Hinge Systems – Spectra and Embeddings

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Abstract

We define the B_2 block-intersection graph of a balanced incomplete block design $(V, \mathcal{B})$ having order n, block size k, and index λ, or BIBD(n, k, λ), to be the graph with vertex set $\mathcal{B}$ in which two vertices are adjacent if and only if their corresponding blocks have exactly two points of V in common. We define an undirected (resp. directed) hinge to be the multigraph with four vertices which consists of two undirected (resp. directed) 3-cycles which share exactly two vertices in common. An undirected (resp. directed) hinge system of order n and index λ is a decomposition of λK_n (resp. λK_n^*) into undirected (resp. directed) hinges. In this paper, we show that each component of the B_2 block-intersection graph of a simple BIBD$(n, 3, 2)$ is 2-edge-connected; this enables us to decompose pure Mendelsohn triple systems and simple 2-fold triple systems into directed and undirected hinge systems, respectively. Furthermore, we obtain a generalisation of the Doyen-Wilson theorem by giving necessary and sufficient conditions for embedding undirected (resp. directed) hinge systems of order n in undirected (resp. directed) hinge systems of order v. Finally, we determine the spectrum for undirected hinge systems for all indices $\lambda \geq 2$ and for directed hinge systems for all indices $\lambda \geq 1$.

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1 Introduction

We define an undirected (resp. directed) *hinge* to be the undirected (resp. directed) multigraph on four vertices which consists of two undirected (resp. directed) 3-cycles which share exactly two vertices. Figure 1 gives two conceivable examples of a directed hinge, but for the purposes of this paper we will consider only directed hinges of the type on the left of the figure. For the undirected cases, we often omit the word 'undirected' when referring to graphs and graph decompositions.

A *hinge system* of order n and index λ is a decomposition of the edges of λK_n, the complete multigraph of order n having λ edges between each pair of vertices, into hinges. Similarly, a *directed hinge system* of order n and index λ is a decomposition of the arcs of λK_n^*, the complete directed multigraph on n vertices having λ arcs from each vertex to each other vertex, into directed hinges.

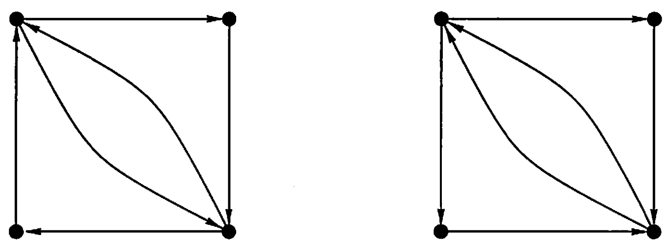

Figure 1: Examples of Directed Hinges

A *balanced incomplete block design* of order n, having block size k and index λ, or BIBD(n, k, λ), is a pair $(V, \mathcal{B})$, where V is a set of n points and $\mathcal{B}$ is a collection of k-subsets of V known as *blocks* such that every pair of points in V occurs in exactly λ blocks of $\mathcal{B}$. Any BIBD$(n, 3, \lambda)$ is also referred to as a *λ-fold triple system* (or TS(n, λ)) and corresponds to a decomposition of λK_n into 3-cycles (or triples). The *block-intersection graph* of a BIBD $(V, \mathcal{B})$ is the graph with vertex set $\mathcal{B}$ in which two vertices are adjacent if and only if their corresponding blocks share at least one point of V. We define the *B_i block-intersection graph* of $(V, \mathcal{B})$ to be the graph with vertex set $\mathcal{B}$ in which two vertices are adjacent if and only if their corresponding blocks share exactly i points in common.

A *Mendelsohn triple system* of order n, MTS(n), is an ordered pair (V, C), where V is a set of n points and C is a collection of cyclic triples defined on V such that each ordered pair of vertices in V is contained in exactly one cyclic triple of C. A cyclic triple (x, y, z) covers the ordered pairs (x, y), (y, z), and (z, x). Many times a MTS(n) is thought of as a decomposition of the edges of the complete directed graph K_n^* into directed 3-cycles. Removing the orientation from the cycles in the MTS(n) produces a triple system of order n and index 2 known as the *underlying 2-fold triple system*. A MTS(n) is said to be *pure* if its underlying 2-fold triple system contains no repeated triples, while a triple system is called *simple* if it contains no repeated triples.

We say that a directed hinge system of order n is *derived* from a MTS(n), (V, C), if the directed 3-cycles of C can be paired together to form a directed hinge system.

In this paper we focus on establishing the spectrum for directed hinge systems of all indices, as well as for undirected hinge systems of all indices $\lambda \geq 2$. The first of our main results applies to directed hinge systems:

Theorem 1.1 *A directed hinge system of order n and index 1 can be derived from any pure MTS(n) for all $n \equiv 0$ or 1 (mod 3), where $n \geq 4$, and $n \neq 6$.*

Theorem 1.2 *A directed hinge system of order n and index λ exists if and only if $n \geq 4$ and*

(a) $\lambda \equiv 0$ (mod 3),

(b) $\lambda \equiv 1$ (mod 3) and $n \equiv 0$ or 1 (mod 3) (but if $\lambda = 1$ then $n \neq 6$), or

(c) $\lambda \equiv 2$ (mod 3) and $n \equiv 0$ or 1 (mod 3),

An undirected (resp. directed) hinge system (V, B) of order n is said to be *embedded* in an undirected (resp. directed) hinge system (V', B') of order v if $V \subseteq V'$ and $B \subseteq B'$. A number of Doyen-Wilson type results have been established for various combinatorial objects: necessary and sufficient conditions for the embedding of Steiner triple systems of all indices [8, 20], Mendelsohn triple systems [18], pure Mendelsohn triple systems [16, 19], extended triple systems of all indices [10, 14], bowtie systems [2], and 5-cycle systems [6] have been obtained. We add to this collection by proving the following result concerning embedding hinge systems:

Theorem 1.3 *Let $n \geq 4$. A directed hinge system of order n and index 1 can be embedded in a directed hinge system of order v and index 1 if and only if*

(a) $v \equiv 0$ or 1 (mod 3),

(b) $v = n$ or $v \geq 2n + 1$, and

(c) $v \neq 6$.

