ is necessary for unique completion. Let

$$\begin{aligned} G &= \{(i,j;i+j(\text{mod } n)) \mid 0 \le i \le n/2-1, n/2 \le j \le n-1, \\ &\qquad n/2+1 \le i+j \le n\}, \text{ and} \\ H &= \{(i,j;i+j(\text{mod } n)) \mid n/2 \le i \le n-1, 0 \le j \le n/2-1, \\ &\qquad n/2+1 \le i+j \le n\}. \end{aligned}$$

In the Appendix, Lemma 22, verifies that $F_n \setminus \{(0,n/2;0),(n/2,0;0)\}$ is UC to the partial latin square $A_n \setminus (G \cup H)$. Note that $R(G) \cap$

$R(H) = \emptyset$ and $C(G) \cap C(H) = \emptyset$. Using Lemma 3 it can be seen that $(P_1 \otimes F_n) \setminus \{(0, n/2; 0), (n/2, 0; 0)\}$ is UC to the partial latin square

$$\begin{aligned}(L_1 \times A_n) \quad \setminus \quad & \{(0, n/2; 0), (n/2, 0; 0), (i, j+n; k+n), \\ & (i+n, j; k+n), (i+n, j+n; k) \mid (i, j; k) \in \\ & (G \cup H) \setminus \{(0, n/2; 0), (n/2, 0; 0)\}\}.\end{aligned}$$

Hence if there exists latin trades $\mathcal{I} \subset (L_1 \times A_n)$ or $\mathcal{J} \subset (L_1 \times A_n)$ such that $\mathcal{I} \cap (P_1 \otimes F_n) = \{(0, n/2; 0)\}$ and $\mathcal{J} \cap (P_1 \otimes F_n) = \{(n/2, 0; 0)\}$ then

$$\begin{aligned}\mathcal{I} & \subseteq \{(0, n/2; 0), (i, j+n; k+n), (i+n, j; k+n), (i+n, j+n; k) \\ & \quad \mid (i, j; k) \in G \setminus \{(0, n/2; 0)\}\}, \\ \mathcal{J} & \subseteq \{(n/2, 0; 0), (i, j+n; k+n), (i+n, j; k+n), (i+n, j+n; k) \\ & \quad \mid (i, j; k) \in H \setminus \{(n/2, 0; 0)\}\}.\end{aligned}$$

Since $\mathcal{I}$ meets block position $(1, 1)$ in precisely one entry and $R(\mathcal{I}) \cap R(\mathcal{J}) = \emptyset$ and $C(\mathcal{I}) \cap C(\mathcal{J}) = \emptyset$ it follows that $proj_{A_n}(\mathcal{I})$ is an intercalate. But $proj_{A_n}(\mathcal{I})$ must be a latin trade contained in G. However, A_n agrees with B_n in all cells except $(0,0), (0, n/2), (n/2, 0)$, $(n/2, n/2)$ and since $n \equiv 2 (\mathrm{mod}\ 4)$, G does not contain an intercalate. Thus we have a contradiction and no such latin trade $\mathcal{I}$ exists. Similarly it can be shown that there is no latin trade $\mathcal{J}$ in $L_1 \times A_n$. The result is now immediate.

7 Appendix

Define $P_n = P_1 \otimes P_{n-1}$, $n \geq 2$. Starting with the partial latin square P_3 of size $4^3 - 3^3 = 37$, we may obtain a distinct critical set $E_3 \subset L_3$ of size $4^3 - 3^3 - 2.2 - 1 - 3 + 3 = 32$ by deleting the entries $(6, 6; 1)$, $(7, 7; 1)$, $(7, 5; 3)$, $(1 + 4t, 3 + 4t; 3)$, $(2 + 4t, 1 + 4t; 2)$, $(3 + 8s, 2 + 8s; 4)$, $t = 0, 1$ and $s = 0$ and inserting entries $(6, 8; 3), (7, 6; 4), (8, 7; 2)$. This critical set is given in Table 4 and the example will be generalized in Lemma 21.

