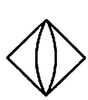
The Metamorphosis of 2-Fold Triple Systems Into 2-Fold 4-Cycle Systems

Mario Gionfriddo
Departimento di Matematica
Università di Catania
95125 Catania
ITALIA
email: gionfriddo@dipmat.unict.it

C. C. Lindner
Department of Discrete and
Statistical Sciences
Auburn University
Auburn, Alabama 36849
USA
email: lindncc@mail.auburn.edu

Abstract

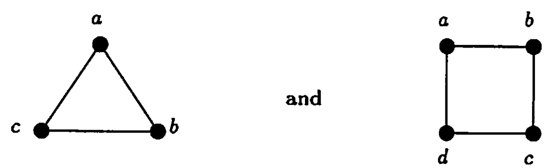
Let $c^* =$  . If we remove the double edge the result is a 4-cycle. Let (S,T) be a 2-fold triple system without repeated triples and (S,C^*) a pairing of the triples into copies of c^*. If C is the collection of 4-cycles obtained by removing the double edges from each copy of c^* and F is a reassembly of these double edges into 4-cycles, then $(S, C \cup F)$ is a 2-fold 4-cycle system. We show that the spectrum for 2-fold triple systems having a *metamorphosis* into a 2-fold 4-cycle system as described above is precisely the set of all $n \equiv 0, 1, 4$ or $9 \pmod{12} \geq 5$.

JCMCC **46** (2003), pp. 129-139

1 Introduction

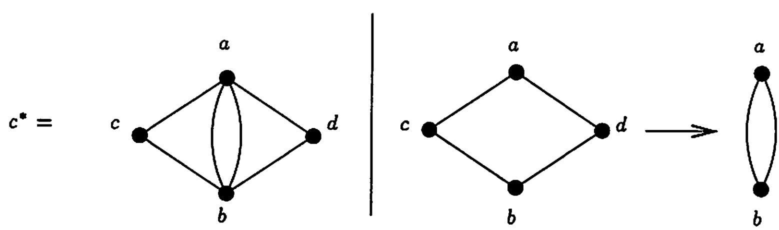
A 2-fold k-cycle system of order n is a pair (S, C), where C is a collection of edge disjoint k-cycles which partitions the edge set of $2K_n$ with vertex set S. ($2K_n$ = 2 copies of the complete undirected graph K_n.) It is well-known that the spectrum for 2-fold 3-cycle systems is the set of all $n \equiv 0$ or 1 (mod 3) and that the spectrum for 2-fold 4-cycle systems is the set of all $n \equiv 0$ or 1 (mod 4). (See [1] for example.)

In what follows we will refer to 3-cycles as triples and 2-fold 3-cycle systems as 2-fold triple systems. We will denote the cycles



by any cyclic shift of (a, b, c) or (b, a, c) *and* any cyclic shift of (a, b, c, d) or (b, a, d, c). We will denote the undirected edge with vertices a and b by $\{a, b\}$.

Now given a pair of triples (a, b, c) and (a, b, d), $c \neq d$, if we remove the double edge $2\{a, b\}$ the result is the 4-cycle (a, d, b, c).



The following problem is immediate: under what conditions can we construct a 2-fold triple system (S, T) without repeated triples so that the triples can be paired to form copies of the graph c^* (see diagram above), remove the double edges from each copy of c^*, and rearrange the deleted double edges into 4-cycles? If C is the partial 4-cycle system obtained by deleting the double edges and F is a rearrangement of the deleted edges into 4-cycles, then $(S, C \cup F)$ is a 2-fold 4-cycle system, called a *metamorphosis* of (S, T). (The algorithm of going from (S, T) to $(S, C \cup F)$ is also called a *metamorphosis*.) So using this vernacular, we wish to know the spectrum of 2-fold triple systems having a metamorphosis into a 2-fold 4-cycle system.

Example 1.1 (metamorphosis of a 2-fold triple system of order 9 into a 2-fold 4-cycle system of order 9).

