On Group-Magic Trees, Double Trees and Abbreviated Double Trees *

Sin-Min Lee and Linda Valdés
San José State University
San José, CA 95192
E-mails: lee@mathcs.sjsu.edu and valdes@mathcs.sjsu.edu

Yong-Song Ho
Nan-Chiau High School, Singapore

Abstract: For $k > 0$, we call a graph G = (V,E) *k*-magic if there exists a labeling l: E(G) → Z_k* such that the induced vertex set labeling l^+: V(G) → Z_k, defined by $l^+(v) = \Sigma\{l(u,v): (u,v) \in E(G)\}$ is a constant map. We denote the set of all k such that G is k-magic by IM(G). We call this set the ***integer-magic spectrum*** of G. We investigate these sets for trees, double trees and abbreviated double trees. We define group-magic spectrum for G similarly. Finally we show that a tree is k-magic, $k > 2$, if and only if it is k-label reducible.

1. Introduction.

Magic squares are popular mathematical puzzles that appeared more than a thousand years ago in different countries. In the early 1960s people try to generalize the concept of magic squares to magic graphs. A natural generalization is view a n-order magic square as an adjacency matrix of a complete bipartite graph K(n,n) with the magic labeling.

The original concept of a magic graph is due to J. Sedlacek [19,20], who defined it to be a graph with real-valued edge labelings such that (i) distinct edges have distinct nonnegative labels, and (ii) the sum of the labels of the edges incident with a particular vertex is the same for all vertices.

Let A be an additive abelian group in which we denote $A^* = A - \{0\}$, where 0 is the zero element. Any mapping l: E(G) → A^* will be called a labeling. Given a labeling of the edge set of G we define a vertex set labeling

l^+: V(G) → A as follows:

$$l^+(v) = \Sigma\{l(u,v): (u,v) \in E(G)\}$$

A graph G is called *A-magic* if there is a labeling l: E(G) → A^* such that for each vertex v the sum of the labels of the edges incident with v are all equal;

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i.e., $l^+(v) = c$ for some fixed c in A. In general, a graph G may admit more than one labeling as an A-magic graph. The following are two Klein 4 group-magic labelings of $K_4 \setminus e$ (see Figure 1)

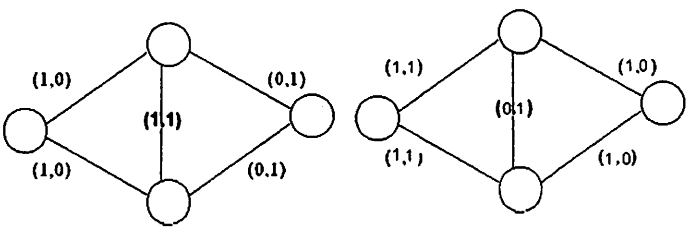
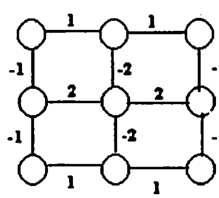


Figure 1

When the group $A \cong Z_k$, we shall refer to a Z_k-magic graph as k-magic. Graphs that are k-magic have been studied in [12,13,16,23]. Doob [3,4,5] considered A-magic graphs where A is an Abelian group. At present, given an abelian group, no general efficient algorithm is known for finding A-magic labelings for general graphs.

When a graph has an A-magic labeling and $A \cong Z$, we say the graph is Z-magic. Some special classes of Z-magic graphs have been considered in the literature (see [17, 22,23, 24]). A graph G=(V,E) is called *magic* [8, 9, 10, 11, 14, 16, 18, 26, 27] if there exits a mapping l: E(G) $\rightarrow$ {1,2,...,} that induces a constant vertex set labeling l^+: V(G) $\rightarrow$ N, $l^+(v)=\Sigma \{l(u,v) : (u,v) \text{ in } E(G)\}$.