Lemma 21 *For all $n \geq 3$ define $\mathcal{E}_n$ to be*

$$(P_n \setminus \{(2^n - 2, 2^n - 2; 1), (2^n - 1, 2^n - 1; 1), (2^n - 1, 2^n - 3; 3),$$

1	2		4	5	6	7	
	1	4	3	6	5		
3		1	2	7		5	
4	3	2	1				
5	6	7		1	2		
6	5						3
7		5			4		
						2	

Table 4: The critical set $\mathcal{E}_3$

$$(2+4t, 1+4t; 2), (1+4t, 3+4t; 3), (3+8s, 2+8s; 4) \mid$$
$$0 \le t \le 2^{n-2}-1, 0 \le s \le 2^{n-3}-1\}) \cup$$
$$\{(2^n-2, 2^n; 3), (2^n-1, 2^n-2; 4), (2^n, 2^n-1; 2)\}.$$

Then $\mathcal{E}_n$ is a critical set in L_n of size $4^n - 3^n - 2^{n-1} - 2^{n-3}$. In addition, for all entries $(i,j;k) \in \mathcal{E}_n \setminus \{(2^n-3, 2^n-2; 2)\}$ there exists an intercalate $I \subset L_n$ such that $I \cap \mathcal{E}_n = \{(i,j;k)\}$.

Proof: Note $\mathcal{E}_n \subset L_n$. Divide L_n into four block positions $(1,1)$, $(1,2), (2,1), (2,2)$ and correspondingly divide $\mathcal{E}_n$ into four block positions.

The partial latin square $\mathcal{E}_n$ was obtained from P_n by deleting symbols $1,2,3,4$ from certain cells and inserting the symbols $2,3,4$ in three cells in the intersection of the last three rows and three columns.

Let N represent a latin square of order 2^n such that $\mathcal{E}_n \subseteq N$. We proceed with the following steps to show that $P_n \subset N$.

1. For all columns $c \neq 2^n-3$ where $c \equiv 1 (\text{mod } 4)$, if cell (r,c) is empty in $\mathcal{E}_n$, then $r = c+1$ or $r \equiv 0 (\text{mod } 4)$. For $r \equiv 0 (\text{mod } 4)$, symbol 2 occurs in row r of $\mathcal{E}_n$. Hence for all $0 \le t \le 2^{n-2}-2$, $(2+4t, 1+4t; 2) \in N$. Given this and $(2^n-2, 2^n; 3) \in \mathcal{E}_n$, $(2^n-1, 2^n; 2), (2^n-2, 2^n-3; 2) \in N$.

2. For all rows $r \neq 2^n-3$ where $r \equiv 1 (\text{mod } 4)$, if cell (r,c) is empty in $\mathcal{E}_n$, then $c = r+2$ or $c \equiv 0 (\text{mod } 4)$. For $c \equiv$

$0 \pmod 4$, symbol 3 occurs in column c of $\mathcal{E}_n$. Hence for all $0 \le t \le 2^{n-2} - 2$, we have $(1 + 4t, 3 + 4t; 3) \in N$, implying $(2^n - 3, 2^n - 1; 3), (2^n, 2^n - 2; 3), (2^n - 1, 2^n - 3; 3) \in N$.

3. For all $i = 1, \ldots, 2^n - 3$, $(i, i; 1), (2^n - 3, 2^n - 2; 2), (2^n - 2, 2^n; 3), (2^n - 1, 2^n - 2; 4) \in \mathcal{E}_n$. Hence $(2^n - 2, 2^n - 2; 1), (2^n - 3, 2^n - 3; 1), (2^n, 2^n; 1) \in N$.