Denote the graph consisting of the triples (a,b,c) and (a,b,d) by $< a,b,c,d >$, $< a,b,d,c >$, $< b,a,c,d >$, or $< b,a,d,c >$. Then $T = \{< 1,2,5,6 >, < 1,3,9,5 >, < 1,4,8,9 >, < 2,3,7,9 >, < 2,4,9,8 >, < 3,4,6,7 >, < 5,6,9,4 >, < 5,7,4,2 >, < 5,8,3,9 >, < 6,7,1,9 >, < 6,8,2,3 >, < 7,8,9,1 >\}$ is a 2-fold triple system of order 9. Remove the double edges from each of the pairs in T and reassemble into the six 4-cycles $(1,2,3,4), (1,2,4,3), (1,3,2,4), (5,6,8,7), (5,6,7,8), (5,7,6,8)$. □

Since the spectrum for 2-fold triple systems is the set of all $n \equiv 0$ or 1 (mod 3) and the spectrum for 2-fold 4-cycle systems is the set of all $n \equiv 0$ or 1 (mod 4), the *necessary* conditions for the existence of a 2-fold triple system of order n having a metamorphosis into a 2-fold 4-cycle system is $n \equiv 0, 1, 4$, or 9 (mod 12). We will show that these necessary conditions are in fact sufficient.

We will organize our work into 5 sections: one each for $n \equiv 0, 1, 4$, or 9 (mod 12) followed by a summary.

The following theorem due to Dominique Sotteau will be used repeatedly in the constructions which follow. So we might as well state it here.

Theorem 1.2 (D. Sotteau [4]) *Necessary and sufficient conditions for the complete bipartite graph $K_{m,n}$ to be partitioned into $2k$-cycles are: (i) m and n are even, (ii) $k \geq m$ and n, and (iii) $2k \mid mn$.* □

2 $\mathbf{n \equiv 0}$ (mod 12)

We begin with two examples.

Example 2.1 ($\mathbf{n = 12}$.) We use the same notation as in Example 1.1. Let $T = \{< 1,2,3,4 >, < 3,4,1,2 >, < 5,6,7,8 >, < 7,8,5,6 >, < 9,10,11,12 >, < 11,12,9,10 >, < 1,5,9,10 >, < 1,6,10,9 >, < 2,5,10,9 >, < 2,6,9,10 >, < 1,7,11,12 >, < 1,8,12,11 >, < 2,7,12,11 >, < 2,8,11,12 >, < 3,11,5,6 >, < 3,12,5,6 >, < 4,12,5,6 >, < 4,11,5,6 >, < 7,9,3,4 >, < 7,10,3,4 >, < 8,10,3,4 >, < 8,9,3,4 >\}$. Delete the double edges and reassemble into the 11 4-cycles $(1,2,6,5)$, $(1,2,5,6)$, $(1,5,2,6)$, $(1,8,2,7)$, $(1,8,2,7)$, $(3,4,12,11)$, $(3,4,11,12)$, $(3,11,4,12)$, $(7,8,10,9)$, $(7,8,9,10)$, $(7,9,8,10)$. □

Example 2.2 ($\mathbf{n = 24}$.) Let $X = \{1,2,3,4,5,6,7,8\}$ and let $(X, \circ)$ a commutative quasigroup with holes $H = \{\{1,2\},\{3,4\},\{5,6\},\{7,8\}\}$ of size 2. (See [2].) Set $S = X \times \{1,2,3\}$ and define a collection of triples T by:

(1) For each x and y belonging to different holes of H define 6 triples $((x,1),(y,1),(x\circ y,2))$, $((x,1),(y,1),(x\circ y,3))$, $((x,2),(y,2),(x\circ y,3))$, $((x,2),(y,2),(x\circ y,1))$, $((x,3),(y,3),(x\circ y,1))$, $((x,3),(y,3),(x\circ y,2))$ and place these triples in T.

(2) Let (Q,t) be the 2-fold triple system given by $Q=\{1,2,3,4,5,6\}$ and $t=\{(1,2,4),\ (1,2,6),\ (2,3,6),\ (2,3,5),\ (3,4,1),\ (3,4,6),\ (4,5,6),\ (4,5,2),\ (1,5,3),\ (1,5,6)\}$. *Note* that we can pair these triples so that the 5 paired double edges are $2\{1,2\}, 2\{2,3\}, 2\{3,4\}, 2\{4,5\}$, and $2\{5,1\}$. For each hole $h=\{i,j\}\in H$, place a copy of (Q,t) on $h\times\{1,2,3\}$ where $1,2,3,4,5,6$ are renamed $(i,1),(j,1),(i,2),(j,2),(i,3)$, and $(j,3)$ respectively, and place these triples in T.

