GRAPHS WITH THE 3-E.C. ADJACENCY PROPERTY CONSTRUCTED FROM AFFINE PLANES

C.A. BAKER, ANTHONY BONATO, AND JULIA M.N. BROWN

ABSTRACT. A graph G is 3-e.c. if for each distinct triple S of vertices, and each subset T of S, there is a vertex not in S joined to the vertices of T and to no other vertices of S. Few explicit examples of 3-e.c. graphs are known, although almost all graphs are 3-e.c. We provide new examples of 3-e.c. graphs arising as incidence graphs of partial planes resulting from affine planes. We also present a new graph operation that preserves the 3-e.c. property.

1. INTRODUCTION

In this paper, we present new examples of graphs with the 3-*e.c. adjacency property*. For a positive integer n, a graph is n-*existentially closed* or n-*e.c.*, if for each n-subset S of vertices, and each subset T of S, there is a vertex not in S joined to each of the vertices of T and to no vertex in $S \backslash T$. Therefore, a graph is 3-e.c. if for each triple of distinct vertices S there are eight vertices joined to the vertices of S in all possible ways. Graphs with the n-e.c. property were first explicitly studied in Caccetta et al. [10] (who referred to them as graphs with property $P(n)$). However, the earlier work of Erdős and Rényi [13] proved that almost all graphs are n-e.c. As discussed in [5], few explicit examples of n-e.c. graphs are known. The *Paley graph* of order q, where $q \equiv 1 \pmod 4$ is a prime power, is the graph with vertices the elements of the finite field with q elements $GF(q)$, where two distinct vertices x and y are joined if $x - y$ is a square in $GF(q)$. In [6, 7] it was shown that sufficiently large Paley graphs are n-e.c. Adjacency properties of Paley graphs have also been studied by Ananchuen and Caccetta [2], who proved among other things that a Paley graph with at least 29 vertices is 3-e.c. Constructions of 1-e.c. and 2-e.c. graphs were given in [8], and it was shown there that a 3-e.c. graph must have at least 20

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vertices. Constructions of 3-e.c. graphs using Hadamard matrices were given in [9]. Recent constructions, using probability theory, of many non-isomorphic n-e.c. graphs are given in [12].

In the present article, new 3-e.c. graphs are constructed using finite affine planes. In particular, we consider partial planes formed by deleting exactly half the parallel classes of affine planes of order q, where q is odd. A 3-e.c. graph is formed by taking the points of the partial plane as vertices and joining two vertices if the points are on a line of the partial plane; see Theorem 1. The 3-e.c. property can be unwieldy to verify for a graph with large order because of the number of cases that must be checked. The advantage of using an incidence structure with strong regularity properties to construct 3-e.c. graphs is that there are fewer cases to verify: the triple of points need only be among a small number of geometric configurations (see the proof of Theorem 1). We note that no restriction is placed on the properties of the affine planes used.

In addition, we show that many vertices may be deleted from our graphs and the 3-e.c. property is preserved; see Corollary 1. Paley graphs of order q^2 are recovered in our construction; see Lemma 1. We show that for orders 7 and 9, our methods produce two non-isomorphic 3-e.c. graphs of orders 49 and 81, respectively. See Corollaries 2 and 3. We close with the introduction of a new 3-e.c. preserving operation which gives examples of 3-e.c. graphs that are not Paley graphs. See Theorem 4.

All graphs considered are finite and simple. For a graph G, the vertex set of G is written $V(G)$, and the edge set is written $E(G)$. Edges are written xy, and we say that x and y are *joined*. Given a fixed vertex x, the *neighbour set* of x is the set of vertices joined to x, written $N(x)$. A *non-neighbour* of x is a vertex not joined to and not equal to x, and the *co-neighbour set* of x is the set of all non-neighbours of x, written $N^c(x)$. The vertices that are not in a set S of vertices will be written $\overline{S}$ (this should not be confused with the complement of G, which is written $\overline{G}$). The complete graph, or clique, of order n is written K_n.

Throughout, $q \geq 7$ will be odd and our affine plane A will be of order q. That is, A is a 2-$(q^2, q, 1)$ design (with "blocks" called "lines"), and hence satisfies the property that given any point x and line ℓ, there is a unique line $L(x, \ell)$ parallel to ℓ that goes through x. As is well known, such a plane has q^2 points, $q^2 + q$ lines, and each line contains exactly q points. The relation of parallelism on the set of lines is an equivalence relation, and the equivalence classes are called *parallel classes*. Each parallel class contains q lines, and there are $q + 1$ parallel classes. A partial plane results from an affine plane A if we delete some set of lines of A. If P is a partial plane resulting from A, then the *incidence graph* of P is the graph with vertices equal to the points of A, with two vertices joined if they are joined by a line of P.

If a point x is on the line ℓ, then we write $xI\ell$. Each pair of non-parallel distinct lines ℓ and m intersect in a unique point, which we will write $\ell \wedge m$. Each pair of distinct points x, y is joined by a unique line that we write as xy. (This notation conflicts with our earlier notation for edges of a graph. We keep both notations since they are standard.) If two lines ℓ and m are parallel, then we write $\ell || m$.

2. THE GRAPHS $G(q, \mathcal{U}, A)$

Fix A an affine plane of order $q \geq 7$, and consider the partial plane that results from deleting the lines of some fixed set of half of the parallel classes of A. The set of lines of this partial plane will be denoted by $\mathcal{U}$ and the set of deleted affine lines (which are non-lines of the partial plane) will be denoted $\mathcal{U}'$. Hence, $|\mathcal{U}| = |\mathcal{U}'| = \frac{q(q+1)}{2}$. The graph $G = G(q, \mathcal{U}, A)$ is the incidence graph of this partial plane. It is not hard to check that G is *strongly regular*: G is regular of degree $\frac{q^2-1}{2}$, each pair of joined vertices has $\frac{q^2-5}{4}$ common neighbours, and each pair of non-joined vertices has $\frac{q^2-1}{4}$ common neighbours. We summarize this information by saying that G is a $SRG(q^2, \frac{q^2-1}{2}, \frac{q^2-5}{4}, \frac{q^2-1}{4})$.

Our main theorem is the following.

Theorem 1. *The graph $G = G(q, \mathcal{U}, A)$ is 3-e.c. and is a $SRG(q^2, \frac{q^2-1}{2}, \frac{q^2-5}{4}, \frac{q^2-1}{4})$.*

Proof. For each triple x, y, z of distinct vertices in $V = V(G)$, we need only show that each of the eight sets

$$S_1(x) \cap S_2(y) \cap S_3(z),$$

where $S_i \in \{N, N^c\}$, is non-empty. We actually prove the stronger fact that

$$|S_1(x) \cap S_2(y) \cap S_3(z)| \geq \frac{q-1}{4} > 0. \tag{2.1}$$

(Observe that $\frac{q-1}{4}$ may not be an integer, in which case $|S_1(x) \cap S_2(y) \cap S_3(z)| \geq \lceil \frac{q-1}{4} \rceil$.) Choose three distinct vertices $x, y, z \in V$. Since the lines (and the parallel classes) of A are divided evenly between $\mathcal{U}$ and $\mathcal{U}'$, the proofs that we give will still hold if we interchange $\mathcal{U}$ and $\mathcal{U}'$, so we need only consider the following cases.