4. Now consider row $2^n - 3$. If $(2^n - 3, c)$ is empty, then $c \equiv 0 \pmod 4$. For $c \equiv 0 \pmod 4$, $c \ne 2^n$, symbol 4 occurs in column c. Thus $(2^n - 3, 2^n; 4) \in N$ and similarly $(2^n, 2^n - 3; 4) \in N$. For column $2^n - 1$, if cell $(r, 2^n - 1)$ is empty, then $r \equiv 0, 2 \pmod 4$. For $r \equiv 0 \pmod 4$ symbol 4 occurs in row r. For $r \equiv 2 \pmod 4$ and $r \ne 2^n - 2$, symbol 4 occurs in row r. Thus $(2^n - 2, 2^n - 1; 4) \in N$. For all rows r, $r \equiv 3 \pmod 8$, if cell (r, c) is empty, then $c = r - 1$ or $c \equiv 0, 6 \pmod 8$. For every column c of $\mathcal{E}_n$, where $c \equiv 0, 6 \pmod 8$, symbol 4 occurs in column c and so, for all $0 \le s \le 2^{n-3} - 1$, $(3+8s, 2+8s; 4) \in N$.

We have shown that $P_n \subset N$ and since P_n has a UC to L_n, $\mathcal{E}_n$ is a UC to L_n.

To complete the proof we must show that for each entry $(i, j; k) \in \mathcal{E}_n$ there exists a latin trade $I \subset L_n$ such that $I \cap \mathcal{E}_n = \{(i, j; k)\}$. If $k \in \{5, \ldots, 2^n\}$ and $(i, j; k) \in L_n$, for some i, j, then $(i, j; k) \in \mathcal{E}_n$ if and only if $(i, j; k) \in P_n$. Lemma 8 implies that for each entry $(i, j; k) \in P_n$, $k \in \{5, \ldots, 2^n\}$, there exists an intercalate $I = \{(i, j; k), (i, j'; k'), (i', j; k'), (i', j'; k)\}$ with $k' \in \{5, \ldots, 2^n\}$ such that $I \cap P_n = \{(i, j; k)\}$. Thus for all such $(i, j; k) \in \mathcal{E}_n$, there exists an intercalate I, such that $I \cap \mathcal{E}_n = \{(i, j; k)\}$. Similarly we can see that for $1 \le i \le 2^n$, there exists an intercalate $I \subset L_n$ such that $(i, i; 1) \in I$ and $I \cap \mathcal{E}_n = \{(i, i; 1)\}$.

It is easy to see that there exist required intercalates for the entries $(2^n - 2, 2^n; 3)$, $(2^n - 1, 2^n - 2; 4)$ and $(2^n, 2^n - 1; 2)$. The necessary intercalates for the remaining entries in $\mathcal{E}_n \setminus \{(2^n - 3, 2^n - 2; 2)\}$ are given in Table 5 and so for all entries $(i, j; k) \in \mathcal{E}_n \setminus \{(2^n - 3, 2^n - 2; 2)\}$ there exists an intercalate I such that $I \cap \mathcal{E}_n = \{(i, j; k)\}$.

Finally, if I is a latin trade such that $I \cap \mathcal{E}_n = \{(2^n - 3, 2^n - 2; 2)\}$ then