Then (S,T) is a 2-fold triple system of order 24.

Now pair up the type (1) triples so that $((x,i),(y,i),(x\circ y,i+1))$, and $((x,i),(y,i),(x\circ y,i+2))$, are paired and remove the double edges $2\{(x,i),(y,i)\}$. Pair the type (2) triples as follows: For each hole $h=\{i,j\}\in H$ pair the triples as in (2) and remove the 5 double edges $2\{(i,1),(j,1)\}, 2\{(j,1),(i,2)\}, 2\{(i,2),(j,2)\}, 2\{(j,2),(i,3)\}, 2\{(i,3),(i,1)\}$. This gives a partial 2-fold 4-cycle system (S,C). We now rearrange the deleted edges into a collection of 4-cycles F as follows:

Define a 2-fold 4-cycle system on $X\times\{1\}$ so that $((1,1),(3,1),(5,1),(7,1))$ and $((2,1),(4,1),(6,1),(8,1))$ are 4-cycles. Define a 2-fold 4-cycle system on $X\times\{2\}$ so that $((1,2),(3,2),(5,2),(7,2))$ and $((2,2),(6,2),(4,2),(8,2))$ are 4-cycles. Define a 2-fold 4-cycle system on $(X\setminus H)\times\{3\}$ so that $((1,3),(3,3),(5,3),(7,3))$ and $((1,3),(5,3),(3,3),(7,3))$ are 4-cycles.

Now rearrange the edges $2\{(1,1),(1,3)\}\cup 2\{(3,1),(3,3)\}\cup 2\{(5,1),(5,3)\}\cup 2\{(7,1),(7,3)\}\cup\{((1,1),(3,1),(5,1),(7,1)),((1,3),(3,3),(5,3),(7,3))\}$ into four 4-cycles; rearrange the edges $2\{(2,1),(1,2)\}\cup 2\{(4,1),(3,2)\}\cup 2\{(6,1),(5,2)\}\cup 2\{(8,1),(7,2)\}\cup\{((2,1),(4,1),(6,1),(8,1)),((1,2),(3,2),(5,2),(7,2))\}$ into four 4-cycles; rearrange the edges $2\{(2,2),(1,3)\}\cup 2\{(4,2),(3,3)\}\cup 2\{(6,2),(5,3)\}\cup 2\{(8,2),(7,3)\}\cup\{((2,2),(6,2),(4,2),(8,2)),((1,3),(5,3),(3,3),(7,3))\}$ into four 4-cycles, and place these twelve 4-cycles in F.

Then $(S,C\cup F)$ is a 2-fold 4-cycle system of order 24 and is a metamorphosis of the 2-fold triple system (S,T). □

The 12n Construction.

Let $12n\geq 36$, set $S=\{1,2,3,\ldots,4n\}\times\{1,2,3\}$ and define a collection of triples T as follows:

(1) Let $H=\{\{1,2,3,4\},\{5,6,7,8\},\ldots,\{4n-3,4n-2,4n-1,4n\}\}$ be a partition of $\{1,2,3,\ldots,4n\}$ into holes of size 4. For each hole $\{a,b,c,d\}\in H$ define the 2-fold triple system of order 12 in Example 2.1 on $\{a,b,c,d\}\times\{1,2,3\}$ and place these triples in T.

(2) Let $Q = \{1, 2, 3, \ldots, 4n\}$ and let $(Q, \circ)$ be a commutative quasigroup with holes H of size 4 (see [2]). For each $a \neq b$ belonging to different holes of H place the 6 triples $\{(a,1),(b,1),(a \circ b, 2)\}$, $\{(a,1),(b,1),(a \circ b,3)\}$, $\{(a,2),(b,2),(a \circ b,3)\}$, $\{(a,2),(b,2),(a \circ b,1)\}$, $\{(a,3),(b,3),(a \circ b,1)\}$, and $\{(a,3),(b,3),(a \circ b,2)\}$ in T.