(1) The vertices x, y, z form a triangle in A with

i) all three sides in $\mathcal{U}$;

ii) exactly two sides in $\mathcal{U}$; without loss of generality $xy, yz \in \mathcal{U}$.

(2) The vertices x, y, z are collinear in A, with $x, y, zI\ell$ and $\ell \in \mathcal{U}$.

We recall that for every $v \in V$,

$$|N(v)| = |N^c(v)| = \frac{q^2-1}{2}.$$

Case 1.i) The three lines xy, xz, yz are distinct and in $\mathcal{U}$.

There are $\frac{q+1}{2} - 3$ lines of $\mathcal{U}\backslash\{xy, xz, L(x, yz)\}$ through x, and they meet yz in distinct points different from y and z. These points are all in $N(x) \cap N(y) \cap N(z)$. Similarly, each of the lines xy and xz also contain $\frac{q+1}{2} - 3$ points of this set. Therefore,

$$|N(x) \cap N(y) \cap N(z)| \geq 3\left(\frac{q-5}{2}\right) > \frac{q-1}{4}.$$

There are $\frac{q+1}{2}$ lines of $\mathcal{U}'$ through x, and they meet yz in distinct points different from y and z. Therefore,

$$|N^c(x) \cap N(y) \cap N(z)| \geq \frac{q+1}{2} > \frac{q-1}{4}.$$

Consider a fixed line $m \in \mathcal{U}\backslash\{xy, xz\}$ with xIm. The $\frac{q+1}{2}$ lines of $\mathcal{U}'$ through y meet m in $\frac{q+1}{2}$ distinct points that are different from x. Similarly, the $\frac{q+1}{2}$ distinct lines of $\mathcal{U}'$ through z meet m in $\frac{q+1}{2}$ distinct points that are different from x. Since m only contains q points including x, the line m must contain at least two points of $N(x) \cap N^c(y) \cap N^c(z)$. Since there are $\frac{q+1}{2} - 2 = \frac{q-3}{2}$ such lines m, we have that

$$|N(x) \cap N^c(y) \cap N^c(z)| \geq 2\left(\frac{q-3}{2}\right) = q - 3 > \frac{q-1}{4}.$$

Consider a fixed line $n \in \mathcal{U}'$, with xIn. The $\frac{q-1}{2}$ lines of $\mathcal{U}'\backslash\{L(y, n)\}$ through y meet n in $\frac{q-1}{2}$ distinct points different from x and $n \wedge yz$. Similarly, the $\frac{q-1}{2}$ lines of $\mathcal{U}'\backslash\{L(z, n)\}$ through z meet n in $\frac{q-1}{2}$ distinct points different from x and $n \wedge yz$. As n contains only q points, there must be at least one point of n in $N^c(x) \cap N^c(y) \cap N^c(z)$. However, there are $\frac{q+1}{2}$ such lines n, so

$$|N^c(x) \cap N^c(y) \cap N^c(z)| \geq \frac{q+1}{2} > \frac{q-1}{4}.$$

The other cases follow by symmetry.

Case 1.ii) Suppose that $xy, yz \in \mathcal{U}$ and $xz \in \mathcal{U}'$.

There are $\frac{q-3}{2}$ lines through z in $\mathcal{U}\backslash\{yz, L(z, xy)\}$. These lines meet xy in $\frac{q-3}{2}$ distinct points different from x and y. Therefore,

$$|N(x) \cap N(y) \cap N(z)| \geq \frac{q-3}{2} > \frac{q-1}{4}.$$

There are $\frac{q-1}{2}$ lines through y in $\mathcal{U}'\backslash\{L(y, xz)\}$. These lines meet xz in $\frac{q-1}{2}$ distinct points different from x and z. Hence,

$$|N^c(x) \cap N^c(y) \cap N^c(z)| \geq \frac{q-1}{2} > \frac{q-1}{4}.$$

There are $\frac{q-1}{2}$ lines through z in $\mathcal{U}'\backslash\{xz\}$. These lines meet xy in $\frac{q-1}{2}$ distinct points different from x and y, so

$$|N(x) \cap N(y) \cap N^c(z)| \geq \frac{q-1}{2} > \frac{q-1}{4}.$$

Choose any line $m \in \mathcal{U}\backslash\{xy\}$ through x. There are $\frac{q-1}{2}$ lines through z in $\mathcal{U}\backslash\{L(z, m)\}$ that meet m in $\frac{q-1}{2}$ distinct points of m different from x. In addition, there are $\frac{q+1}{2}$ lines through y in $\mathcal{U}'$ that also meet m in distinct points different from x. Hence, m contains at least one point in $N(x) \cap N^c(y) \cap N(z)$. However, there are $\frac{q-1}{2}$ such lines m, so

$$|N(x) \cap N^c(y) \cap N(z)| \geq \frac{q-1}{2} > \frac{q-1}{4}.$$

Each such line $m \in \mathcal{U}\backslash\{xy\}$ through x also meets the $\frac{q-1}{2}$ lines of $\mathcal{U}'\backslash\{xz\}$ through z in distinct points different from x, so m contains at least one point of $N(x) \cap N^c(y) \cap N^c(z)$, and as above,

$$|N(x) \cap N^c(y) \cap N^c(z)| \geq \frac{q-1}{2} > \frac{q-1}{4}.$$

Finally, the $\frac{q-3}{2}$ lines through y in $\mathcal{U}\backslash\{xy, yz\}$ meet xz in distinct points, so

$$|N^c(x) \cap N(y) \cap N^c(z)| \geq \frac{q-3}{2} > \frac{q-1}{4}.$$

The remaining cases follow by symmetry.

Case 2. Suppose that x, y, z lie on a line $\ell \in \mathcal{U}$.

The $q-3$ points on ℓ not equaling one of x, y, nor z all lie in $N(x) \cap N(y) \cap N(z)$. Therefore,

$$|N(x) \cap N(y) \cap N(z)| \geq q - 3 > \frac{q-1}{4}.$$

Choose any line m through x in $\mathcal{U}\backslash\{\ell\}$. The $\frac{q+1}{2}$ lines of $\mathcal{U}'$ incident with y meet m in $\frac{q+1}{2}$ distinct points different from x as do the $\frac{q+1}{2}$

lines of $\mathcal{U}'$ through z. Therefore, m contains at least two points of $N(x) \cap N^c(y) \cap N^c(z)$. Since there are $\frac{q-1}{2}$ such lines m, we have that

