Handicap Achievement for Squares

HEIKO HARBORTH AND MARKUS SEEMANN
Technische Universität Braunschweig
D–38023 Braunschweig, Germany
e-mail: h.harborth@tu-bs.de, m.seemann@tu-bs.de

Abstract: Given a polyomino P with n cells, two players A and B alternately color the cells of the square tessellation of the plane. In the case of A-achievement, player A tries to achieve a copy of P in his color and player B tries to prevent A from achieving a copy of P. The handicap number $h(P)$ denotes the minimum number of cells such that a winning strategy exists for player A. For all polyominoes that form a square of $n = s^2$ square cells the handicap number will be determined to be $s^2 - 1$.
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1 Introduction

In an A-achievement game two players A and B alternately color the squares of the Euclidean tessellation of the plane. For a given polyomino P with n squares player A wins if he achieves a copy of P in his color (green) and player B (by red squares) tries to prevent A from winning (for references see [1]).

In [1] the *handicap number* $h(P)$ is introduced as the minimum number of squares that A has to color before he starts to achieve the given polyomino P by an A-achievement game with a winning strategy for A. Polyominoes with $h(P) = 0$ are known as winners. There are 11 winners, one polyomino, called *Snaky*, is undecided, and for all others a winning strategy for A does not exist. In general, it is trivial that $h(P) \leq n - 1$.

Besides exact values of $h(P)$ for some small polyominoes, in [1] it is proved $h(P) = n - 2$ if P consists of n consecutive squares in a line and $h(P) = n - 1$ if P is a generalized "plus sign" with $n = 4s + 1$ squares. For snaky only $h \leq 2$ is known. Here we will prove mainly $h(P) = n - 1$ for $P = P_{s^2}$ being a square of $n = s^2$ square cells.

2 Polyominoes of square shape

The trivial upper bound $h(P) \leq n - 1$ is attained if P is an $s \times s$-square P_{s^2}.

Theorem 1. $h(P_{s^2}) = s^2 - 1$ for $s \neq 2$.

As mentioned above $h(P_{s^2}) \leq s^2 - 1$ is fulfi ed. The proof of $h(P_{s^2}) \geq s^2 - 1$ for $s \geq 3$ is partitioned into two cases when s is odd and even. For $s = 1$ it is trivial and $h(P_{2^2}) = 2$ is proved in [1].

2.1 Proof for odd s

For odd s every P_{s^2} has a central cell. To obtain $h(P_{s^2}) \geq s^2 - 1$ we prove that B is able to exclude every cell of the plane as a central cell of a green P_{s^2} if player A has colored any set of $s^2 - 2$ cells in green before his first move.

We need the following three lemmas. We frequently will use $\frac{s+1}{2} = t$.

Lemma 1. No cell of a $t \times t$-square is possible as a central cell of a green P_{s^2} if this $t \times t$-square contains a red cell.

Proof. Any $s \times s$-square having its central cell inside of a $t \times t$-square covers this $t \times t$-square completely and thus contains a red cell. ∎

Lemma 2. If a rectangle of s rows and t columns contains a red cell then no cell of the middle row is possible as a central cell of a green P_{s^2}.

Proof. Any $s \times s$-square having its central cell inside of the middle row of the rectangle covers it completely and thus contains a red cell. ∎

Lemma 3. If (α) or (β) is fulfilled then player B can color cells inside of a stripe R of t consecutive infinite lines (rows or columns) of square cells such that no cell of R is possible as a central cell of a green P_{s^2}.

(α) There exists a partition of R into $t \times t$-squares, each having at least two uncolored cells.

(β) In R there are i lines, $2 \leq i \leq t$, having at most $si - 2$ green cells.

Proof. If (α) is fulfilled then player B can color one of the two cells within any $t \times t$-square in red and Lemma 1 can be used.