Then (S,T) is a 2-fold triple system of order $12n$.

The metamorphosis of (S,T) into a 2-fold 4-cycle system is as follows:

(1) For each hole belonging to H use the metamorphosis described in Example 2.1.

(2) For each $a \neq b$ belonging to different holes of H remove the 6 edges $2\{(a,i),(b,i)\}$, $i = 1,2,3$, from the triples in (2) in the $12n$ Construction.

Denote the resulting collection of 4-cycles by C. The edges that need to be assembled into 4-cycles are $2(K_{4n} \setminus H)$ with vertex sets $Q \times \{1\}, Q \times \{2\}$, and $Q \times \{3\}$. By Sotteau's Theorem $2(K_{4n} \setminus H)$ can be decomposed into 4-cycles. Place a copy on each of the vertex sets $Q \times \{1\}, Q \times \{2\}$, and $Q \times \{3\}$ in F.

Then $(S, C \cup F)$ is a 2-fold 4-cycle system of order $12n$. □

Lemma 2.3 *There exists a 2-fold triple system of every order $n \equiv 0$ (mod 12) having a metamorphosis into a 2-fold 4-cycle system.* □

3 n ≡ 1 (mod 12)

This is by far the easiest case.

The 2n + 1 Construction

Let $\{x_1, y_1, z_1\}, \{x_2, y_2, z_2\}, \ldots, \{x_{2n}, y_{2n}, z_{2n}\}$ be a solution of Heffter's Difference Problem. (See [3].) So $z_i = \pm(x_i + y_i)$. For each $\{x_i, y_i, z_i\}$ construct the two base blocks $(0, x_i, x_i + y_i)$ and $(0, x_i, -y_i)$. Let (S,T) be the 2-fold triple system obtained by developing the two base blocks $(0, x_i, x_i + y_i)$ and $(0, x_i, -y_i)$, $i = 1, 2, \ldots, 2n$. For each i, pair the triples $(j, x_i + j, x_i + y_i + j)$ and $(j, x_i + j, -y_i + j)$, $j = 0, 1, \ldots, 12n$, and remove the double edge $2\{j, x_i + j\}$ to obtain the 4-cycle $(j, x_i + y_i + j, x_i + j, -y_i + j)$. Denote the resulting collection of 4-cycles by C. The deleted edges have lengths $x_1, x_2, x_3, \ldots, x_{2n}$. Now partition $\{x_1, x_2, \ldots, x_{2n}\}$ into the 2-element subsets $\pi = \{\{x_1, x_2\}, \{x_3, x_4\}, \ldots, \{x_{2n-1}, x_{2n}\}\}$. For each $\{x_i, x_{i+1}\} \in \pi$ form the 4-cycle $(0, x_i, x_i + x_{i+1}, x_{i+1})$, and denote by F the collection of 4-cycles obtained by developing these 4-cycles. Then $(S, C \cup F)$ is a 2-fold 4-cycle system. □

Lemma 3.1 *There exists a 2-fold triple system of every order $n \equiv 1$ (mod 12) having a metamorphosis into a 2-fold 4-cycle system.* □

4 $n \equiv 4 \pmod{12}$

We begin with three examples.

Example 4.1 (n = 16.) Let (K_{16}, B) be the affine plane of order 4 with vertex set $S = \{1,2,3,4\} \times \{1,2,3,4\}$. We can assume that the parallel class of blocks $b_1 = \{1,2,3,4\} \times \{1\}$, $b_2 = \{1,2,3,4\} \times \{2\}$, $b_3 = \{1,2,3,4\} \times \{3\}$, and $b_4 = \{1,2,3,4\} \times \{4\}$ belongs to B. Defining a 2-fold triple system on each block of B gives a 2-fold triple system of order 16.