$$|N(x) \cap N^c(y) \cap N^c(z)| \geq q - 1 > \frac{q-1}{4}.$$

Next, we note that

$$\begin{aligned}
|N(x) \cap N(y) \cap N^c(z)| &= |N^c(z)| - |N^c(z) \cap \overline{N(x)}| - |N^c(z) \cap \overline{N(y)}| \\
&\quad + |N^c(z) \cap \overline{N(x)} \cap \overline{N(y)}| \\
&= |N^c(z)| - |N^c(x) \cap N^c(z)| - |N^c(y) \cap N^c(z)| \\
&\quad + |N^c(x) \cap N^c(y) \cap N^c(z)| \\
&= \frac{q^2-1}{2} - 2\left(\frac{q^2-1}{4}\right) \\
&\quad + |N^c(x) \cap N^c(y) \cap N^c(z)| \\
&= |N^c(x) \cap N^c(y) \cap N^c(z)|,
\end{aligned}$$

the second equality holding since x and y are joined to z. Similarly, $|N^c(x) \cap N(y) \cap N(z)| = |N(x) \cap N^c(y) \cap N(z)| = |N^c(x) \cap N^c(y) \cap N^c(z)|$. Therefore, to finish the remaining cases, we need only show that $|N^c(x) \cap N^c(y) \cap N^c(z)| \geq \frac{q-1}{4}$. To obtain a contradiction, suppose that

$$|N^c(x) \cap N^c(y) \cap N^c(z)| = k < \frac{q-1}{4}.$$

Let

$$\begin{aligned}
\mathcal{B} &= (N^c(x) \cap N^c(y) \cap N^c(z)) \cup (N(x) \cap N(y) \cap N^c(z)) \\
&\cup (N(x) \cap N^c(y) \cap N(z)) \cup (N^c(x) \cap N(y) \cap N(z)).
\end{aligned}$$

Then $|\mathcal{B}| = 4k$.

There exists a line $\ell' \| \ell, \ell' \neq \ell$, that contains no points of $\mathcal{B}$ since $4k < q - 1$ (recall that there are q lines in each parallel class). So

$$\begin{aligned}
|N(x) \cap N(y) \cap \ell'| &= |N(x) \cap N(y) \cap N(z) \cap \ell'| \\
&= |N(x) \cap N(z) \cap \ell'| \\
&= |N(y) \cap N(z) \cap \ell'|.
\end{aligned}$$

By the fact that $|N^c(x) \cap N^c(y) \cap N^c(z) \cap \ell'| = 0$ and the Principle of Inclusion-Exclusion, we have that

$$\begin{aligned}
q = |\ell'| &= |(N(x) \cup N(y) \cup N(z)) \cap \ell'| \\
&= 3|N(x) \cap \ell'| - 3|N(x) \cap N(y) \cap \ell'| + |N(x) \cap N(y) \cap N(z) \cap \ell'| \\
&= 3|N(x) \cap \ell'| - 2|N(x) \cap N(y) \cap \ell'| \\
&= 3\left(\frac{q-1}{2}\right) - 2|N(x) \cap N(y) \cap \ell'|,
\end{aligned}$$

so

$$(2.2)\quad |N(x) \cap N(y) \cap \ell'| = \frac{q-3}{4} \in \mathbb{Z} \text{ and } q \equiv 3 \pmod 4).$$

If $k > 0$, then choose any point $p \in N^c(x) \cap N^c(y) \cap N^c(z) \subseteq \mathcal{B}$. Then the lines px, py, and pz are all in $\mathcal{U}'$. The line px contains p, x and exactly $\frac{q-3}{2}$ points of each of $(N^c(x) \cap N^c(y))\backslash\{p\}$ and $(N^c(x) \cap N^c(z))\backslash\{p\}$; hence, px contains at least one point of $N^c(x) \cap N(y) \cap N(z) \subseteq \mathcal{B}$. Therefore, px contains at least two points of $\mathcal{B}$. Since $4k - 2 < (q-1) - 2 = q - 3$, at least one of the $q - 3$ lines parallel to px, but not incident with x, y, nor z, must contain no points of $\mathcal{B}$. Call this line n. See Figure 1.

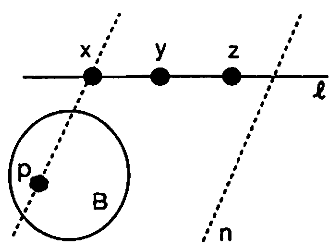

FIGURE 1. The line n in the case when $k > 0$.

If $k = 0$, then let n be any line of $\mathcal{U}'$ that is not through x, y, nor z. In either case, since n contains no points of $\mathcal{B}$, we have that

$$\begin{aligned} q = |n| &= |(N(x) \cup N(y) \cup N(z)) \cap n| \\ &= 3|N(x) \cap n| - 3|N(x) \cap N(y) \cap n| + |N(x) \cap N(y) \cap N(z) \cap n| \\ &= 3|N(x) \cap n| - 2|N(x) \cap N(y) \cap n| \\ &= 3\left(\frac{q+1}{2}\right) - 2|N(x) \cap N(y) \cap n|. \end{aligned}$$

Hence, $|N(x) \cap N(y) \cap n| = \frac{q+3}{4} \in \mathbb{Z}$, so $q \equiv 1 \pmod 4$, which contradicts (2.2). Therefore, $|N^c(x) \cap N^c(y) \cap N^c(z)| \geq \frac{q-1}{4}$, as desired. □

By (2.1) in the proof of Theorem 1, we can remove any set of n vertices from G, where n is an integer satisfying $0 < n \leq \frac{q-5}{4}$, and the resulting graph $G_n = (V_n, E_n)$ (with $|V_n| = q^2 - n$) will remain 3-e.c. (Observe that the first q where this is possible is $q = 9$.) In general, the graphs G_n are no longer regular since removing one vertex, say x, reduces the cardinality of the neighbour sets of precisely half the remaining vertices (those in $N(x)$) by one, but leaves the cardinality of the neighbour sets of the vertices in $N^c(x)$ unchanged.

Corollary 1. *If n is an integer satisfying $0 < n \leq \frac{q-5}{4}$, then the graph formed by deleting any set of n vertices from the graph $G(q,\mathcal{U},A)$ is 3-e.c.*

Let q be the power of an odd prime. We denote the Paley graph of order q^2 by P_{q^2}. As noted in the paragraph before Theorem 1, the graph $G(q,\mathcal{U},A)$ is a $SRG(q^2, \frac{q^2-1}{2}, \frac{q^2-5}{4}, \frac{q^2-1}{4})$, and therefore has the same parameters as P_{q^2}. A natural question, which we answer in the affirmative in the next lemma, is whether P_{q^2} is isomorphic to a $G(q,\mathcal{U},A)$ for some suitable set $\mathcal{U}$ and affine plane A. For q a prime power, $AG(2,q)$ denotes the unique affine plane of order q built from the field of order q.