Furthermore, conditions (a) and (b) are necessary and sufficient for embedding undirected hinge systems of order n *and index* 2 *in undirected hinge systems of order* v *and index* 2.

After presenting our results for directed hinge systems, we proceed to consider undirected hinge systems. Notice that an undirected hinge contains an edge of multiplicity 2, and so there can be no undirected hinge system with index 1. We establish the spectrum for undirected hinge systems:

Theorem 1.4 *An undirected hinge system of order* n *and index* λ *exists if and only if* $\lambda \geq 2$, $n \geq 4$, *and*

(a) $n \equiv 1$ *or* 9 *(mod* 12*) if* $\lambda \equiv 1$ *or* 5 *(mod* 6*),*

(b) $n \equiv 0$ *or* 1 *(mod* 3*) if* $\lambda \equiv 2$ *or* 4 *(mod* 6*),*

(c) $n \equiv 1$ *(mod* 4*) if* $\lambda \equiv 3$ *(mod* 6*), or*

(d) no additional restrictions on n *if* $\lambda \equiv 0$ *(mod* 6*).*

2 Preliminary Results

We start with a lemma found in [3] which is a corollary of Tutte's famous 1-factor theorem.

Lemma 2.1 *Any* 3*-regular graph without cut edges has a* 1*-factor.*

The following lemma is important when embedding (directed) hinge systems.

Lemma 2.2 ([16]) *Any pure Mendelsohn triple system of order* n *can be embedded in a pure Mendelsohn triple system of order* v *if and only if* $n, v \equiv 0$ *or* 1 *(mod* 3*) and* $v \geq 2n+1$, $v \geq 4$, *and* $v \neq 6$.

Let $K_v \setminus K_n$ (resp. $K_v^* \setminus K_n^*$) denote the complete (directed) graph of order v with the edges of a complete (directed) graph of order n removed. Lemma 2.2 leads us to the following proposition.

Proposition 2.3 *Let* $n \geq 4$. *Any MTS*(n) *having exactly* t *repeated triples can be embedded in a MTS*(v) *without introducing any more repeated triples if and only if* $n, v \equiv 0$ *or* 1 *(mod* 3*) and* $v = n$ *or* $v \geq 2n+1$, $v \neq 6$.

Proof. Necessity: Clearly, it is necessary that $n, v \equiv 0$ or $1 \pmod 3$ and $n, v \neq 6$, since a MTS can only exist for these values [12]. Let (V, C) and (V', C') be a MTS(n) and a MTS(v), respectively, such that $V \subseteq V'$ and $C \subseteq C'$. Now every cyclic triple which contains a point in V but is not entirely contained in V must contain two points in $V' \setminus V$. Therefore, for every two arcs which have one endpoint in V and the other in $V' \setminus V$, there must be an arc whose endpoints both lie in $V' \setminus V$. So the total number of arcs which have exactly one endpoint in V and exactly one endpoint in $V' \setminus V$ must be at most twice the number of arcs with both endpoints in $V' \setminus V$. This implies that either $v = n$ or $n(v-n) \leq 2\binom{v-n}{2}$, so $v = n$ or $v \geq 2n+1$.

Sufficiency: Since embedding a MTS(n) in a MTS(v) can be thought of as finding a decomposition of the arcs of $K_v^* \setminus K_n^*$ into cyclic triples, we find that this decomposition does not at all depend upon what the triples of the MTS(n) are. Therefore, since a pure MTS(n) exists for all $n \equiv 0$ or $1 \pmod 3$, where $n \geq 4$ and $n \neq 6$ [1], we find that Lemma 2.2 allows us to embed any MTS(n) having exactly t repeated triples in a MTS(v) without introducing any more repeated triples if $n, v \equiv 0$ or $1 \pmod 3$ and $v \geq 2n+1$, $v \geq 4$, and $v \neq 6$. □

Also, we will use the following result of Shen:

Lemma 2.4 ([17]) *Let (V, B) be a simple λ-fold triple system of order n. Then (V, B) can be embedded in a simple λ-fold triple sytem of order $v > n$ if and only if $n \geq \lambda + 2$, $\lambda(v-1) \equiv 0 \pmod 2$, $\lambda v(v-1) \equiv 0 \pmod 6$, and $v \geq 2n+1$.*

For $\lambda = 2$, we have the following corollary:

Corollary 2.5 *Let $n \geq 4$. Any 2-fold triple system of order n having exactly t repeated triples can be embedded in a 2-fold triple system of order v without introducing any more repeated triples if and only if $v \equiv 0$ or $1 \pmod 3$ and $v = n$ or $v \geq 2n+1$.*

Proof. We can think of embedding a TS(n, 2) in a TS(v, 2) as a decomposition of the edges of $2K_v \setminus 2K_n$ into triples. The decomposition does not depend upon what the triples of the TS(n, 2) are, so we are allowed to replace the original TS(n, 2) with a simple TS(n, 2). The result then follows directly from Lemma 2.4. □

3 Directed Hinge Systems

Theorem 3.1 *A directed hinge system of order n and index 1 can be derived from any pure MTS(n) for all $n \equiv 0$ or $1 \pmod 3$, where $n \geq 4$, and $n \neq 6$.*

Proof. Let (V, C) be a pure MTS(n). These exist for all positive integers $n \equiv 0$ or 1 (mod 3), where $n \neq 1, 3$, or 6 [1]. Let $(V, \mathcal{B})$ be the underlying 2-fold triple system of (V, C), and let G be its block-intersection graph. Every point $x \in V$ induces a clique in G since all vertices corresponding to blocks in $\mathcal{B}$ which contain a point x are adjacent in G. Now consider G_2, the B_2 block-intersection graph of $(V, \mathcal{B})$. Since (V, C) is a pure MTS(n), it follows that each vertex of G has degree 3 in G_2, for each vertex $\{x, y, z\}$ will be adjacent to three other vertices which represent triples, and each of these triples contains exactly one of the pairs (x, y), (x, z), and (y, z). Therefore, G_2 is a 3-regular spanning subgraph of G.