Now pair the triples in each block so that the double edges $2\{(1,i),(2,i)\}$, $2\{(3,i),(4,i)\}$, $i = 1,2,3,4$, are removed from each of the blocks b_1, b_2, b_3, and b_4 and the double edges $2\{(a,1),(b,2)\}$, $2\{(c,3),(d,4)\}$ are removed from each block $\{(a,1),(b,2),(c,3),(d,4)\}$. This gives a partial 2-fold 4-cycle system (S, C). It is straight forward to arrange the deleted edges into a collection of 4-cycles F, giving a 2-fold 4-cycle system $(S, C \cup F)$. □

Example 4.2 (n = 16 with a hole of size 4.) Let $Q = \{1,2,3,4,5\}$, set $S = \{\infty\} \cup (Q \times \{1,2,3\})$, and define a collection of triples T as follows:

(1) Define a 2-fold triple system on $\{\infty\} \cup \{(i,1),(i,2),(i,3)\}$, $i = 1,2,3,4,5$, and place these triples in T.

(2) Let $(Q, \circ)$ be an idempotent commutative quasigroup and for each $a \neq b \in Q$ define the 6 triples $((a,1),(b,1),(a \circ b,2))$, $((a,1),(b,1),(a \circ b,3))$, $((a,2),(b,2),(a \circ b,3))$, $((a,2),(b,2),(a \circ b,1))$, $((a,3),(b,3),(a \circ b,1))$ and $((a,3),(b,3),(a \circ b,2))$ and place these triples in T.

Then (S, T) can be considered to be a 2-fold triple system with the hole $H = \{\infty\} \cup \{(1,1),(1,2),(1,3)\}$ of size 4. The metamorphosis of (S, T) into a 2-fold 4-cycle system is as follows: For each of the blocks $\{\infty\} \cup \{(i,1),(i,2),(i,3)\}$, $i = 2,3,4,5$, pair the triples so that the double edges $2\{\infty,(i,1)\}$, $2\{(i,2),(i,3)\}$ are removed. Also remove the double edges $2\{(a,i),(b,i)\}$, $i = 1,2,3$, in (2). This gives a partial 2-fold 4-cycle system (S, C). The deleted edges are assembled into a collection F of 4-cycles in the following manner. It is straight forward to decompose $2K_6 \backslash 2\{\infty,(1,1)\}$ with vertex set $\{\infty\} \cup (Q \times \{1\})$ into 4-cycles. Place these 4-cycles in F. The remaining edges are copies of $2K_5$ with vertex sets $Q \times \{2\}$ and $Q \times \{3\}$ and the 4 double edges $2\{(2,i),(3,i)\}$, $i = 2,3,4,5$. There is no difficulty in partitioning these edges into 4-cycles. Then $(S, C \cup F)$ is a 2-fold 4-cycle system of order 16 with the hole $H = (\{\infty\} \cup \{(1,1),(2,1),(3,1)\})$ of size 4. □

Example 4.3 (n = 28.) Let (Z_7, P) be the cyclic 2-fold triple system of order 7 with base blocks $(0,1,3)$ and $(0,1,5)$. Set $S = Z_7 \times \{1,2,3,4\}$ and define a collection of triples T on S by:

(1) For each triple $t = (i, j, k) \in P$, let $(\{i, j, k\} \times \{1, 2, 3, 4\}, G(t), B(t))$ be a transversal design with groups $G(t) = \{\{i\} \times \{1, 2, 3, 4\}, \{j\} \times \{1, 2, 3, 4\}, \{k\} \times \{1, 2, 3, 4\}\}$ and place the blocks (= triples) of $B(t)$ in T.

(2) For each $x \in Z_7$ define a 2-fold triple system on $\{x\} \times \{1, 2, 3, 4\}$ and place these triples in T.

Then (S, T) is a 2-fold triple system of order 28.