Lemma 1. *Let q be the power of an odd prime. The graph P_{q^2} is isomorphic to a $G(q,\mathcal{U},AG(2,q))$ for a suitable choice of parallel classes $\mathcal{U}$.*

Proof. Let $K = GF(q^2)$ and let F denote its subfield of order q. Then K is a 2-dimensional vector space V over F. We use the standard vector space construction for $A = AG(2,q)$ (see, for example, p. 66 of [4]). Namely, the set of points are the vectors in V, and the lines are cosets of the form $\vec{v} + \langle\vec{u}\rangle$, where $\vec{u} \in V\backslash\{\vec{0}\}$ and $\vec{0}$ is the zero vector. (Note that this representation of a line is not unique. We have that $\vec{v} + \langle\vec{u}\rangle = \vec{v^*} + \langle\vec{u^*}\rangle$ if and only if $\vec{v^*} - \vec{v} = k\vec{u}$ and $\vec{u^*} = \ell\vec{u}$ for some $k \in F$ and $\ell \in F \setminus \{0\}$.) Incidence is containment, so $\vec{w} I (\vec{v} + \langle\vec{u}\rangle)$ if and only if $\vec{w} = \vec{v} + k\vec{u}$ for some $k \in F$. Then

$$(\vec{v}_1 + \langle\vec{u}_1\rangle) || (\vec{v}_2 + \langle\vec{u}_2\rangle) \text{ iff } \langle\vec{u}_1\rangle = \langle\vec{u}_2\rangle.$$

Hence, each parallel class is determined by a 1-dimensional subspace $\langle\vec{u}\rangle$. In what follows, we use $\langle\vec{u}\rangle$ as a name for the parallel class it determines. Two distinct points, say $\vec{u}$ and $\vec{v}$, are joined by the unique line $\vec{v} + \langle\vec{u} - \vec{v}\rangle$.

Next we note that every element of F is the square of an element in K. To see this, observe first that the multiplicative groups of invertible elements of the fields K and F, called K^* and F^*, respectively, are cyclic groups of order $q^2 - 1$ and order $q - 1$, respectively. Hence, the elements of F^* are $(q+1)$st powers of the elements of K^*. (Note that $q + 1$ is even.) So if $\vec{u}$ is a square in K, then every element of $\langle\vec{u}\rangle$ is also a square in K.

We now choose $\mathcal{U}$ to be the set of lines from the parallel classes in A associated with $\langle\vec{u}\rangle$, where $\vec{u}$ is a square in K. Note that $|\mathcal{U}| = \frac{q(q+1)}{2}$.

Therefore, $\vec{u}$ and $\vec{v}$ are joined in the graph $G(q,\mathcal{U},A)$ if and only if $\vec{v}+\langle\vec{u}-\vec{v}\rangle \in \mathcal{U}$. However, the latter statement is equivalent to the statement that $\vec{u}-\vec{v}$ is a square in K, which precisely defines the edges of the Paley graph P_{q^2}. □

Using known results on adjacency properties of Paley graphs (see [7, 6]), Lemma 1 demonstrates that for a fixed integer n and sufficiently large q, the graph $G(q,\mathcal{U},A)$ is n-e.c. for a suitable choice of $\mathcal{U}$ and A. We conjecture that for a fixed integer n, there is an integer N so that if $q \geq N$, the graph $G(q,\mathcal{U},A)$ is n-e.c. for *all* choices of $\mathcal{U}$ and A.

3. Non-isomorphic 3-e.c. graphs of orders 49 and 81

In Lemma 1 we showed that the Paley graph P_{q^2} is isomorphic to a graph $G(q,\mathcal{U},A)$ for some choice of parallel classes $\mathcal{U}$ and A. In this section, we show that in the cases where q is 7 or 9, our construction also provides a 3-e.c. graph that is not isomorphic to a Paley graph.

Theorem 2. *Up to isomorphism, there are exactly two graphs* $G(7,\mathcal{U}, AG(2,7))$.

Proof. In this proof we represent the 2-dimensional vector space over $F = GF(7)$ by $F \times F$; that is, the set of points of $A = AG(2,7)$ is $F \times F$. Let

$$\mathcal{U}_t = \{\vec{u}+\langle\vec{v}\rangle : \vec{v} \in \{(1,0),(0,1),(1,1),(1,t)\}\},$$

where $2 \leq t \leq 6$. Since an automorphism of A that maps $\mathcal{U}$ to $\mathcal{U}_t$ induces a graph isomorphism from $G(7,\mathcal{U},A)$ to $G(7,\mathcal{U}_t,A)$, it is sufficient for the proof that there are *at most* two isomorphism types to show that for any parallel class $\mathcal{U}$, there is an affine plane isomorphism which either maps $\mathcal{U}$ to $\mathcal{U}_3$ or maps $\mathcal{U}$ to $\mathcal{U}_6$.

It is well known and easy to verify that any three distinct lines through $\vec{0}$ can be mapped by an affine plane isomorphism of the form

$$(x,y) \mapsto (ax+cy, bx+dy),$$

with $ad-bc \neq 0$, to

$$\{\langle(1,0)\rangle, \langle(0,1)\rangle, \langle(1,1)\rangle\}.$$

Thus, each parallel class $\mathcal{U}$ in A can be mapped to $\mathcal{U}_t$ for some $t \in \{2,3,4,5,6\}$. Now observe that $(x,y) \mapsto (5x+y,y)$ is an automorphism of A which maps $\mathcal{U}_2$ to $\mathcal{U}_6$, $(x,y) \mapsto (6x+y,y)$ is an automorphism of A which maps $\mathcal{U}_4$ to $\mathcal{U}_6$, and $(x,y) \mapsto (5x,y)$ is an automorphism of A which maps $\mathcal{U}_5$ to $\mathcal{U}_3$.

In order to prove there are *at least* two isomorphism types, we first prove the following Claim.

Claim 1: A set $S = \{a, b, c, d\}$ of points from A is the set of vertices of a K_4 in G if and only if one of the following holds:
i) S is a set of four points on a line of $\mathcal{U}$;
ii) S consists of three points on a line $k \in \mathcal{U}$ together with a point off line k which is joined to each of the first three points by a line of $\mathcal{U}$;
iii) up to renaming of points, the six lines ab, ac, ad, bc, bd, and cd are distinct lines of $\mathcal{U}$ and $ab||cd$, $ad||bc$.

We call these *configurations of types i), ii), and iii)*, respectively. The proof of the "if" part of Claim 1 is straightforward, so we verify the "only if" part. Fix a set of four points of A that form a clique in G. If the four points are collinear, then we have a configuration of type i). Otherwise, if exactly three points are collinear, then we obtain a configuration of type ii). Suppose that no three points are collinear. Then the six lines joining the points pairwise are distinct. Since there are only four parallel classes in $\mathcal{U}$, then two applications of the Pigeonhole Principle ensures that there must be two pairs of parallel lines among the six lines, so without loss of generality, $ab||cd$ and $ad||bc$, and we obtain a configuration of type iii). Claim 1 follows.

We observe that for each choice of $\mathcal{U}$, there are exactly $28 \cdot \binom{7}{4}$ configurations of type i) and exactly $49 \cdot 4 \cdot 6$ configurations of type ii). To obtain two non-isomorphic graphs of the form $G(7, \mathcal{U}, A)$, we need only show that by choosing appropriate sets of parallel classes $\mathcal{U}$, we obtain different numbers of configurations of type iii), which in turn yields a different number of subgraphs isomorphic to K_4. More explicitly, we show that if $\mathcal{U} = \mathcal{U}_6$, then there is at least one configuration of type iii), and if $\mathcal{U} = \mathcal{U}_3$, then there are no configurations of type iii).