A magic graph is called supermagic if the magic mapping l: E(G) $\rightarrow$ {1,2,...,|E|} is a bijection ([7, 24, 25, 27]). It is well-known that a graph G is magic if and only if each edge of G is contained in a 1-factor (a perfect matching) or a {1,2}-factor ([11, 18, 28]). Some general constructions of magic graphs is considered in [27] and an effective algorithm to find magic labeling is introduced in [26]. Berge [1,2] called a graph ***regularisable***, if a regular multigraph could be obtained from G by adding edges parallel to the edges of G. In fact, a graph is regularisable if and only if it is magic. Figure 2 depicits a



grid graph $P_3 x P_3$ which is Z-magic but not magic.

Figure 2.

Stanley considered Z-magic graphs in [21,22]. He pointed out that the theory of magic labelings could be put into the more general context of linear homogeneous diophantine equations. A generalization of supermagic graphs is introduced by the first author in [14]. The reader is referred to ([15, 21]) for some of the properties and conjectures of edge-magic graphs.

Given a graph G, we denote the set of all $k > 0$ such that G is k-magic by IM(G). A 1-magic graph will refer to a graph that is Z–magic. We call this set the **integer-magic spectrum** of G. Likewise, we denote AM(G)={ A ε **Ab**: G is A-magic} as the **group-magic spectrum** of G. These sets for general graphs were investigated in [15, 16].

Hartnell and Kocay [5] introduced a class of graphs that are formed by stars. For each $k > 1$, they take isomorphic copies of star St(k) with $k + 1$ vertices and connect k pairs of corresponding leaf vertices with edges. The resulting graph is called a double star DS(k). We generalize this construction to any tree T and we use the symbol DT to present the double tree of T. See Figure 3.

Let the tree T = (V,E) with $V(T) = \{v_1,\ldots, v_m\}$ and $E(T) = \{e_1,\ldots,e_{m-1}\}$, and let T* be the mirror image of T with T* = (V,E) with $V(T^*) = \{v_1^*,\ldots, v_m^*\}$ and $E(T^*) = \{e_1^*,\ldots,e_{m-1}^*\}$. We denote $V(DT) = \{v_1,\ldots, v_m, v_1^*,\ldots, v_m^*\}$ and $E(DT) = \{e_1,\ldots,e_{m-1}, e_1^*,\ldots,e_{m-1}^*\}\cup\{(v,v^*)$: v and v* are end vertices in T and T*, respectively}.See Figure 3.

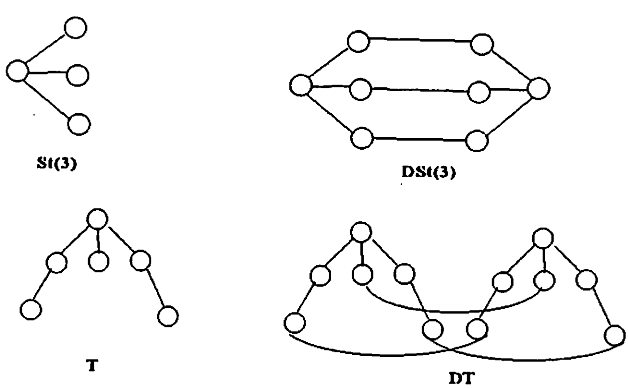


Figure 3.

There are many trees T with IM(T) = ∅, e.g., $K_{1,2}$. However, for any tree T, the double tree DT always has a non-empty IM set. In this paper we investigate IM sets of double trees and abbreviated double trees. In the last section we give a characterization of trees which are k-magic.

2. All double trees DT are Z-magic.

We have the following

Observation 1. A graph is 2-magic if and only if the degree of every vertex is of the same parity (see [15]).