Now pair the type (1) triples so that $((i, x), (i + 1, y), (i + 3, z))$ and $((i, x), (i+1, y), (i+5, z))$ are paired for all $i \in Z_7$ and remove the 16 double edges $2\{(i, x), (i + 1, y)\}$ for all $i \in Z_7$. Pair the type (2) triples as follows: For each 2-fold triple system on $\{i\} \times \{1, 2, 3, 4\}$ pair the triples so that the 2 double edges $2\{(i, 1), (i, 2)\}$ and $2\{(i, 3), (i, 4)\}$ are removed. This gives a partial 2-fold 4-cycle system (S, C). We rearrange the deleted edges into a collection of 4-cycles F as follows: Let $(K_{4,4}; B(i, i + 1))$ be the bipartite 4-cycle system with parts $\{i\} \times \{1, 2, 3, 4\}$ and $\{i + 1\} \times \{1, 2, 3, 4\}$ where $B(i, i{+}1) = \{((i, 1), (i{+}1, 1), (i, 2), (i{+}1, 2)), ((i, 3), (i{+}1, 3), (i, 4), (i{+}1, 4)), ((i, 1), (i{+}1, 3), (i, 2), (i{+}1, 4)), ((i, 3), (i{+}1, 1), (i, 4), (i{+}1, 2))\}$. Decompose $2B(i, i + 1) \cup \{2\{(i, 1), (i, 2)\} \cup 2\{(i + 1, 3), (i + 1, 4)\}\}$ into 9 4-cycles as follows: $2\{B(i, i{+}1) \backslash ((i, 1), (i{+}1, 3), (i, 2), (i{+}1, 4))\} \cup \{((i, 1), (i{+}1, 3), (i{+}1, 4), (i, 2)), ((i, 1), (i{+}1, 3), (i, 2), (i{+}1, 4)), ((i, 1), (i, 2), (i{+}1, 3), (i{+}1, 4))$. For each $(i, i + 1)$, $i \in Z_7$, place these 9 4-cycles in F.

Then $(S, C \cup F)$ is a 2-fold 4-cycle system, completing the metamorphosis of (S, T). □

The 12n + 4 Construction.

Let $12n + 4 \geq 40$ and let $S = \{1, 2, 3, 4\} \cup (Q \times \{1, 2, 3\})$, where Q is a set of size $4n$. Let $(Q, \circ)$ be a commutative quasigroup with holes $H = \{h_1, h_2, \ldots, h_n\}$ of size 4. Define a collection of triples T as follows:

(1) Define a copy of the 2-fold triple system of order 16 in Example 4.1 on $\{1, 2, 3, 4\} \cup (h_1 \times \{1, 2, 3\})$ and place these triples in T.

(2) For each hole $h_i \in H$, $i = 2, 3, \ldots, n$, place a copy of the 2-fold triple system in Example 4.2 on $\{1, 2, 3, 4\} \cup (h_i \times \{1, 2, 3\})$, where $\{1, 2, 3, 4\}$ is the hole of size 4, and place these triples in T.

(3) For each $a \neq b \in Q$ in different holes of H, place the 6 triples $\{(a, 1), (b, 1), (a \circ b, 2)\}$, $\{(a, 1), (b, 1), (a \circ b, 3)\}$, $\{(a, 2), (b, 2), (a \circ b, 3)\}$, $\{(a, 2), (b, 2), (a \circ b, 1)\}$, $\{(a, 3), (b, 3), (a \circ b, 1)\}$, and $\{(a, 3), (b, 3), (a \circ b, 2)\}$ in T.

The metamorphosis of (S, T) into a 2-fold 4-cycle system is as follows: For triples of types (1) and (2) use the pairing of triples in Examples 4.1 and 4.2 and remove the double edges. For triples of type (3) remove the double edges $2\{(a, i), (b, i)\}$, $i = 1, 2, 3$. This gives a partial 2-fold 4-cycle system

(S, C). Now reassemble the deleted edges into the following collection F of 4-cycles. Reassemble the deleted edges in (1) and (2) according to the metamorphosis in Examples 4.1 and 4.2 and place these 4-cycles in F. Sotteau's Theorem guarantees that the deleted edges in (3) can be reassembled into 4-cycles. Place these 4-cycles in F. Then $(S, C \cup F)$ is a 2-fold 4-cycle system. □

Lemma 4.4 *There exists a 2-fold triple system of every order $n \equiv 4 \pmod{12}$ having a metamorphosis into a 2-fold 4-cycle system.* □

5 $n \equiv 9 \pmod{12}$

The recursive construction for $n \equiv 9 \pmod{12}$ begins with $n = 45$, so we will need examples for $n = 9, 21$, and 33. Since we already have an example for $n = 9$ (Example 1.1), what remains are examples for $n = 21$ and 33.