$$I = \{(2^n - 3, 2^n - 2; 2), (2^n - 3, 2^n - 1; 3), (2^n - 3, 2^n; 4),$$

$(i,j;k) \in \mathcal{E}_n$, $0 \le t \le 2^{n-2}-2$, $0 \le s \le 2^{n-3}-2$	Entries which together with $(i,j;k)$ form an intercalate I in L_n such that $\mathcal{E}_n \cap I = \{(i,j;k)\}$
$(1+4t, 2+4t; 2)$	$(1+4t, 2^n; 2^n-4t), (2^n-1, 2+4t; 2^n-4t),$ $(2^n-1, 2^n; 2)$
$(3+4t, 4+4t; 2)$	$(3+4t, 2^n; 2^n-2-4t), (2^n-1, 2^n; 2),$ $(2^n-1, 4+4t; 2^n-2-4t)$
$(4+4t, 3+4t; 2)$	$(4+4t, 2^n-3; 2^n-4t), (2^n-2, 2^n-3; 2),$ $(2^n-2, 3+4t; 2^n-4t)$
$(2+4t, 4+4t; 3)$	$(2+4t, 2^n-1; 2^n-4t), (2^n-3, 2^n-1; 3),$ $(2^n-3, 4+4t; 2^n-4t)$
$(3+4t, 1+4t; 3)$	$(3+4t, 2^n-2; 2^n-4t), 2^n, 1+4t; 2^n-4t),$ $(2^n, 2^n-2; 3)$
$(4+4t, 2+4t; 3)$	$(4+4t, 2^n-3; 2^n-4t), (2^n-1, 2^n-3; 3),$ $(2^n-1, 2+4t; 2^n-4t)$
$(1+4t, 4+4t; 4)$	$(1+4t, 2^n; 2^n-4t), (2^n-3, 4+4t; 2^n-4t),$ $(2^n-3, 2^n; 4)$
$(2+4t, 3+4t; 4)$	$(2+4t, 2^n-1; 2^n-4t), (2^n-2, 2^n-1; 4),$ $(2^n-2, 3+4t; 2^n-4t)$
$(4+4t, 1+4t; 4)$	$(4+4t, 2^n-3; 2^n-4t), (2^n, 1+4t; 2^n-4t),$ $(2^n, 2^n-3; 4)$
$(7+8s, 6+8s; 4)$	$(7+8s, 2^n-6; 2^n-8s), (2^n-5, 2^n-6; 4),$ $(2^n-5, 6+8s; 2^n-8s)$

Table 5: Necessary intercalates

$(2^n-2, 2^n-3; 2), (2^n-2, 2^n-2; 1), (2^n-2, 2^n-1; 4),$
$(2^n-1, 2^n-3; 3), (2^n-1, 2^n-1; 1), (2^n-1, 2^n; 2),$
$(2^n, 2^n-3; 4), (2^n, 2^n-2; 3), (2^n, 2^n; 1)\ \}.$

Note that $|\mathcal{E}_n| = 4^n - 3^n - 2.2^{n-2} - 2^{n-3} - 3 + 3 = 4^n - 3^n - 2^{n-1} - 2^{n-3}$.

The last two results are required for the proof of Lemma 20.

Lemma 22 *Let $n \equiv 2 (mod\ 4)$ and let F_n and A_n be partial latin squares as defined in Section 6. If there exists latin trades $I, J \subset A_n$ such that $I \cap F_n = \{(0, n/2; 0)\}$ and $J \cap F_n = \{(n/2, 0; 0)\}$ then*

$$I \subseteq \{(i, j; i + j (mod\ n)) \mid 0 \le i \le n/2 - 1, n/2 \le j \le n - 1,\\ n/2 + 1 \le i + j \le n\},\ and$$
$$J \subseteq \{(i, j; i + j (mod\ n)) \mid n/2 \le i \le n - 1, 0 \le j \le n/2 - 1,\\ n/2 + 1 \le i + j \le n\}.$$

In addition, $|I| > 4$ and $|J| > 4$, $R(I) \cap R(J) = \emptyset$ and $C(I) \cap C(J) = \emptyset$.

Proof: It will be shown that if there exists a partial latin square A of order n, such that $(F_n \setminus \{(0, n/2; 0), (n/2, 0; 0)\}) \subseteq A$ then

$$A \subseteq A_n \setminus (\{(i, j; i + j (\text{mod } n)) \mid 0 \le i \le n/2 - 1,\\ n/2 \le j \le n - 1, n/2 + 1 \le i + j \le n\} \cup\\ \{(i, j; i + j (\text{mod } n)) \mid n/2 \le i \le n - 1,\\ 0 \le j \le n/2 - 1, n/2 + 1 \le i + j \le n\}).$$

Assume $(F_n \setminus \{(0, n/2; 0), (n/2, 0; 0)\}) \subseteq A$ then

- for $k = n/2 - 1, \dots, 1$, for $i = k+1, \dots, n-1$, $(i, n+k-i; k) \in A$;
- for $k = n/2$, $(0, 0; n/2) \in A$, and for $i = 1, \dots, n/2 - 1$, $(i, k - i; k) \in A$;
- for $k = n/2 + 1, \dots, n - 1$, for $i = 0, \dots, k - n/2$, if $(i, j; k) \in A$ then $j \ge n/2$ and so for $i' = k - n/2 + 1, \dots, n/2 - 1$, $(i', k - i'; k) \in A$.