Consider the clique in G induced by the vertex $x \in V$. This clique contains some edges which also occur in G_2. These edges of G_2 form a 2-factor within the x clique, since every vertex $\{x, y, z\}$ is adjacent in G_2 to some vertex $\{x, y, a\}$ (the unique neighbor which contains the pair (x, y)) and to another vertex $\{x, z, b\}$ (the unique neighbor which contains the pair (x, z)). Note also that the third neighbor of $\{x, y, z\}$ in G_2 contains the pair (y, z) but not the vertex x since (V, C) is pure. Thus, every edge in G_2 is contained in a cycle in G_2 within some clique of G. Therefore, no component of G_2 contains a cut edge, so by Lemma 2.1, G_2 contains a 1-factor.

Let F be a 1-factor in G_2. For every edge $\{\{a, b, c\}, \{a, d, b\}\} \in F$, let the triples $\{a, b, c\} \in C$ and $\{a, d, b\} \in C$ form a directed hinge. Clearly, since the cycles in C partition the arcs of K_n^*, we have a partition of the arcs of K_n^* into directed hinges. Therefore, if $n \equiv 0$ or 1 (mod 3), where $n \neq 1, 3$ or 6, we can derive a directed hinge system of order n from a pure MTS(n). □

Corollary 3.2 *An undirected hinge system of order n and index 2 can be derived from any pure MTS(n) for all $n \equiv 0$ or 1 (mod 3), where $n \geq 4$, and $n \neq 6$.*

Proof. To derive an undirected hinge system of order n and index 2, we may simply derive a directed hinge system of order n and index 1 from a pure MTS(n) and subsequently remove the orientation from the arcs of the directed hinges. □

Theorem 3.3 *A directed hinge system of order n and index λ exists if and only if $n \geq 4$ and*

(a) $\lambda \equiv 0$ (mod 3),

(b) $\lambda \equiv 1$ (mod 3) and $n \equiv 0$ or 1 (mod 3) (but if $\lambda = 1$ then $n \neq 6$), or

(c) $\lambda \equiv 2$ (mod 3) and $n \equiv 0$ or 1 (mod 3),

Proof. Necessity. Clearly, since a directed hinge contains four vertices, we must have $n \geq 4$. Furthermore, the number of arcs in a directed hinge must divide the number of arcs in λK_n^*. Hence it is necessary that $6 \mid \lambda n(n-1)$, which is always true when $\lambda \equiv 0 \pmod 3$, but for $\lambda \equiv 1$ or $2 \pmod 3$ is only true when $n \equiv 0$ or $1 \pmod 3$.

In addition, the existence of a directed hinge system of order 6 and index 1 would imply the existence of a MTS(6) (simply break each hinge apart to produce pairs of cyclic triples). Since there is no MTS(6), there cannot be a directed hinge system of order 6 and index 1.

Sufficiency. By Theorem 3.1, a directed hinge system of order n and index 1 can be derived from any pure MTS(n) for all $n \equiv 0$ or $1 \pmod 3$, where $n \geq 4$, and $n \neq 6$.

We must now consider indices $\lambda > 1$. If $\lambda \geq 2$, $n \equiv 0$ or $1 \pmod 3$, and $n \neq 6$ then we may obtain a directed hinge decomposition of λK_n^* by taking λ copies of a directed hinge decomposition of K_n^*.

Two cases now remain: $n = 6$ with $\lambda > 1$, and $\lambda \equiv 0 \pmod 3$ with $n \equiv 2 \pmod 3$.

Consider the case in which $n = 6$ and $\lambda > 1$. In the appendix we present directed hinge decompositions of $2K_6^*$ and $3K_6^*$. Thus, if λ is even such that $\lambda = 2k$, we can obtain a directed hinge decomposition of λK_6^* by taking k copies of a directed hinge decomposition of $2K_6^*$. And if λ is odd such that $\lambda = 2k + 3$, we can obtain a directed hinge decomposition of λK_6^* by taking one copy of a directed hinge decomposition of $3K_6^*$ and k copies of a directed hinge decomposition of $2K_6^*$.

Now consider the case in which $\lambda \equiv 0 \pmod 3$ and $n \equiv 2 \pmod 3$. Given a directed hinge decomposition of $3K_n^*$, we could obtain a directed hinge decomposition of λK_n^* by simply taking $\lambda/3$ copies of a hinge decomposition of $3K_n^*$. We now complete the proof by presenting a directed hinge decomposition of $3K_n^*$. As ingredients, we use directed hinge decompositions of $3K_5^*$, $3K_8^*$, $3K_{11}^*$, $3K_{20}^*$, and $3(K_5^* \setminus K_2^*)$, all of which are provided in the appendix. We therefore also assume that $n = 14, 17$ or $n \geq 23$.