Example 5.1 (n = 21.) Let $Q = \{1,2,3,4,5,6,7\}$ and let $(Q, \circ_1)$ and $(Q, \circ_2)$ be the following pair of commutative quasigroups.

$\circ_1$	1	2	3	4	5	6	7
1	1	3	2	7	6	5	4
2	3	2	1	5	4	7	6
3	2	1	3	6	7	4	5
4	7	5	6	4	2	3	1
5	6	4	7	2	5	1	3
6	5	7	4	3	1	6	2
7	4	6	5	1	3	2	7

$\circ_2$	1	2	3	4	5	6	7
1	1	3	2	6	7	4	5
2	3	2	1	7	6	5	4
3	2	1	3	5	4	7	6
4	6	7	5	4	3	1	2
5	7	6	4	3	5	2	1
6	4	5	7	1	2	6	3
7	5	4	6	2	1	3	7

Set $S = Q \times \{1,2,3\}$ and define a collection of triples T as follows:

(1) Define Example 1.1 on $\{1,2,3\} \times \{1,2,3\}$.

(2) For each $a \neq b \in Q$, not both in $\{1,2,3\}$, define the 2 triples $((a,1),(b,1),(a \circ_1 b, 2))$ and $((a,1),(b,1),(a \circ_2 b, 2))$ and place these triples in T.

(3) Let $\alpha = (123)(4567)$ and for each $a \neq b \in Q$, not both in $\{1,2,3\}$, define the 2 triples $((a,2),(b,2),(a \circ_1 b, 3))$ and $((a,2),(b,2),((a \circ_1 b)\alpha, 3))$ and place these triples in T.

(4) For each $a \neq b \in Q$, not both in $\{1,2,3\}$, define the 2 triples $((a,3),(b,3),(a \circ_1 b,1))$ and $((a,3),(b,3),((a \circ_1 b)\alpha^{-1},1))$ and place these triples in T.

(5) For each $a \in Q \setminus \{1,2,3\}$, place the 2 triples $((a,1),(a,2),(a,3))$ and $((a,1),(a,2),(a\alpha,3))$ in T.

Then (S,T) is a 2-fold triple system of order 21.

The metamorphosis is the following:

(i) Use the metamorphosis in Example 1.1 for the type (1) triples.

(ii) Pair the type (2), (3), and (4) triples so that the double edges $2\{(a,i),(b,i)\}$ are removed.

(iii) Pair the type (5) triples so that the double edges $2\{(4,1),(4,2)\}$, $2\{(5,1),(5,2)\}$, $2\{(6,1),(6,2)\}$, and $2\{(7,1),(7,2)\}$ are removed.

In what follows K_7 has vertex set Q and K_3 has vertex set $\{1,2,3\}$. We will need the following two examples: (a) a decomposition of $E(2K_7) \setminus E(2K_3)$ into nine 4-cycles, and (b) a decomposition of $E(2K_7) \setminus E(2K_3)$ into eight 4-cycles with leave the two double edges $2\{4,5\}$ and $2\{6,7\}$.

(a) $\{(3,4,7,6),(3,5,6,7),(3,4,5,7),(3,5,4,6),(4,5,6,7),(1,4,2,5),$ $(1,4,2,5),(1,6,2,7),(1,6,2,7)\}$.

(b) $\{(3,4,6,5),(3,7,4,6),(3,4,7,5),(3,6,5,7),(1,4,2,5),(1,4,2,5),$ $(1,6,2,7),(1,6,2,7)\}$.

Now, rearrange the deleted edges $E(2K_7) \setminus E(2K_3)$ with vertex set $Q \times \{3\}$ into a copy of (a). Rearrange the deleted edges $E(2K_7) \setminus E(2K_3)$ with vertex sets $Q \times \{1\}$ and $Q \times \{2\}$ into copies of (b) with leaves $2\{(4,1),(5,1)\}$, $2\{(6,1),(7,1)\}$, $2\{(4,2),(5,2)\}$, and $2\{(6,2),(7,2)\}$. It is straightforward to arrange these edges along with the deleted edges in (iii) into four 4-cycles, completing the metamorphosis. □

Example 5.2 (n = 33.) Let X be a set of size 8 and $(\{1,2,3,4\},F)$ the 2-fold triple system of order 4. Set $S = \{\infty\} \cup (X \times \{1,2,3,4\})$ and define a collection of triples T by:

(1) For each $i = 1,2,3,4$, define a copy of the 2-fold triple system of order 9 in Example 1.1 on $\{\infty\} \cup (X \times \{i\})$ and place these triples in T.