To see this, note that if $\mathcal{U} = \mathcal{U}_6$, then the four points $(0,0), (1,0)$, $(1,1)$, $(0,1)$ form the required configuration. Consider now the case when $\mathcal{U} = \mathcal{U}_3$, and suppose, in order to obtain a contradiction, that there is a configuration of type iii) present in $G(7, \mathcal{U}_3, A)$. We select two distinct parallel classes $\langle \vec{v_1} \rangle$ and $\langle \vec{v_2} \rangle$ in $\mathcal{U}_3$ to be the parallel classes containing the lines ab and ad, respectively. Then $\vec{b} = \vec{a} + k_1\vec{v_1}$ and $\vec{d} = \vec{a} + k_2\vec{v_2}$, for some scalars $k_1, k_2 \neq 0$. Further, $\vec{a} + k_1\vec{v_1} + m_2\vec{v_2} = \vec{b} + m_2\vec{v_2} = \vec{c} = \vec{d} + m_1\vec{v_1} = \vec{a} + k_2\vec{v_2} + m_1\vec{v_1}$, for some scalars m_1 and m_2. Since $\vec{v_1}$ and $\vec{v_2}$ are linearly independent, we have that $k_1 = m_1$ and $k_2 = m_2$. Therefore, the parallel class containing the line ac is $\langle \vec{c} - \vec{a} \rangle = \langle k_1\vec{v_1} + k_2\vec{v_2} \rangle$, and the parallel class containing bd is $\langle \vec{d} - \vec{b} \rangle = \langle k_2\vec{v_2} - k_1\vec{v_1} \rangle$. Using the fact that $\langle \vec{c} - \vec{a} \rangle \in \mathcal{U}_3$, direct computation then yields the following table:

$\langle \vec{v_1} \rangle$ and $\langle \vec{v_2} \rangle$	$\langle \vec{c} - \vec{a} \rangle$	$\langle \vec{d} - \vec{b} \rangle$
$\langle(1,0)\rangle, \langle(0,1)\rangle$	$\langle(1,1)\rangle$	$\langle(1,6)\rangle$
	$\langle(1,3)\rangle$	$\langle(1,4)\rangle$
$\langle(1,0)\rangle, \langle(1,1)\rangle$	$\langle(0,1)\rangle$	$\langle(1,4)\rangle$
	$\langle(1,3)\rangle$	$\langle(1,2)\rangle$
$\langle(1,0)\rangle, \langle(1,3)\rangle$	$\langle(0,1)\rangle$	$\langle(1,5)\rangle$
	$\langle(1,1)\rangle$	$\langle(1,4)\rangle$
$\langle(0,1)\rangle, \langle(1,1)\rangle$	$\langle(1,0)\rangle$	$\langle(1,2)\rangle$
	$\langle(1,3)\rangle$	$\langle(1,6)\rangle$
$\langle(0,1)\rangle, \langle(1,3)\rangle$	$\langle(1,0)\rangle$	$\langle(1,6)\rangle$
	$\langle(1,1)\rangle$	$\langle(1,5)\rangle$
$\langle(1,1)\rangle, \langle(1,3)\rangle$	$\langle(0,1)\rangle$	$\langle(1,2)\rangle$
	$\langle(1,0)\rangle$	$\langle(1,5)\rangle$

Since none of the lines in the third $\langle \vec{d} - \vec{b} \rangle$ column gives a parallel class in $\mathcal{U}_3$, we obtain a contradiction. □

Corollary 2. *The graph $G(7, \mathcal{U}, AG(2,7))$, for a suitable choice of $\mathcal{U}$, is a $SRG(49, 24, 11, 12)$ that is not isomorphic to P_{49}.*

We now consider the case $q = 9$.

Theorem 3. *Up to graph isomorphism, there are exactly two graphs $G(9, \mathcal{U}, AG(2,9))$.*

Proof. In this proof, we represent the elements of $F = GF(9)$ by $\{a+bi : a, b \in \mathbb{Z}_3\}$ and use the operations defined by

$$(a+bi) + (c+di) = (a+c) + (b+d)i,$$
$$(a+bi)(c+di) = (ac - bd) + (ad + bc)i,$$

where arithmetic between elements of $\mathbb{Z}_3$ is the modulo 3 arithmetic of $\mathbb{Z}_3$. Then $A = AG(2,9)$ can be viewed as the two-dimensional vector space over F; thus, the set of points of A is $F \times F$. The line which is the coset

$$\vec{v} + \langle \vec{u} \rangle = (v_1, v_2) + \langle (u_1, u_2) \rangle$$

is the set of all solutions (x, y) of the equation: $x = v_1$, if $u_1 = 0$, and $y = (u_2(u_1)^{-1})x + (u_1v_2 - u_2v_1)(u_1)^{-1}$, if $u_1 \neq 0$. Observe that these equations (of the form $y = mx + k$ or $x = j$) are unique. In the first case, we say the line has *slope* ∞, and in the second case we say the line has *slope* $u_2(u_1)^{-1}$. Lines are parallel if and only if they have the same slope. Thus, there is a one-to-one correspondence between the set of parallel classes and the set $\{\infty\} \cup F$ of slopes. In this proof, we name a parallel class by its slope, so if $B \subseteq F \cup \{\infty\}$, then we can view B as a set of slopes, the corresponding set of parallel classes, or as the set of all lines contained in these parallel classes, whichever is appropriate.

First we prove that there are *at most* two non-isomorphic graphs $G = G(9, \mathcal{U}, A)$. Since an automorphism of A that maps $\mathcal{U}$ to some $\mathcal{U}^*$ induces a graph isomorphism from $G(9,\mathcal{U},A)$ to $G(9,\mathcal{U}^*,A)$, it is sufficient to show that for any set $\mathcal{U}$ of parallel classes, there is an automorphism of A that maps $\mathcal{U}$ to one of $\mathcal{U}_1 = \{\infty, 0, 1, -1, i\}$ or $\mathcal{U}_2 = \{\infty, 0, 1, 1+i, -1+i\}$.

It is well known and easy to verify that any three distinct slopes (and therefore, any three distinct parallel classes) can be mapped by an automorphism of A of the form $(x, y) \mapsto (ax + cy, bx + dy)$, for some $ad - bc \neq 0$, to the slopes $\infty, 0, 1$. Therefore, we need only show that any one of the $\binom{7}{2} = 21$ slope sets of the form $\{\infty, 0, 1, m, n\}$ for $m, n \neq \infty, 0, 1$ with $m \neq n$, can be mapped to either $\mathcal{U}_1$ or $\mathcal{U}_2$. Note that the mapping of points, defined by

$$(x, y) \mapsto (ax + cy + e, bx + dy + f)$$

for $ad - bc \neq 0$, induces the mapping of slopes, or *slope map*, defined by

$$m \mapsto \begin{cases} (b + dm)(a + cm)^{-1} & \text{if } m \neq -ac^{-1} \text{ or } c = 0; \\ \infty & \text{otherwise;} \end{cases}$$

$$\infty \mapsto \begin{cases} dc^{-1} & \text{if } c \neq 0; \\ \infty & \text{otherwise.} \end{cases}$$