Let $v = (n-2)/3$ and let $S = \{s_1, s_2, \ldots, s_v\}$. Let $V = (S \times \{0,1,2\}) \cup \{\infty_1, \infty_2\}$ represent the vertex set of $3K_n^*$. On the $3K_5^*$ subgraph of $3K_n^*$ induced by the vertex set $(\{s_1\} \times \{0,1,2\}) \cup \{\infty_1, \infty_2\}$ place a directed hinge decomposition of $3K_5^*$. On each $3(K_5^* \setminus K_2^*)$ subgraph that is induced by the vertex set $(\{s_i\} \times \{0,1,2\}) \cup \{\infty_1, \infty_2\}$, for $2 \leq i \leq v$, in which the vertex set $\{\infty_1, \infty_2\}$ induces the $3K_2^*$ whose edges are missing, we place a directed hinge decomposition of $3(K_5^* \setminus K_2^*)$. Now let L be an idempotent self-orthogonal latin square of order v; since $n \notin \{8, 11, 20\}$, we have $v \notin \{2, 3, 6\}$ and so such a latin square is known to exist [4, 5, 7]. We assume L to be on the symbol set $\{1, 2, \ldots, v\}$ and we let $L(i,j)$ denote the symbol contained in cell (i,j) of L. For each cell (i,j) for which $i < j$, we take 3 copies of the directed hinge having the two ordered triples $((s_i, z), (s_j, z), (s_{L(i,j)}, z+1))$

and $((s_i, z), (s_{L(j,i)}, z+1), (s_j, z))$, for $z \in \{0,1,2\}$, where we take addition modulo 3. Notice that arcs such as $((s_i, 0), (s_j, 1))$ will be used by hinges that are generated from the cells (i,k) and (k,i) where either $L(i,k) = j$ or $L(k,i) = j$. □

Corollary 3.4 *If $n \equiv 0$ or 1 (mod 3) and $n \geq 4$, then there exists an undirected hinge system of order n and index 2.*

Proof. If $n \equiv 0$ or 1 (mod 3), $n \geq 4$, and $n \neq 6$, then we obtain an undirected hinge system of order n and index 2 by simply removing the orientation on the arcs of a directed hinge system of order n and index 1.

Now consider the case $n = 6$. Let a hinge consisting of the triples $\{x, y, z\}$ and $\{x, y, a\}$ be denoted by $(x, y, z) - (x, y, a)$. Let $V = \{1, \ldots, 6\}$, and let $W = \{(1,4,2)-(1,4,5), (5,6,2)-(5,6,4), (1,6,2)-(1,6,3), (3,5,1)-(3,5,2), (3,4,2)-(3,4,6)\}$. Then the hinges in W form an undirected hinge system of order 6. □

Now that we have established the spectrum for directed hinge systems of index 1, we focus on finding necessary and sufficient conditions for embedding (directed) hinge systems in larger (directed) hinge systems:

Theorem 3.5 *Let $n \geq 4$. A directed hinge system of order n and index 1 can be embedded in a directed hinge system of order v and index 1 if and only if*

(a) $v \equiv 0$ or 1 (mod 3),

(b) $v = n$ or $v \geq 2n + 1$, and

(c) $v \neq 6$.

Furthermore, conditions (a) and (b) are necessary and sufficient for embedding undirected hinge systems of order n and index 2 in undirected hinge systems of order v and index 2.

Proof. Throughout this proof, whenever an index is not explicitly stated, it is assumed that the index is 1 (resp. 2) when referring to directed (resp. undirected) hinge systems.

Necessity. By Theorem 3.3, the spectrum for directed hinge systems of order v and index 1 is precisely the set of all $v \equiv 0$ or 1 (mod 3), $v \neq 1, 3$, or 6. Furthermore, any hinge system can trivially be embedded in itself. However, embedding an undirected (resp. directed) hinge system (V, W) of order n in an undirected (resp. directed) hinge system (V', W') of order v can be thought of as finding a decomposition of the edges of $2K_v \setminus 2K_n$ (resp. $K_v^* \setminus K_n^*$) into undirected (resp. directed) hinges. Now, for every two edges connecting vertices in V to vertices in $V' \setminus V$, there must be an edge entirely contained in $V' \setminus V$. There are $n(v-n)$ edges which connect vertices in V to vertices in $V' \setminus V$, so there must be at least $n(v-n)/2$ edges which have both ends in the set $V' \setminus V$. This implies that $n(v-n)/2 \leq \binom{v-n}{2}$, so $v \geq 2n+1$. Clearly, if $n \geq 4$ and $v \geq 2n+1$, then $v \geq 9$, so v can never be 6. However, it is worth mentioning that an undirected hinge system of order 6 can trivially be embedded in itself, so condition (c) is not necessary for embedding undirected hinge systems.

Sufficiency. Let (V, W) be an undirected (resp. directed) hinge system of order n. Since $n \equiv 0$ or 1 (mod 3) and $n \geq 4$ (and $n \neq 6$ in the directed case), we can form a MTS(n) (V, C) or a TS(n, 2) by breaking each undirected (resp. directed) hinge apart to form pairs of undirected (resp. directed) 3-cycles. By Proposition 2.3, if $n \geq 4$, any MTS(n) (V, C) can be embedded in a MTS(v) (V', C') without introducing any more repeated triples if and only if $v \equiv 0$ or 1 (mod 3) and $v = n$ or $v \geq 2n+1$, where $v \neq 6$. (By Corollary 2.5, the same is true when embedding a TS(n, 2) in a TS(v, 2), except that v can be 6). Let $(V, \mathcal{B})$ and $(V', \mathcal{B}')$ denote the underlying 2-fold triple systems of (V, C) and (V', C'), respectively. Let G and G_2 denote the block-intersection graph and B_2 block-intersection graph, respectively, of $(V', \mathcal{B}')$. Every new triple that is introduced when embedding (V, C) in (V', C') contains at most one vertex from V. Therefore, in G_2, no vertex which corresponds to a block in $\mathcal{B}' \setminus \mathcal{B}$ will be adjacent to any vertex which corresponds to a block in $\mathcal{B}$. Consider the vertices which correspond to blocks in $\mathcal{B}' \setminus \mathcal{B}$. In G_2 there is a 3-regular subgraph which spans these vertices, since every vertex $\{x, y, z\} \in \mathcal{B}' \setminus \mathcal{B}$ is adjacent to three other vertices in $\mathcal{B}' \setminus \mathcal{B}$ which correspond to distinct triples which each contain exactly one of the pairs (x, y), (y, z), and (x, z). Now consider the clique in G induced by a vertex $x \in V' \setminus V$. Every vertex $\{x, y, z\} \in \mathcal{B}' \setminus \mathcal{B}$ in this clique will have degree 2 in G_2 within this clique. Therefore, every edge in G_2 which contains vertices in $\mathcal{B}' \setminus \mathcal{B}$ is contained within a cycle within some clique in G. Therefore, no component of G_2 which contains only vertices in $\mathcal{B}' \setminus \mathcal{B}$ contains a cut edge. So by Lemma 2.1, the subgraph in G_2 induced by the vertices in $\mathcal{B}' \setminus \mathcal{B}$ contains a 1-factor.