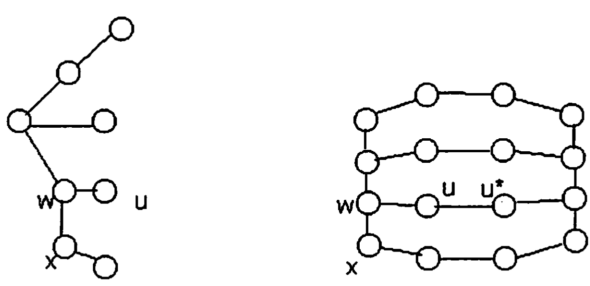
Observation 2. A double tree has vertices of even degree.

Theorem 1. The double tree DT is Z-magic for any tree T.

Proof. We show that every edge of DT is in a perfect matching, thereby proving the statement.

Choose *any* edge of the tree T and either of its vertices to be the root. Place this edge into a set S. Using a depth-first search algorithm take the next edge the algorithm yields that is not adjacent to any edge already in S. Place this edge into the set S and continue placing edges in S in this manner until the algorithm terminates. The corresponding edges of the mirror image tree T* are also placed in S. At this point in the algorithm every vertex of T is incident with an edge in S except, possibly, some end vertices. If any end vertex v of the tree T is not incident with any edge in S, we place (v,v^*) into S. This produces a perfect matching.

We have now shown *any* edge of the subgraphs T or T* in DT is in a perfect matching. To show that *any* edge (u,u^*) is in a perfect matching, we consider an adjacent edge (u,w) and any edge incident to the vertex w, say (w,x). The choice of (w,x) as an edge in a perfect matching forces the edge (u,u^*) into the set S. If there is no w in V(T) or x in V(T), we have the trivial cases DT $\cong$ K_2 or DT $\cong C_4$, respectively. See Figure 4.



Tree T

Double tree DT

Figure 4.

3. Double trees DT with N – {2} ⊂ IM(DT).

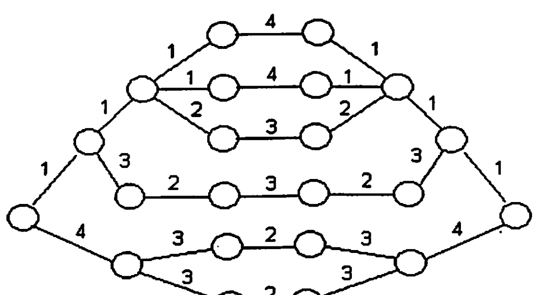
Theorem 2. If T is a tree then N – {2} ⊆ IM(DT).

Proof. We will show that the double tree DT has a labeling l: E(DT) → Z_k^*, k > 2, in which $l^+(v) = 0$ for all $v \in$ V(DT).

It is easily shown that the equation $\sum_{i=1}^{d} x_i = c$, has a solution for all $c \in Z_k^*$, such that $x_i \in Z_k^*$, $k > 2$, and $d \geq 1$ and a solution for all $c \in Z_k$, $k > 2$, and $d > 1$. Take any vertex in T with degree d greater than one, and call it the root r. Label the d edges incident with r with the labels $x_i \neq 0$ such that $\sum_{i=1}^{d} x_i = 0$. Use a depth first algorithm to visit every vertex $v \in V(T) \backslash \{$end vertices$\}$exactly once. There is exactly one edge incident with each of these vertices with a label, say t. Label the remaining incident edges with labels x_i such that $\sum_{i=1}^{\deg(v)-1} x_i = k - t \neq 0$, where $\deg(v) - 1 \geq 1$. It follows that $l^+(v) = k - t + t = 0$.

The mirror image of T, T*, is labeled identically. Let u_i and u_i^* be end vertices in T and T* respectively and (w, u_i) and (w^*, u_i^*) be the incident edges. If $l(w, u_i) = l(w^*, u_i^*) = c$, label the edge (u_i, u_i^*) with the element $k - c$ providing us with $l^+(u) = l^+(u^*) = 0$. See Figure 5.