If (β) is fulfilled then we can assume that at least one $t \times t$-square of every partition of R into $t \times t$-squares has at most one uncolored cell since otherwise (α) can be used. There are t different partitions into $t \times t$-squares and we can assume that each partition has one square Q_j, $j = 1, 2, \ldots, t$, with at least $ti - 1$ green cells in the considered i lines. No pair of these squares exists with an empty intersection since otherwise R would contain at least $2(ti - 1)$ green cells in the i lines and thus more than $si - 2$, a contradiction to (β). It follows that the t squares Q_j have to cover an $s \times t$-rectangle R_1. The leftmost and rightmost of these squares Q_j have in common exactly one column C of t cells. Since each Q_j has at most one uncolored cell and since the i lines contain at most $si - 2$ green cells, there remain no green cells outside of R_1 and in these i lines, and exactly two uncolored cells occur in these i lines and inside of R_1. Then player B can color one of these two cells in red and Lemma 2 excludes all cells of column C as central cells. If R on both sides of C is partitioned into $t \times t$ squares then all of them contain at least two uncolored cells (at least in the i lines). Then Lemma 1 completes the proof in the case of (β). ∎

We now are ready to exclude every cell of the plane as a central cell of a green P_{s^2}.

We consider partitions of the plane into horizontal and vertical stripes of width t. We can assume that every partition contains one stripe that fulfills neither (α) nor (β) since otherwise Lemma 3 can be used. Horizontal and vertical there are t different partitions each with one stripe H_j or V_j, respectively, having at least $si - 1$ green cells in every set of i lines of H_j and V_j with $1 \leq j \leq t$ and $2 \leq i \leq t$. These t stripes have pairwise at least one line of cells in common since otherwise there are altogether at least $2(st - 1)$ green cells, that is, more than the given number of $s^2 - 2$ green cells. Thus the stripes H_j and V_j form stripes H and V, respectively, of width $t + t - 1 = s$.

If $s \geq 5$ then $t-1 \geq 2$ and H so as V contains at least $st-1+s(t-1)-1 = s^2 - 2$ green cells (since (β) is not fulfilled), that means, all green cells are within H and V, that is, within an $s \times s$-square Q being the intersection of H and V. The two uncolored cells of Q exclude the central cell of Q. Around this central cell we partition the plane into $t \times t$-squares as indicated in Figure 1. Since each of these $t \times t$-squares contains at least two uncolored cells, player B can color one cell in red and Lemma 1 can be applied to exclude all other cells of the plane.

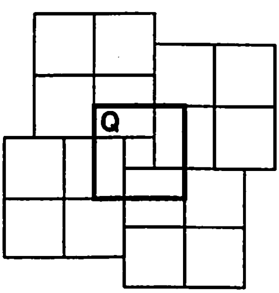

Figure 1. Partition of the plane into $t \times t$-squares around the central cell of Q.

If $s = 3$ then H and V have the width 3, both stripes of width $t = 2$ contain at least five cells, and the middle line of cells contains three or four green cells. Thus there remain only two possibilities, either all seven green cells are within the intersection Q of H and V or five of them are within Q and one in each of the two middle lines as in Figure 2. In both cases a partition as in Figure 1 guarantees at least two uncolored cells in the 2×2-squares and the proof is finished as before. ∎

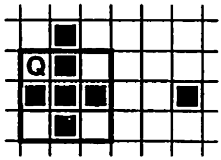

Figure 2. Five green cells within Q and one in each of the two middle lines.

2.2 Proof for even s

For even s every P_{s^2} has a central lattice point and we exclude every lattice point of the plane as the central lattice point of a green P_{s^2} if player A has colored any set of $s^2 - 2$ cells in his color before his first move.

Again, we need three lemmas and we will use $\frac{s}{2} = t$.

Lemma 4. No lattice point of a $t \times t$-square is possible as a central lattice point of a green P_{s^2} if this $t \times t$-square contains a red cell.

Proof. Any $s \times s$-square having its central lattice point as a lattice point of a $t \times t$-square covers this $t \times t$-square completely and thus contains a red cell. ∎

Lemma 5. If a rectangle of s rows and t columns contains a red cell then no lattice point of the middle row of lattice points is possible as a central lattice point of a green P_{s^2}.

Proof. Any $s \times s$-square having its central lattice point as one of the middle row of lattice points covers all lattice points of the $s \times t$-rectangle and thus contains a red cell. ∎

Lemma 6. If (α) or (β) is fulfilled then player B can color cells inside of a stripe R of t consecutive infinite lines (rows or columns) of square cells such that no lattice point of R is possible as a central lattice point of a green P_{s^2}.