Let F be the 1-factor just obtained. Now the blocks of $\mathcal{B}$ were already paired into hinges, so we use F to pair the remaining blocks into hinges. For each edge $\{\{x, y, z\}, \{x, y, a\}\} \in F$, pair the triples $\{x, y, z\}$ and $\{x, y, a\}$ to form a hinge. Clearly, we have a partition of the blocks in $\mathcal{B}' \setminus \mathcal{B}$ into hinges, and this produces an embedding of $(V, \mathcal{B})$ in $(V', \mathcal{B}')$. Since $(V, \mathcal{B})$ and $(V', \mathcal{B}')$ were derived from (V, C) and (V', C'), respectively, we have the desired embedding of the directed hinge system (V, C) in (V, C'). Clearly, the same procedure can be used for embedding undirected hinge systems of order n in undirected hinge systems of order v. □

4 Undirected Hinge Systems

Our goal in this section is to establish the spectrum for undirected hinge sytems of order n, for all indices $\lambda \geq 2$. We begin with a few lemmas which support the main result.

Lemma 4.1 *If $n \equiv 1 \pmod 4$ and $n \geq 4$, then there exists an undirected hinge decomposition of $3K_n$.*

Proof. We consider n modulo 48 and present a method for constructing hinge decompositions of $3K_n$ from decompositions of smaller graphs. Required for this construction are hinge decompositions of $3K_5$, $3K_9$, $3K_{17}$, $3K_{21}$, $3K_{41}$, $3K_{45}$, $3K_{89}$, $3K_{93}$, $3K_{4,4,4}$, and $3K_{8,8,8}$, each of which is presented in the appendix of this paper.

Case 1: $n \equiv$ 5, 13, 29, or 37 (mod 48). Let $v = (n-1)/4$ and observe that $v \equiv 1$ or $3 \pmod 6$. Hence there exists a Steiner triple system of order v, so let $(S, \mathcal{T})$ be a Steiner triple system of order v such that $S = \{s_1, s_2, \ldots, s_v\}$. Let $V = (S \times \{0,1,2,3\}) \cup \{\infty\}$ represent the vertex set of $3K_n$.

Each triple $(s_i, s_j, s_k) \in \mathcal{T}$ corresponds to the subset $\{s_i, s_j, s_k\} \times \{0,1,2,3\}$ of V. By partitioning this subset into three parts, namely $\{s_i\} \times \{0,1,2,3\}$, $\{s_j\} \times \{0,1,2,3\}$, and $\{s_k\} \times \{0,1,2,3\}$, we establish a correlation between each triple in $\mathcal{T}$ and a subgraph $3K_{4,4,4}$ of $3K_n$. On each of these $3K_{4,4,4}$ subgraphs we place a hinge decomposition.

Now, each of $(\{s_x\} \times \{0,1,2,3\}) \cup \{\infty\}$, for $x \in \{i,j,k\}$, corresponds to a $3K_5$ subgraph onto which we place a hinge decomposition.

Case 2: $n \equiv$ 1, 9, 25 or 33 (mod 48). Let $v = (n-1)/4$ and observe that $v \equiv 0$ or $2 \pmod 6$. Hence there exists a maximum partial Steiner triple system of order v whose leave is a 1-factor. Without loss of generality, we let $(S, \mathcal{T})$ be a maximum partial Steiner triple system of order v such that $S = \{s_1, s_2, \ldots, s_v\}$ and such that the edges $\{s_1, s_2\}, \{s_3, s_4\}, \ldots, \{s_{v-1}, s_v\}$ comprise the 1-factor leave of $(S, \mathcal{T})$. Let $V = (S \times \{0,1,2,3\}) \cup \{\infty\}$ represent the vertex set of $3K_n$.

Each triple $(s_i, s_j, s_k) \in \mathcal{T}$ corresponds to the subset $\{s_i, s_j, s_k\} \times \{0,1,2,3\}$ of V. By partitioning this subset into three parts, namely $\{s_i\} \times \{0,1,2,3\}$, $\{s_j\} \times \{0,1,2,3\}$, and $\{s_k\} \times \{0,1,2,3\}$, we establish a correlation between each triple in $\mathcal{T}$ and a subgraph $3K_{4,4,4}$ of $3K_n$. On each of these $3K_{4,4,4}$ subgraphs we place a hinge decomposition.

Now, for each edge $\{s_{2i-1}, s_{2i}\}$ of the 1-factor leave of $(S, \mathcal{T})$, consider the subgraph $3K_9$ of $3K_n$ induced by the vertex set $(\{s_{2i-1}, s_{2i}\} \times \{0,1,2,3\}) \cup \{\infty\}$. On each such $3K_9$ subgraph we place a hinge decomposition.

Case 3: $n \equiv 17 \pmod{48}$. Let $v = (n-1)/8$ and observe that $v \equiv 2 \pmod 6$. Hence there exists a maximum partial Steiner triple system of order v whose leave is a 1-factor. Without loss of generality, we let $(S, \mathcal{T})$ be a maximum partial Steiner triple system of order v such that $S = \{s_1, s_2, \ldots, s_v\}$ and such that the edges $\{s_1, s_2\}, \{s_3, s_4\}, \ldots, \{s_{v-1}, s_v\}$ comprise the 1-factor leave of $(S, \mathcal{T})$. Let $V = (S \times \{0,1,2,3,4,5,6,7\}) \cup \{\infty\}$ represent the vertex set of $3K_n$.