This set of 720 distinct slope maps forms a group under composition that we name $\mathcal{G}$. The order three slope map $f \in \mathcal{G}$ defined by $m \mapsto (1 - m)^{-1}$, fixes -1 and has the action

$$\begin{aligned} \infty &\mapsto 0 \mapsto 1 \mapsto \infty, \\ i &\mapsto (-1 - i) \mapsto (1 + i) \mapsto i, \\ -i &\mapsto (-1 + i) \mapsto (1 - i) \mapsto -i. \end{aligned}$$

Using f and f^2, we can map any one of the 21 distinct slope sets $\mathcal{U}$ containing $\infty, 0, 1$ to one of the slope sets $\mathcal{U}_1, \mathcal{U}_2$,

$$\begin{aligned} \mathcal{U}_3 &= \{\infty, 0, 1, i, -1+i\},\ \mathcal{U}_4 = \{\infty, 0, 1, 1-i, -1+i\}, \\ \mathcal{U}_5 &= \{\infty, 0, 1, 1+i, -1-i\},\ \mathcal{U}_6 = \{\infty, 0, 1, i, -i\}, \\ \mathcal{U}_7 &= \{\infty, 0, 1, -1, -i\}. \end{aligned}$$

(More formally, the subgroup of $\mathcal{G}$ generated by f acts on the set of all $\mathcal{U}$ containing $\infty, 0, 1$; it has 7 orbits with 3 elements each, and $\mathcal{U}_1, \ldots, \mathcal{U}_7$ are representatives of these orbits.) The following table, which gives a slope map in the first column, and the corresponding action on certain slope sets in the second column, completes the proof of the *at most two* part.

slope map	action on slope sets
$m \mapsto (1+i)m$	$\mathcal{U}_3$ to $\mathcal{U}_2$
$m \mapsto (m-1+i)m^{-1}$	$\mathcal{U}_4$ to $\mathcal{U}_1$
$m \mapsto (-1+i)m+1$	$\mathcal{U}_5$ to $\mathcal{U}_1$
$m \mapsto im$	$\mathcal{U}_6$ to $\mathcal{U}_1$
$m \mapsto -m$	$\mathcal{U}_7$ to $\mathcal{U}_1$

To prove that there are *at least* two non-isomorphic graphs, we first prove the following Claim.

Claim 2: A set S of 9 points in A is the set of vertices of a K_9 in G if and only if

i) The set S consists of 9 points on a line of $\mathcal{U}$; or

ii) The set S is an image of the 9 element set

$$\{(i,j) : i,j \in \{-1,0,1\}\}$$

under the group (with the operation of composition) of all mappings

$$(x,y) \mapsto (ax+cy+e, bx+dy+f)$$

with $a,b,c,d,e,f \in F$ and $ad-bc \neq 0$.

To prove Claim 2, notice first that the proof of the "if" part is straightforward. Therefore, we may assume that S is the set of vertices of a K_9 in G, and that S in not the set of all points on some line of $\mathcal{U}$. We show that S is a set with the properties of part ii) of Claim 2.

A line of $\mathcal{U}$ will be called a *j-secant* if it has exactly j points of S on it. Under our assumptions, a j-secant exists only for $j < 9$. So for any j-secant there is a point r of S off the j-secant. Then there are lines through r from j parallel classes of $\mathcal{U}$ and these classes do not contain the j-secant. Thus, $j+1 \leq 5$ and so $j \leq 4$.

Suppose there is a point s of S which has either two 4-secants through it or has both a 4-secant and a 3-secant through it. By using an automorphism $(x,y) \mapsto (ax+cy+e, bx+dy+f)$ of A, we may assume that $(0,0),(1,0),(a,0),(0,1),(0,b),(0,d)$ are in S with $a \neq 0,1$, $b \neq 0,1,d$, $d \neq 0,1,b$. Observe that this implies that $\infty, 0 \in \mathcal{U}$. There are six other field elements which arise as slopes of lines between these points. These field elements are $-1,-b,-d$ (which are distinct and nonzero so that $\mathcal{U} = \{\infty,0,-1,-b,-d\}$) and $-a^{-1}, -a^{-1}b, -a^{-1}d$. Since $a \neq 1$, we have that $a^{-1} = b$ or d. Without loss of generality, $a^{-1} = b$. Then $\{b^2, bd\} = \{1,d\}$. Since $b \neq 1$, we must have that $bd = 1$ and $b^2 = d$.

Thus, $d = b^{-1}$ and so $b^2 = b^{-1}$. But then $b^3 = 1$. However, in F (which is of characteristic 3) this implies that $b = 1$, a contradiction.

Now let y be a point of S and for a fixed j, let y_j be the number of j-secants through y. Using $y_j = 0$ for $j > 4$ and counting first all secants through y, and then second all edges with vertex y, we obtain the linear system

$$\begin{aligned} y_1 + y_2 + y_3 + y_4 &= 5 \\ y_2 + 2y_3 + 3y_4 &= 8. \end{aligned}$$

The previous paragraph implies that

$$y_4 = 0 \text{ or } (y_4 = 1 \text{ and } y_3 = 0).$$

Solving this linear system for nonnegative integer solutions gives $y_4 = 0$ (so there are no 4-secants) and $(y_1, y_2, y_3) = (1, 0, 4)$ or $(0, 2, 3)$.

Consider a fixed $m \in \mathcal{U}$. Let m_i be the number of i-secants in the parallel class m, and let e_m denote the number of edges of G which lie on a line of m. By counting lines in m, points of S, and edges on secants we obtain the linear system

$$\begin{aligned} m_0 + m_1 + m_2 + m_3 &= 9 \\ m_1 + 2m_2 + 3m_3 &= 9 \qquad (3.1) \\ m_2 + 3m_3 &= e_m. \end{aligned}$$

Since the total number of edges between vertices of S is 36 and the number of parallel classes is 5, there must exist an m with $e_m \geq 8$. Assume that $e_m \geq 8$ for this m. Solving the linear system (3.1) for nonnegative integer solutions gives $(m_0, m_1, m_2, m_3) = (6, 0, 0, 3)$ with $e_m = 9$. Consider $\mathcal{U} \setminus \{m\}$. Here the total number of edges is $36 - 9 = 27$ in four parallel classes so we may choose an $n \in \mathcal{U} \setminus \{m\}$ with $e_n \geq 7$. Solving the linear system (3.1) for nonnegative integer solutions gives that $(n_0, n_1, n_2, n_3) = (5, 1, 1, 2)$ with $e_n = 7$, or $(n_0, n_1, n_2, n_3) = (6, 0, 0, 3)$ with $e_n = 9$.