(2) For each triple $t = \{i,j,k\} \in F$, let $(X \times \{i,j,k\}, G(t), B(t))$ be a $TD(8,3)$ with groups $G(t) = \{X \times \{i\}, X \times \{j\}, X \times \{k\}\}$ and place the blocks (= triples) of $B(t)$ in T.

Then (S,T) is a 2-fold triple system of order 33.

Now pair off the type (1) triples as in Example 1.1 and use the metamorphosis in Example 1.1.

Pair up the type (2) triples so that $((x,1),(y,2),(z,3))$ and $((x,1),(y,2),(w,4))$ are paired *and* $((x,3),(y,4),(u,1))$ and $((x,3),(y,4),(v,2))$ are paired and remove the double edges $2\{(x,1),(y,2)\}$ and $2\{(x,3),(y,4)\}$, $x,y \in X$. Sotteau's Theorem guarantees that these double edges can be reassembled into 4-cycles. If (S,C) is the partial 4-cycle system obtained by deleting the double edges from (1) and (2) and F is the reassembly of these double edges into 4-cycles as described above, then $(S, C \cup F)$ is a 2-fold 4-cycle system. □

The 12n + 9 Construction. Let $12n+9 \geq 45$ and let $S = Z \cup (Q \times \{1,2,3\})$, where Q is a set of size $4n \geq 12$ and Z is a set of size 9. Let $(Q,\circ)$ be a commutative quasigroup with holes $H = \{h_1, h_2, h_3, \ldots, h_n\}$ of size 4. Define a collection of triples T as follows:

(1) Define a copy of the 2-fold triple system in Example 5.1 on $Z \cup (h_1 \times \{1,2,3\})$ and place these triples in T.

(2) Notice that the 2-fold triple system of order 21 in Example 5.1 has a subsystem of order 9. We can consider the vertex set to be Z. For each $h_i \in H, i = 2,3,\ldots,n$, place a copy of Example 5.1 on $Z \cup (h_i \times \{1,2,3,\})$, where Z is a hole of size 9.

(3) For each $a \neq b \in Q$ in different holes of H, place the 6 triples $\{(a,1),(b,1),(a\circ b,2)\}$, $\{(a,1),(b,1),(a\circ b,3)\}$, $\{(a,2),(b,2),(a\circ b,3)\}$, $\{(a,2),(b,2),(a\circ b,1)\}$, $\{(a,3),(b,3),(a\circ b,1)\}$, and $\{(a,3),(b,3),(a\circ b,2)\}$ in T.

Then (S,T) is a 2-fold triple system of order $12n+9$. The metamorphosis of (S,T) into a 2-fold 4-cycle system is as follows: for triples of types (1) and (2) use the pairing in Example 5.1 and remove these double edges. For triples of type (3) remove the double edges $2\{(a,i),(b,i)\}$, $i = 1,2,3$. This gives a partial 2-fold 4-cycle system (S,C). Now reassemble the deleted edges into the following collection F of 4-cycles. Reassemble the deleted edges in (1) and (2) according to Example 5.1 and place these 4-cycles in F. Use Sotteau's Theorem to reassemble the deleted edges in (3) into 4-cycles. Place these 4-cycles in F. Then $(S, C \cup F)$ is a 2-fold 4-cycle system, completing the metamorphosis. □

Lemma 5.3 *There exists a 2-fold triple system of every order $n \equiv 9 \pmod{12}$ having a metamorphosis into a 2-fold 4-cycle system.* □

6 Summary

Collecting together Lemmas 2.3, 3.1, 4.4, and 5.3 gives the following theorem.

Theorem 6.1 *The spectrum for 2-fold triple systems having a metamorphosis into a 4-cycle system is precisely the set of all* $n \equiv 0, 1, 4,$ *and* 9 (mod 12). □

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