Each triple $(s_i, s_j, s_k) \in \mathcal{T}$ corresponds to the subset $\{s_i, s_j, s_k\} \times \{0,1,2,3,4,5,6,7\}$ of V. By partitioning this subset into three parts, namely $\{s_i\} \times \{0,1,2,3,4,5,6,7\}$, $\{s_j\} \times \{0,1,2,3,4,5,6,7\}$, and $\{s_k\} \times \{0,1,2,3,4,5,6,7\}$, we establish a correlation between each triple in $\mathcal{T}$ and a subgraph $3K_{8,8,8}$ of $3K_n$. On each of these $3K_{8,8,8}$ subgraphs we place a hinge decomposition.

Now, for each edge $\{s_{2i-1}, s_{2i}\}$ of the 1-factor leave of $(S, \mathcal{T})$, consider the subgraph $3K_{17}$ of $3K_n$ induced by the vertex set $(\{s_{2i-1}, s_{2i}\} \times \{0,1,2,3,4,5,6,7\}) \cup \{\infty\}$. On each such $3K_{17}$ subgraph we place a hinge decomposition.

Case 4: $n \equiv 21 \pmod{48}$. The case of $n = 21$ is handled in the appendix, so we assume that $n \geq 69$. Let $v = (n-1)/4$ and let $S = \{s_1, s_2, \ldots, s_v\}$. Let $V = (S \times \{0,1,2,3\}) \cup \{\infty\}$ represent the vertex set of $3K_n$.

On the $3K_{21}$ subgraph of $3K_n$ induced by the vertex set $(\{s_1, s_2, \ldots, s_5\} \times \{0,1,2,3\}) \cup \{\infty\}$, place a hinge decomposition of $3K_{21}$.

On each of the $3K_5$ subgraphs induced by $(\{s_i\} \times \{0,1,2,3\}) \cup \{\infty\}$, for $i \in \{6, 7, \ldots, v\}$, place a hinge decomposition of $3K_5$.

Now notice that $v - 5 \equiv 0 \pmod{6}$. Hence, by a result of Rees [15], there exists a decomposition of K_{v-5} into t 1-factors and $k - t$ triangle-factors provided that $(v-5)/2 + 1 \leq k \leq (v-5) - 2$. In particular, we want $k = (v-1)/2$ and $t = 5$, which satisifies the hypothesis of Rees' result whenever $n \geq 53$. Let $\{F_1, F_2, \ldots, F_5\}$ be the set of 1-factors in such a decomposition of K_{v-5} where $\{s_6, s_7, \ldots, s_v\}$ are the vertices of K_{v-5}. Now for $1 \leq i \leq 5$ and for each edge $(s_j, s_k) \in F_i$, we place a hinge decomposition of $3K_{4,4,4}$ on the subgraph of $3K_n$ induced by the sets of vertices $\{s_i\} \times \{0,1,2,3\}$, $\{s_j\} \times \{0,1,2,3\}$, and $\{s_k\} \times \{0,1,2,3\}$.

Lastly, on each triangle, (s_i, s_j, s_k), of the decomposition of K_{v-5}, we place a hinge decomposition of $3K_{4,4,4}$ on the subgraph of $3K_n$ induced by the sets of vertices $\{s_i\} \times \{0,1,2,3\}$, $\{s_j\} \times \{0,1,2,3\}$, and $\{s_k\} \times \{0,1,2,3\}$.

Case 5: $n \equiv 41 \pmod{48}$. The cases of $n = 41$ or 89 are handled in the appendix, so we assume that $n \geq 137$. Let $v = (n-1)/8$ and let $S = \{s_1, s_2, \ldots, s_v\}$. Let $V = (S \times \{0,1,2,3,4,5,6,7\}) \cup \{\infty\}$ represent the vertex set of $3K_n$.

On the $3K_{41}$ subgraph of $3K_n$ induced by the vertex set $(\{s_1, s_2, \ldots, s_5\} \times \{0, 1, \ldots, 7\}) \cup \{\infty\}$, place a hinge decomposition of $3K_{41}$.

On each of the $3K_9$ subgraphs induced by $(\{s_i\} \times \{0,1,2,3,4,5,6,7\}) \cup \{\infty\}$, for $i \in \{6, 7, \ldots, v\}$, place a hinge decomposition of $3K_9$.

Now notice that $v - 5 \equiv 0 \pmod{6}$. Hence, by a result of Rees [15], there exists a decomposition of K_{v-5} into t 1-factors and $k-t$ triangle-factors provided that $(v-5)/2+1 \leq k \leq (v-5)-2$. In particular, we want $k = (v-1)/2$ and $t = 5$, which satisifies the hypothesis of Rees' result whenever $n \geq 105$. Let $\{F_1, F_2, \ldots, F_5\}$ be the set of 1-factors in such a decomposition of K_{v-5} where $\{s_6, s_7, \ldots, s_v\}$ are the vertices of K_{v-5}. Now for $1 \leq i \leq 5$ and for each edge $(s_j, s_k) \in F_i$, we place a hinge decomposition of $3K_{8,8,8}$ on the subgraph of $3K_n$ induced by the sets of vertices $\{s_i\} \times \{0,1,2,3,4,5,6,7\}$, $\{s_j\} \times \{0,1,2,3,4,5,6,7\}$, and $\{s_k\} \times \{0,1,2,3,4,5,6,7\}$.

Lastly, on each triangle, (s_i, s_j, s_k), of the decomposition of K_{v-5}, we place a hinge decomposition of $3K_{8,8,8}$ on the subgraph of $3K_n$ induced by the sets of vertices $\{s_i\} \times \{0,1,2,3,4,5,6,7\}$, $\{s_j\} \times \{0,1,2,3,4,5,6,7\}$, and $\{s_k\} \times \{0,1,2,3,4,5,6,7\}$.