In the case that $T \equiv K_2$, the double tree is isomorphic to C_4, the cycle with 4 edges. Label K_2 and K_2^* with the element 1 and the remaining two edges of DK_2 with the element k-1. Note that that this graph has $IM(DK_2) = N$.



A 5-magic labeling of a double tree
Figure 5

4. Double trees DT with IM(DT) = N.

.*Corollary 3.* IM(DT) = N if and only if the degree of every vertex in $V(T) \backslash \{$end vertices$\}$ is of even parity .

Proof. If every vertex in $V(T) \backslash \{$end vertices$\}$ is of even parity, then the parity of every vertex in DT is even. Label each edge with the element $1 \in Z_2$. Then $l^+(v) = 0$ for all $v \in V(DT)$.

5. Double trees DT with A-magic labelings

Corollary 4. All double trees DT have A-magic labelings for all finitely generated abelian groups except for the group $A \cong Z_2$. The double trees DT have A-magic labelings for all finitely generated abelian groups if and only if every vertex is of even parity.

Proof. Since we know that there is a k-magic labeling for all double trees when $k > 2$, by choosing labels from a subgroup B of a finitely generated group A, such that $B \cong Z_k$, $k > 2$, an A-magic labeling is found. We need only be concerned then with those abelian groups $A \cong Z_2 \times Z_2$.

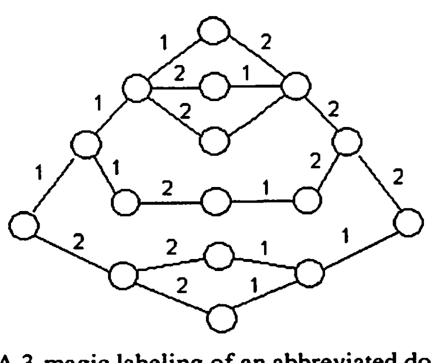
If every vertex of DT has even degree we are done. Therefore, choose a vertex in T with odd degree greater than 1 as the root r. It can be easily shown that the equation $\sum_{i=1}^{d} x_i = c$ has a solution for all $c \in Z_2 \times Z_2^*$, such that $x_i \in Z_2 \times Z_2^*$, and $d \geq 1$ and a solution for all $c \in Z_2 \times Z_2$ when $d > 1$. The proof of the corollary continues as in Theorem 2.

6. Abbreviated Double trees ADT with A-magic labelings

Let the tree $T = (V,E)$ with $V(T) = \{v_1,\ldots, v_m,u_1,\ldots,u_s\}$ where $\{u_1,\ldots,u_s\}$ are end vertices and $E(T) = \{e_1,\ldots,e_{m-1}\}$, and let T^* be the mirror image of T with $T^* = (V,E)$ with $V(T^*) = \{v_1^*,\ldots, v_m^*, u_1^*,\ldots,u_s^*\}$ and $E(T^*) = \{e_1^*,\ldots,e_{m-1}^*\}$. We denote $V(ADT) = \{v_1,\ldots, v_m, v_1^*,\ldots, v_m^*, x_1,\ldots,x_s\}$ where the end vertices in T have been identified with those in T^*, and $E(ADT) = \{e_1,\ldots,e_{m-1}, e_1^*,\ldots,e_{m-1}^*\}$. We call the resulting graph an *abbreviated double tree* ADT.

Theorem 5. If T is a tree then $N \setminus \{1,2\} \subseteq IM(ADT)$.

Proof. Label the edges of the tree T with elements from Z_k^*, $k > 2$, as in the proof of Theorem 2. Label the corresponding edges of the tree T^* with the elements $k - l(u,w) \in Z_k^*$, where $(u,w) \in E(T)$. For $v \in V(T)\setminus\{\text{end vertices}\}$ or $v \in V(T^*)\setminus\{\text{end vertices}\}$, $l^+(v) = 0$ as in the labeling in the proof of Theorem 2. For $x_i \in V(ADT)$, $l^+(x_i) = l(x_i,v_a) + l(x_i,v_a^*) = l(x_i,v_a) - l(x_i,v_a) = 0$. See Figure 6.