(α) There exists a packing of R by disjoint $t \times t$-squares such that all lattice points of R are covered and each $t \times t$-square contains at least two uncolored cells.

(β) At most $2t^2 - 2$ cells of R are green.

Proof. If (α) is fulfilled then Lemma 4 can be used.

If (β) is fulfilled and (α) is not then at least one $t \times t$-square of each of the $t + 1$ different packings of R by disjoint $t \times t$-squares covering all lattice points has at most one uncolored cell. Two of these squares, Q_1 and Q_2, do not have any cells in common. Together, Q_1 and Q_2 have at least $2(t^2 - 1) = 2t^2 - 2$ green cells. Therefore, Q_1 and Q_2 each has exactly $t^2 - 1$ green cells and all remaining cells of R are uncolored. The squares

Q_1 and Q_2 have a side in common since otherwise (α) is fulfilled. Now by Lemma 5 the lattice points of the common side of Q_1 and Q_2 are excluded. The remaining lattice points of R are covered by packings of disjoint $t \times t$-squares and Lemma 4 can be applied since $t \geq 2$. ∎

To exclude every lattice point of the plane we consider vertical and horizontal packings of the plane by disjoint stripes of width t such that all lattice points are covered. It suffices to assume that each of the $t+1$ packings contains one stripe that fulfills neither (α) nor (β) of Lemma 6. Two of these stripes do not have any cells in common. Together, each pair of stripes has $2(2t^2 - 1) = s^2 - 2$ green cells, that is, all cells outside of the stripes are uncolored. The pairs of stripes have one line of lattice points in common since otherwise one of the $t+1$ packings exists where all stripes fulfill (α) of Lemma 6. Thus all $s^2 - 2$ green cells are inside of that $s \times s$-square Q being the intersection of the horizontal and vertical pair of stripes. By the two uncolored cells of Q the central lattice point of Q is excluded. All surrounding lattice points can be covered by a packing of disjoint $t \times t$-squares each having at least two uncolored cells (see Figure 3) and then they are excluded by Lemma 4. ∎

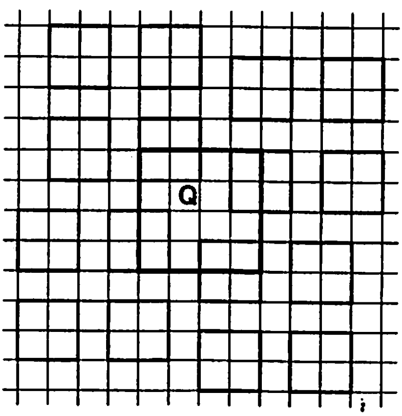

Figure 3. Packing of the plane by disjoint $t \times t$-squares around the central lattice point of Q.

3 Some related polyominoes

Exact handicap numbers for some further classes of polyominoes P with n cells can be determined using Theorem 1. If one, two, or three square cells are added to an $s \times s$-square as in Figures 4, 5, or 6 (the dashed lines are symmetry lines) then the lower bounds of the handicap numbers follow from

Theorem 1 and the arguments for the upper bounc. are straightforward to obtain $h(P) = n - 2$, $n - 3$, or $n - 4$, respectively.

In general, it may be a question whether polyominoes P with n cells do exist having $h(P) = n - c$ for any c.

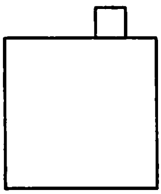

Figure 4. $h(P) = n - 2$.

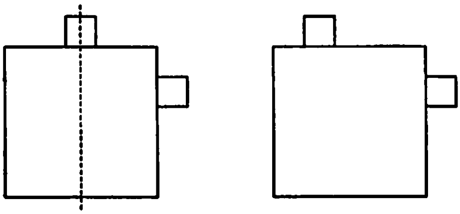

Figure 5. $h(P) = n - 3$.

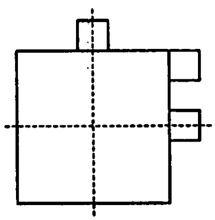

Figure 6. $h(P) = n - 4$.

Reference

[1] F. Haray, H. Harborth, and M. Seemann: *Handicap achivement for polyominoes*. Congr. Numer. **145** (2000) 65–80.