Case 6: $n \equiv 45 \pmod{48}$. The cases of $n = 45$ or 93 are handled in the appendix, so we assume that $n \geq 141$. Let $v = (n-1)/4$ and let $S = \{s_1, s_2, \ldots, s_v\}$. Let $V = (S \times \{0,1,2,3\}) \cup \{\infty\}$ represent the vertex set of $3K_n$.

On the $3K_{45}$ subgraph of $3K_n$ induced by the vertex set $(\{s_1, s_2, \ldots, s_{11}\} \times \{0,1,2,3\}) \cup \{\infty\}$, place a hinge decomposition of $3K_{45}$.

On each of the $3K_5$ subgraphs induced by $(\{s_i\} \times \{0,1,2,3\}) \cup \{\infty\}$, for $i \in \{12, 13, \ldots, v\}$, place a hinge decomposition of $3K_5$.

Now notice that $v - 11 \equiv 0 \pmod 6$. Hence, by a result of Rees [15], there exists a decomposition of K_{v-11} into t 1-factors and $k-t$ triangle-factors provided that $(v-11)/2+1 \leq k \leq (v-11)-2$. In particular, we want $k = (v-1)/2$ and $t = 11$, which satisifies the hypothesis of Rees' result whenever $n \geq 99$. Let $\{F_1, F_2, \ldots, F_{11}\}$ be the set of 1-factors in such a decomposition of K_{v-11} where $\{s_{12}, s_{13}, \ldots, s_v\}$ are the vertices of K_{v-11}. Now for $1 \leq i \leq 11$ and for each edge $(s_j, s_k) \in F_i$, we place a hinge decomposition of $3K_{4,4,4}$ on the subgraph of $3K_n$ induced by the sets of vertices $\{s_i\} \times \{0,1,2,3\}$, $\{s_j\} \times \{0,1,2,3\}$, and $\{s_k\} \times \{0,1,2,3\}$.

Lastly, on each triangle, (s_i, s_j, s_k), of the decomposition of K_{v-11}, we place a hinge decomposition of $3K_{4,4,4}$ on the subgraph of $3K_n$ induced by the sets of vertices $\{s_i\} \times \{0,1,2,3\}$, $\{s_j\} \times \{0,1,2,3\}$, and $\{s_k\} \times \{0,1,2,3\}$.
□

Before proceeding further, we need an existence result for latin squares. An *almost-self-orthogonal latin square of order* $2n$ *with holes of size* 2 is a latin square of order $2n$ such that cell (i,j) contains the same symbol as cell (j,i) if and only if

(a) $i = j$,

(b) i is odd and $j = i+1$, or

(c) i is even and $j = i - 1$.

Almost-self-orthongal latin squares with holes of size 2 are similar in concept to *frame self-orthogonal latin squares* (see [7]), but differ in that the holes are filled with symbols (and also in that the symbols placed in the holes make each hole symmetric). An almost-self-orthogonal latin square of order 6 having holes of size 2 is illustrated in Figure 2. Given a latin square, L, we let $L(i,j)$ and $L(j,i)$ denote the symbols contained in cells (i,j) and (j,i) of L, respectively.

1 2 2	5 6 6	3 4 4
1	5	3
6 5 5	3 4 4	1 2 2
6	3	1
4 3 3	2 1 1	5 6 6
4	2	5

Figure 2: An Almost-Self-Orthogonal Latin Square of Order 6 with Holes of Size 2

Lemma 4.2 *An almost-self-orthogonal latin square of order* $2n$ *with holes of size* 2 *exists for all* $n \geq 3$.

Proof. Begin with an idempotent latin square, L, of order n, on the symbol set $\{1, 2, \ldots, n\}$; such latin squares are known to exist for all $n \geq 3$. For each cell of L we will create a two by two latin sub-square of our desired almost-self-orthogonal latin square. For each cell (i,j) of L that is either on or above the main diagonal of L, create the two by two sub-square shown below:

$2L(i,j)-1$	$2L(i,j)$
$2L(i,j)$	$2L(i,j)-1$

For each cell (i,j) of L that is below the main diagonal of L, create the two by two sub-square shown below:

$2L(i,j)$	$2L(i,j)-1$
$2L(i,j)-1$	$2L(i,j)$

The resultant $2n$ by $2n$ latin square will be almost-self-orthogonal with holes of size 2. Furthermore, observe that the i^{th} hole of this almost-self-orthogonal latin square contains only the symbols $2i-1$ and $2i$, such that the symbols on the main diagonal of this almost-self-orthogonal latin square consist only of odd numbers. □

Lemma 4.3 *If $n \equiv 2 \pmod 3$ and $n \geq 4$, then there exists an undirected hinge decomposition of* $6K_n$.

Proof. We consider n modulo 12 and present a method for constructing hinge decompositions of $6K_n$ from decompositions of smaller graphs. Required for this construction are hinge decompositions of $6K_8$, $6K_{11}$, $6K_{14}$, $6(K_8 \setminus K_2)$, and $6(K_{11} \setminus K_5)$, each of which is presented in the appendix of this paper.

Case 1: $n \equiv 2$ or $8 \pmod{12}$. The cases of $n = 8$ or 14 are handled in the appendix, so we assume that $n \geq 20$. Let $v = (n-2)/3$ and let $S = \{s_1, s_2, \ldots, s_v\}$. Let $V = (S \times \{0,1,2\}) \cup \{\infty_1, \infty_2\}$ represent the vertex set of $6K_n$.

On the $6K_8$ subgraph of $6K_n$ induced by the vertex set $(\{s_1, s_2\} \times \{0,1,2\}) \cup \{\infty_1, \infty_2\}$, place a hinge decomposition of $6K_8$.