A 3-magic labeling of an abbreviated double tree

Figure 6

Corollary 6. If T is a tree in which IM(ADT) = N \{1} then the degree of every vertex in V(T)\{end vertices} is of even parity.

Corollary 7. IM(ADT) = N if and only if ADT ≅ C_{2n}, i.e., T ≅ P_n, the path with n edges.

Proof. Let T be a tree that is not a path. Suppose a matching exists in ADT and find such a matching. Choose a vertex r with deg(r) > 2. Name the adjacent vertices such that (r,v_1) is an edge in the matching. Choose two other edges incident with r, and call them (r,v_2) and (r,v_3). We say that an alternating path in a graph with a matching is one that alternates between an edge in the matching and an edge that is not. Find a shortest alternating path from r to r^* where the first edge in the path is (r,v_2); find another shortest alternating path from r to r^* where the first edge in the path is (r,v_3). Both of these paths have an even number of edges and are nonintersecting implying the last two edges in the paths are in the matching. But, these two edges are both incident with r^* yielding a contradiction.

7. A necessary and sufficient condition for trees to be k-magic, $k > 2$.

We now consider the problem from another direction. That is, can we find families of graphs that are k-magic?

We define a type of reducibility, k-label reducibility, $k > 1$, by the following algorithm. A graph is k-label reducible if the algorithm returns TRUE.

Algorithm for k-label reducibility

1. If a tree T ≅ K_1, or T ≅K1,s , 2<s<k,and (s-1,k)≠1, Then
   Return True.
2. Delete any and all sets of k leaves incident with the same vertex. Call the new tree T. Continue with the deletion process until no such sets of k leaves can be found.
3. If there is a vertex u of degree 2 incident with a leaf,
   Return False.
4. Let $v \in V(T)$ with deg(v) = t, $2 < t \leq k$, that is incident with exactly $t - 1$ leaves. Delete the $t - 1$ leaves, and replace the remaining edge (v,w) with $k - t + 2$ edges incident with the vertex w. Call the new tree T.
5. If T ≅ K_1, or T ≅K1,s , 2<s<k,and (s-1,k)≠1, return True
   Else go to step 2. See Figure 7.

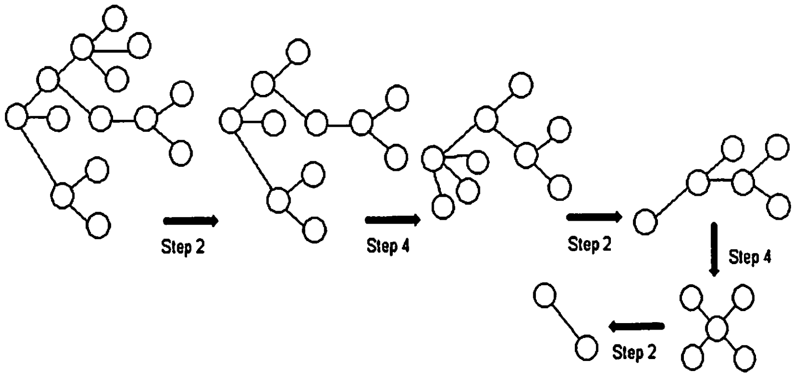
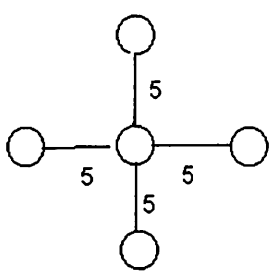


Figure 7. Algorithm for 3-label reducibility

Theorem 8. A tree is k-magic, $k > 2$, if and only if it is k-label reducible.
Proof. Assume a tree T is k-magic. We will show the algorithm for k-label reducibility returns TRUE.

We note that each of the leaves must be labeled with the same element c.