As indicated earlier, there is an automorphism of A

$$(x, y) \longmapsto (ax + cy + e, bx + dy + f)$$

which maps m to ∞ and n to 0. Let g be the point of intersection between a 3-secant through ∞ and a 3-secant through 0. Let h be the point of intersection between another 3-secant through ∞ and another 3-secant through 0 (we use here the facts that $m_3, n_3 \geq 2$). Since all points of S are on the three 3-secants in the parallel class with slope ∞, it follows that the points g and h are in S. Also, the slope of the line through g and h is neither ∞ nor 0. So some mapping $(x, y) \longmapsto (ax + e, dy + f)$, with $a, d \neq 0$, maps the point g to $(0, 0)$

and point h to $(1,1)$. Then $1 \in \mathcal{U}$. Also, $(1,0)$ and $(0,1) \in S$ and so $-1 \in \mathcal{U}$.

Suppose the third point of S on the line with equation $y = 0$ is $(a,0)$ with $a \neq 0,1$. Then $(a,1) \in S$ and the slope of the line through $(0,0)$ and $(a,1)$ is a^{-1}, and slope of line through $(0,1)$ and $(a,0)$ is $-a^{-1}$. So $\{\infty, 0, 1, -1, a^{-1}, -a^{-1}\} \subseteq \mathcal{U}$. Now $a \neq 0,1$ and $|\mathcal{U}| = 5$ implies $a = -1$. Suppose the third point of S on the line with equation $x = 0$ is $(0,b)$ with $b \neq 0,1$. Then $(1,b) \in S$ and the slope of the line through $(0,0)$ and $(1,b)$ is b, and slope of line through $(1,0)$ and $(0,b)$ is $-b$. So $\{\infty, 0, 1, -1, b, -b\} \subseteq \mathcal{U}$. Now, $b \neq 0,1$ and $|\mathcal{U}| = 5$ implies $b = -1$. We have now determined 8 points of S, and the ninth point is $(-1,d)$ for some $d \neq 0,1$. The slope of the line through $(0,0)$ and $(-1,d)$ is $-d$. The slope of the line through $(1,0)$ and $(-1,d)$ is $d(-2)^{-1} = d$ (because $3 = 0$ in $GF(9) = GF(3^2)$). So $\{\infty, 0, 1, -1, d, -d\} \subseteq \mathcal{U}$. Now, $d \neq 0,1$ and $|\mathcal{U}| = 5$ implies $d = -1$. Hence, S is the desired set of points in part ii) of Claim 2.

For all choices of $\mathcal{U}$, the graph $G(9,\mathcal{U},A)$ has exactly 45 pairwise distinct subgraphs isomorphic to a K_9 of the type in item i) of Claim 2. A graph $G(9,\mathcal{U},A)$ has a subgraph isomorphic to K_9 of the type in item ii) of Claim 2 if and only if $\{\infty, 0, 1, -1\} \subseteq \mathcal{U}$. This is the case for $\mathcal{U}_1$ (defined earlier in the proof). To prove that there are at least two non-isomorphic graphs of the form $G(9,\mathcal{U},A)$, we need only show that some choice of $\mathcal{U}$ contains no $\mathcal{G}$ image of the set $\{\infty, 0, 1, -1\}$ of slopes.

Recall that the group $\mathcal{G}$ has order 720. The stabilizer in $\mathcal{G}$ of $\{\infty, 0, 1, -1\}$ includes all 24 of the elements of $\mathcal{G}$ with coefficients in the order 3 subfield of F. So this stabilizer has order at least 24. So the orbit of $\{\infty, 0, 1, -1\}$ under $\mathcal{G}$ has size at most $720/24 = 30$. Hence, the cardinality of the set of all $\mathcal{U}$ which include $\{\infty, 0, 1, -1\}$ is at most $30 \cdot 6 = 180 < 252 = \binom{10}{5}$, which equals the cardinality of the set of all the $\mathcal{U}$. Therefore, some set of parallel pencils $\mathcal{U}$ contains no $\mathcal{G}$ image of $\{\infty, 0, 1, -1\}$. □

Corollary 3. *The graph $G(9,\mathcal{U},AG(2,9))$, for a suitable choice of $\mathcal{U}$, is a $SRG(81,40,19,20)$ that is not isomorphic to P_{81}.*

We conjecture that for each odd $q \geq 11$ such that q is the order of an affine plane, there is a suitable choice of $\mathcal{U}$ and A so that there are at least two non-isomorphic graphs $G(q,\mathcal{U},A)$.

4. A NEW 3-E.C. PRESERVING OPERATION

The *symmetric difference* of G and H is the graph with vertices $V(G) \times V(H)$ and (a,b) is joined to (c,d) if and only if a is joined to c in

G and b is not joined to d in H or a is not joined to c in G and b is joined to d in H. The symmetric difference operation is a commutative binary operation on graphs. In [8], it was proved that symmetric difference preserves 3-e.c., but other well-known graph operations, such as join, Cartesian product, and categorical product, do not. We present the following new graph operation, and we prove in Theorem 4 that it preserves the 3-e.c. property.

Definition 1. *Let G be a graph with $V(G) = \{x_1, \ldots, x_m\}$, and let H be a graph with $V(H) = \{1, \ldots, n\}$. Fix $1 \le k \le n$. Let G_i be an isomorphic copy of G, disjoint from G and the other G_j if $i \neq j$, with vertex x_k called x_{ik}, where $1 \le k \le m$.*

We define a graph $H(G)$ to have vertices $\bigcup_{1\le i \le n} V(G_i)$ and edges those of each G_i and the following edges between the G_i. If i and j are joined in H, then x_{ik} is joined to the neighbours of x_{jk} in G_j and x_{jk} is joined to the neighbours of x_{ik} in G_i. If i and j are not joined in H, then x_{ik} is joined to the non-neighbours of x_{jk} in G_j excluding x_{jk} and x_{jk} is joined to the non-neighbours of x_{ik} in G_i excluding x_{ik}.

For example, if P_3 is the path with three edges, then $P_3(P_3)$ is the graph in Figure 2. Observe that for any graph H, $H(K_1)$ is isomorphic to $\overline{K_{|V(H)|}}$ and $K_1(H)$ is isomorphic to H.

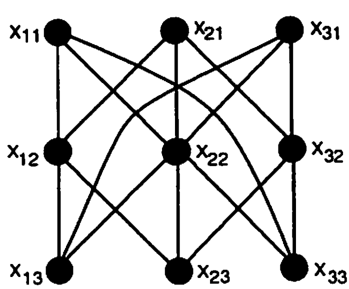

FIGURE 2. The graph $P_3(P_3)$.

Our new operation is distinct from the symmetric difference operation defined earlier. For example, the reader may verify that $\overline{K_2}(K_2)$ is isomorphic to the disjoint union of two edges, $K_2(\overline{K_2})$ is isomorphic to $\overline{K_4}$, while the symmetric difference of K_2 and $\overline{K_2}$ is isomorphic to the 4-cycle.

We now introduce some notation that was first used in [8]. For a positive integer n, an *n-e.c. problem* in G is a $2\times n$ matrix $\begin{pmatrix} x_1 & \ldots & x_n \\ i_1 & \ldots & i_n \end{pmatrix}$, where $S = \{x_1, \ldots, x_n\}$ is an n-element subset of $V(G)$, and for $1 \le j \le n$, $i_j \in \{0,1\}$. A *solution* to this problem is a vertex $y \in V(G) \setminus S$ so that y is joined to x_j if $i_j = 1$ and y is not joined to x_j if $i_j = 0$. Observe that a graph G is n-e.c. if and only if each n-e.c. problem in G has a solution.