We have shown that if there exists a latin trade $I \subset A_n$ such that $I \cap F_n = \{(0, n/2; 0)\}$ then I is bounded by rows 0 and $n/2 - 1$, and columns $n/2$ and $n - 1$. In addition, $(i, j) \in S_I$ if and only if $n/2 + 1 \leq i + j \leq n$. Similarly, if there exists a latin trade $J \subset A_n$ such that $J \cap F_n = \{(n/2, 0; 0)\}$ then J is bounded by columns 0 and $n/2 - 1$, and rows $n/2$ and $n - 1$. Further, $(i, j) \in S_J$ if and only if $n/2 + 1 \leq i + j \leq n$. Using the structure of B_n it can be shown that if $n \equiv 2 \pmod 4$, then $|I| > 4, |J| > 4$. Thus $R(I) \cap R(J) = \emptyset$ and $C(I) \cap C(J) = \emptyset$.

Lemma 23 *Let P_1, L_1, F_n and A_n be partial latin squares as defined in Section 6. Then there exists a latin trade $\mathcal{I} \subset (L_1 \times A_n)$ such that $\mathcal{I} \cap (P_1 \otimes F_n) = \{(n/2, n/2; n/2)\}$.*

Proof: To verify this we use the existence of latin trades in B_n as documented in **[6]**. That is, for all positive even integers n there exists a latin trade denoted $H_{n/2-4,2}$ in $B_{n/2-2}$ with the property that $H_{n/2-4,2}$ is contained in the subarray defined by the intersection of rows 0 to $n/2-4$ with columns 0 to 2 and contains the entries $(0,0;0), (0,2;2), (n/2-4, 0; n/2-4), (n/2-4, 2; 0)$. Let $H'_{n/2-4,2}$ denote the disjoint mate of $H_{n/2-4,2}$. The required latin interchange $\mathcal{I}$ and disjoint mate $\mathcal{I}'$ are given below:

$$\begin{aligned}
\mathcal{I} = {} & \{(n/2, 0; 0), (n/2, n/2; n/2), (n/2, n+1; 3n/2+1), \\
& (n/2, 3n/2-1; 2n-1)\} \cup \{(3n/2-1, n/2; 2n-1), \\
& (3n/2-1, n+1; n/2)\} \cup \{(3n/2+1, 0; 3n/2+1), \\
& (3n/2+1, n/2-2; 2n-1), (3n/2+1, 3n/2-1; 0)\} \cup \\
& \{(j+3n/2-1, i+2; k+3n/2+1) \mid \\
& (i, j; k) \in H_{n/2-4,2} \setminus \{(n/2-4, 2; 0)\}\} \\
I' = {} & \{(n/2, 0; 3n/2+1), (n/2, n/2; 2n-1), (n/2, n+1; n/2), \\
& (n/2, 3n/2-1; 0)\} \cup \{(3n/2-1, n/2-2; 2n-1), \\
& (3n/2-1, n/2; n/2), (3n/2-1, n+1; 3n/2+1)\} \cup \\
& \{(3n/2+1, 0; 0), (3n/2+1, 3n/2-1; 2n-1)\} \cup \\
& \{(j+3n/2-1, i+2; k+3n/2+1) \mid \\
& (i, j; k) \in H'_{n/2-4,2} \setminus \{(n/2-4, 0; 0)\}\}.
\end{aligned}$$

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