1. $T \cong K_1$ is k-magic, $k \geq 2$, and the algorithm returns TRUE. If $T \cong K_{1,s}$, $2<s<k$ is k-magic then every edge must be labeled with the same element c implying the vertex of degree s is labeled sc=c and $(s-1)c \equiv 0 \pmod{k}$. Since $c \neq 0$ then $(s-1,k) \neq 1$, and the algorithm returns TRUE. See Figure 8.



A Z_{15}- magic labeling of $K_{1,4}$

Figure 8.

2. If there are k leaves incident to the same vertex, the sum of the labels of these leaves equals 0, so they can be removed without affecting the labeling. The new tree T has a k-magic labeling. Note that if k=2, then $T \cong K_1$.

3. The algorithm could not produce a vertex u of degree 2 incident to a leaf. The leaf must be labeled with the element c, and the other edge incident with the vertex u could only be labeled with the element 0 to yield the sum c. But, 0 is not a possible choice for a label of any edge. This contradicts the assumption that T is k-magic.

4. Let $v \in V(T)$ with $deg(v) = t$, $2 < t \leq k$. Since there are exactly $t - 1$ leaves incident with v the sum of the values of the labels of these leaves is $c(t - 1)\ (mod\ k)$ implying the remaining edge (v,w) incident with v must have the label $c - c(t - 1) = c(2 - t)\ (mod\ k)$. Replacing the remaining edge with $(2 - t)\ (mod\ k)$ edges incident with the vertex then yields a new tree T with $l^{+}(w) = c$. The new tree T has a k-magic labeling.

5. With each return to Step 2 and consequently Step 4 , the diameter of T decresases, or FALSE is returned in Step 3. If a vertex u of degree 2 incident to a leaf is not found in Step 3, then the algorithm ends where T is a tree of diameter 1 or 2. As in 1) above , this is only when $T \cong K_1$ or $T \cong K_{1,s}$, 2<s<k, and $(s-1,k)\neq 1$. The algorithm returns TRUE. .

If the tree T is k-label reducible, then reversing the algorithm will yield an k-magic labeling. See Figure 9.

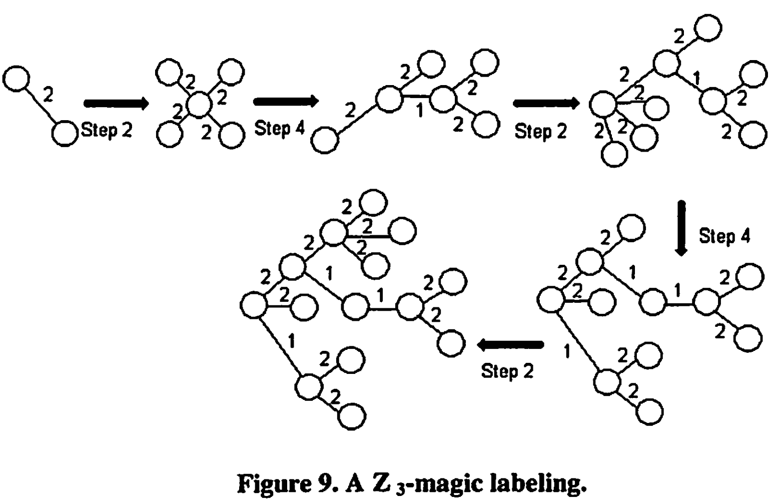


Figure 9. A Z_3-magic labeling.

References

[1] C. Berge, Regularisable graphs I, Discrete Math. **23** (1978),85-89.

[2] C. Berge, Regularisable graphs II, Discrete Math. **23** (1978),91-95

[3] M. Doob, On the construction of magic graphs, *Proc. Fifth S.E. Conference on Combinatorics*, Graph Theory and Computing(1974), 361-374.

[4] M. Doob, Generalizations of magic graphs, *Journal of Combinatorial Theory, Series B*, **17**(1974), 205-217.