Theorem 4. *If G and H are 3-e.c. graphs, then the graph $H' = H(G)$ is 3-e.c.*

Proof. Fix a, b, and c in $V(H')$, and consider the 3-e.c. problem

$$\begin{pmatrix} a & b & c \\ \cdot & \cdot & \cdot \\ i_1 & i_2 & i_3 \end{pmatrix} \tag{4.1}$$

in H', where $i_j \in \{0,1\}$ are fixed. We consider cases depending on the location of a, b, and c. To ease notation, we can always assume that by reordering $V(H)$ that a, b, and c are in G_1, G_2, or G_3.

Case 1. The vertices a, b, and c are in three distinct G_i. Without loss of generality, suppose that a is in G_1, b is in G_2, and c is in G_3, and that $a = x_{1r}$, $b = x_{2s}$, and $c = x_{3t}$.

Case 1.i) $|\{r,s,t\}| = 3$.

In this case, since H is 3-e.c. we may choose a vertex k in H joined to each of 1, 2, and 3. Let u be a solution to $\begin{pmatrix} x_{kr} & x_{ks} & x_{kt} \\ \cdot & \cdot & \cdot \\ i_1 & i_2 & i_3 \end{pmatrix}$ in G_k. Then u is a solution of (4.1) in H'.

Case 1.ii) $|\{r,s,t\}| = 2$. We assume without loss of generality that $r = s$.

By the 3-e.c. property for H, we may choose a vertex k in H joined to each of 1, 2, and 3, and we may choose $\ell \notin \{1,2,3\}$ to be a vertex joined to 1 and 3 but not to 2. A solution of $\begin{pmatrix} x_{kr} & x_{kt} \\ \cdot & \cdot \\ i_1 & i_3 \end{pmatrix}$ in G_k solves (4.1) in H' when (i_1, i_2, i_3) is an element of the set $\{(1,1,1), (0,0,0), (1,1,0), (0,0,1)\}$. (That is, when $i_1 = i_2$.) The remaining cases are solved by considering solutions of $\begin{pmatrix} x_{\ell r} & x_{\ell t} \\ \cdot & \cdot \\ i_1 & i_3 \end{pmatrix}$ in G_ℓ.

Case 1.iii) $|\{r,s,t\}| = 1$.

Using the 3-e.c. property for H, we can find vertices k_1, k_2, k_3, and k_4 of $V(H) \setminus \{1,2,3\}$ with the following properties: k_1 is joined to $1,2,3$; k_2 is not joined and not equal to 1 but joined to $2,3$; k_3 is joined to 1 and 3 but not joined and not equal to 2; and k_4 is joined to $1, 2$ and not joined and not equal to 3.

Let u be a vertex of G_{k_1} that is joined to x_{k_1r} and let v be a vertex of G_{k_1} that is not joined and not equal to x_{k_1r}. Then u solves (4.1) with $(i_1, i_2, i_3) = (1, 1, 1)$, and v solves (4.1) with $(i_1, i_2, i_3) = (0, 0, 0)$. In a similar fashion, neighbours and non-neighbours of x_{k_2r} in G_{k_2} solve (4.1) when (i_1, i_2, i_3) is one of $(0, 1, 1)$ and $(1, 0, 0)$; neighbours and non-neighbours of x_{k_3r} in G_{k_3} solve (4.1) when (i_1, i_2, i_3) is one of $(1, 0, 1)$ and $(0, 1, 0)$; and neighbours and non-neighbours of x_{k_4r} in G_{k_4} solve (4.1) when (i_1, i_2, i_3) is one of $(1, 1, 0)$ and $(0, 0, 1)$.

Case 2. The vertices a, b, and c are in exactly two G_i.

Without loss of generality, we may suppose that $a = x_{1r}$, $b = x_{2s}$, and $c = x_{2t}$. The cases when the indices r, s, and t are all distinct, and when $r = s$, are similar to Cases 1.i) and 1.ii), respectively, and so are omitted. The case when the vertices a, b, and c are all in the same G_i is obvious, since each G_i is 3-e.c. by hypothesis. □

By Theorem 4, we may construct 3-e.c. graphs that are not Paley graphs by considering graphs of the form $H(G)$, where H is some graph $H(q, \mathcal{U}, A)$, and G is some graph $G(r, \mathcal{U}', A')$ with $r \geq 7$ relatively prime to q.

References

[1] B. Alspach, C.C. Chen, and K. Heinrich, Characterization of a class of triangle-free graphs with a certain adjacency property, *J. Graph Theory* **15** (1991) 375-388.

[2] W. Ananchuen and L. Caccetta, On the adjacency properties of Paley graphs, *Networks* **23** (1993) 227-236.

[3] W. Ananchuen and L. Caccetta, A note on graphs with a prescribed adjacency property, *Bull. Austral. Math. Soc.* **51** (1995) 5–15.

[4] E. Artin, *Geometric Algebra*, Interscience Publishers Inc., New York, 1966.

[5] A. Blass and F. Harary, Properties of almost all graphs and complexes, *J. Graph Theory* **3** (1979) 225-240.

[6] A. Blass, G. Exoo, and F. Harary, Paley graphs satisfy all first-order adjacency axioms, *J. Graph Theory* **5** (1981) no. 4 435–439.

[7] B. Bollobás, A. Thomason, Graphs which contain all small graphs, *European J. Combin.* **2** (1981) no. 1 13-15.

[8] A. Bonato, K. Cameron, On an adjacency property of almost all graphs, *Discrete Math.* **231** (2001) 103-119.

[9] A. Bonato, W.H. Holzmann, H. Kharaghani, Hadamard matrices and strongly regular graphs with the 3-e.c. adjacency property, *Electron. J. Combin.* **8** (2001) no. 1, 9 pp.

[10] L. Caccetta, P. Erdős, and K. Vijayan, A property of random graphs, *Ars Combin.* **19** (1985) 287–294.

[11] P.J. Cameron, The random graph, in: *Algorithms and Combinatorics* 14 (eds. R.L. Graham and J. Nešetřil), Springer Verlag, New York (1997) 333–351.
[12] P.J. Cameron, D. Stark, A prolific construction of strongly regular graphs with the n-e.c. property, submitted to *Discrete Math.*
[13] P. Erdős, and A. Rényi, Asymmetric graphs, *Acta Math. Acad. Sci. Hungar.* 14 (1963) 295–315.

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, 67 YORK STREET, SACKVILLE, NB, CANADA, E4L 1E6
E-mail address: cbaker@mta.ca

DEPARTMENT OF MATHEMATICS, WILFRID LAURIER UNIVERSITY, WATERLOO, ON, CANADA, N2L 3C5
E-mail address: abonato@wlu.ca

DEPARTMENT OF MATHEMATICS AND STATISTICS, YORK UNIVERSITY, N520 ROSS BUILDING, 4700 KEELE STREET, TORONTO, ONTARIO, CANADA M3J 1P3
E-mail address: juliamnb@YorkU.CA