[5] M. Doob, Characterizations of regular magic graphs, *Journal of Combinatorial Theory, Series B*, **25**(1978), 94-104.

[6] B.L. Hartnell and W. Kocay, On minimal neighborhood connected graphs, Discrete Math. **92** (1991), 95-105.

[7] N. Hartsfield and G. Ringel, Supermagic and antimagic graphs, *Journal of Recreational Mathematics*, **21**(1989), 107-115.

[8] R.H. Jeurissen, The incidence matrix and labelings of a graph, *Journal of Combinatorial Theory, Series B*, **30**(1981), 290-301.

[9] R.H. Jeurissen, Disconnected graphs with magic labelings, *Discrete math.*, **43**(1983), 47-53.

[10] R.H. Jeurissen, Pseudo-magic graphs, *Discrete math.*, **43**(1983),207-214.

[11] S. Jezny and M. Trenkler, Characterization of magic graphs, *Czechoslovak Mathematical Journal*, **33**(108), (1983), 435-438.

[12] M.C.Kong,S-M Lee and Hugo Sun, On magic strength of graphs, Ars Combinatoria **45** (1997), 193-200.

[13] S-M Lee, F. Saba and G. C. Sun, Magic strength of the k-th power of paths, *Congressus Numerantium*, **92**(1993), 177-184.

[14] S-M Lee, E. Seah and S.K. Tan, On edge-magic graphs, *Congressus Numerantium*, **86**(1992), 179-191

[15] S-M Lee, W.M. Pigg and T.J. Cox, On edge-magic cubic graphs conjecture, *Congressus Numerantium*, **105**(1994), 214-222.

[16] S-M Lee, Hugo Sun and Ixin Wen, On group-magic graphs, *The Journal of Combinatorial Mathematics and Combinatorial Computing* 38 (2001),197-207.

[17] S-M. Lee and Henry Wong, On group-magic spectra of kth-power of paths, Manuscript..

[18] L.Sandorova and M. Trenkler, On a generalization of magic graphs, in "Combinatorics 1987", *Proc. 7th Hungary Colloq. Eger/Hu;ng. Colloquia Mathematica Societatis Janos Bolyai,* **52**(1988), 447-452.

[19] J. Sedlacek, On magic graphs, *Math. Slov*, **26**(1976), 329-335.

[20] J. Sedlacek, Some properties of magic graphs, in "Graphs, Hypergraph, Block Syst. 1976, Proc. Symp. Comb. Anal., Zielona Gora (1976), 247-253.

[210] W. C. Shiu, P.E.B. Lam and Sin-Min Lee, Edgemagicness of the composition of a cycle with a null graph, *Congressus Numerantium* **132**,(1988),9-18.

[22] R.P. Stanley, Linear homogeneous diophantine equations and magic labelings of graphs, *Duke Math. J.*, **40**(1973), 607-632.

[23] R.P. Stanley, Magic labeling of graphs, symmetric magic squares, systems of parameters and Cohen-Macaulay rings, *Duke Math. J.*, **40**(1976), 511-531.

[24] B.M. Stewart, Magic graphs, *Canadian Journal of Mathematics,* **18**(1966),1031-1059.

[25] B.M. Stewart, Supermagic complete graphs, *Canadian Journal of Mathematics,* **19**(1967), 427-438.

[26] G. C. Sun, J. Guan and Sin-Min Lee, A labeling algorithm for magic graph, *Congressus Numerantium* **102** (1994), 129-137.

[27] G. C. Sun and S-M Lee, Constructions of magic graphs, Congressus Numerantium **103** (1994), 243-251.

[28] M. Trenkler, Some results on magic graphs, in "Graphs and other Combinatorial Topics", Proc. Of the 3rd Czechoslovak Symp., Prague, 1983, edited by M. Fieldler, Teubner-Texte zur Mathematik Band, 59(1983), Leipzig, 328-332.