On each $6(K_8 \setminus K_2)$ subgraph induced by the vertex set $(\{s_{2i-1}, s_{2i}\} \times \{0,1,2\}) \cup \{\infty_1, \infty_2\}$, for $2 \leq i \leq v/2$, in which the vertex set $\{\infty_1, \infty_2\}$ induces the $6K_2$ whose edges are missing, we place a hinge decomposition of $6(K_8 \setminus K_2)$.

Now let L be an almost-self-orthogonal latin square of order v having holes of size 2. We further require that L is on the symbol set $\{1, 2, \ldots, v\}$, that symbols $2i-1$ and $2i$ are the symbols of the i^{th} hole of L, and that the symbols of the main diagonal of L consist only of odd numbers. By Lemma 4.2, such a latin square is known to exist for $v \geq 6$ (i.e. $n \geq 20$).

For each cell (i,j) for which $i < j$ and $L(i,j) \neq L(j,i)$, we will create six hinges of $6K_n$ (these cells are precisely those not contained in the holes of size 2). Using the notation introduced in the proof of Corollary 3.4, we take 6 copies of each of the hinges $((s_i, z), (s_j, z), (s_{L(i,j)}, z+1)) - ((s_i, z), (s_j, z), (s_{L(j,i)}, z+1))$, for $z \in \{0,1,2\}$, where we take addition modulo 3. Notice that edges such as $\{(s_i, 0), (s_j, 1)\}$ will be used by hinges that are generated from the cells (i,k) and (k,i) where either $L(i,k) = j$ or $L(k,i) = j$.

Case 2: $n \equiv 5 \pmod{12}$. Notice that if $n \equiv 5 \pmod{12}$ then $n \equiv 1 \pmod 4$ and so, by Lemma 4.1, there exists a hinge decomposition of $3K_n$. To obtain a hinge decomposition of $6K_n$, simply combine two copies of a hinge decomposition of $3K_n$.

Case 3: $n \equiv 11 \pmod{12}$. The case of $n = 11$ is handled in the appendix, so we assume that $n \geq 23$. Let $v = (n-5)/3$ and let $S = \{s_1, s_2, \ldots, s_v\}$. Let $V = (S \times \{0,1,2\}) \cup \{\infty_1, \infty_2, \ldots, \infty_5\}$ represent the vertex set of $6K_n$.

On the $6K_{11}$ subgraph of $6K_n$ induced by the vertex set $(\{s_1, s_2\} \times \{0,1,2\}) \cup \{\infty_1, \infty_2, \ldots, \infty_5\}$, place a hinge decomposition of $6K_{11}$.

On each $6(K_{11} \setminus K_5)$ subgraph induced by the vertex set $(\{s_{2i-1}, s_{2i}\} \times \{0,1,2\}) \cup \{\infty_1, \infty_2, \ldots, \infty_5\}$, for $2 \leq i \leq v/2$, in which the vertex set

$\{\infty_1, \infty_2, \ldots, \infty_5\}$ induces the $6K_5$ whose edges are missing, we place a hinge decomposition of $6(K_{11} \setminus K_5)$.

Now let L be an almost-self-orthogonal latin square of order v having holes of size 2. We further require that L is on the symbol set $\{1, 2, \ldots, v\}$, that symbols $2i-1$ and $2i$ are the symbols of the i^{th} hole of L, and that the symbols of the main diagonal of L consist only of odd numbers. By Lemma 4.2, such a latin square is known to exist for $v \geq 6$ (i.e. $n \geq 23$).

For each cell (i,j) for which $i < j$ and $L(i,j) \neq L(j,i)$, we will create six hinges of $6K_n$ (these cells are precisely those not contained in the holes of size 2). Using the notation introduced in the proof of Corollary 3.4, we take 6 copies of each of the hinges $((s_i, z), (s_j, z), (s_{L(i,j)}, z+1)) - ((s_i, z), (s_j, z), (s_{L(j,i)}, z+1))$, for $z \in \{0, 1, 2\}$, where we take addition modulo 3. Notice that edges such as $\{(s_i, 0), (s_j, 1)\}$ will be used by hinges that are generated from the cells (i,k) and (k,i) where either $L(i,k) = j$ or $L(k,i) = j$. □

Theorem 4.4 *An undirected hinge system of order n and index λ exists if and only if $\lambda \geq 2$, $n \geq 4$, and*

(a) $n \equiv 1$ or 9 (mod 12) if $\lambda \equiv 1$ or 5 (mod 6),

(b) $n \equiv 0$ or 1 (mod 3) if $\lambda \equiv 2$ or 4 (mod 6),

(c) $n \equiv 1$ (mod 4) if $\lambda \equiv 3$ (mod 6), or

(d) no additional restrictions on n if $\lambda \equiv 0$ (mod 6).

Proof. Necessity: Clearly it is necessary that $n \geq 4$ since a hinge contains 4 vertices. Now observe that λK_n contains $\lambda n(n-1)/2$ edges and that a hinge contains 6 edges. For a hinge decomposition to exist, it is therefore necessary that 6 divide $\lambda n(n-1)/2$. Also, since each vertex of a hinge has even degree, it is necessary that the degree, $\lambda(n-1)$, of each vertex of λK_n is even.

If $\lambda \equiv 0$ (mod 6), then 6 always divides $\lambda n(n-1)/2$, and $\lambda(n-1)$ is even. Therefore, there is no restriction placed on n other than that $n \geq 4$.

If $\lambda \equiv 1$ or 5 (mod 6), then 6 must divide $n(n-1)/2$. Hence, 12 must divide $n(n-1)$ and so $n \equiv 0$, 1, 4, or 9 (mod 12). Since $\lambda(n-1)$ must be even, we must have odd n. Therefore, $n \equiv 1$ or 9 (mod 12). Also, $\lambda \neq 1$ since a hinge contains a multiple edge.

If $\lambda \equiv 2$ or 4 (mod 6), then 3 must divide $n(n-1)/2$. Therefore, $n \equiv 0$ or 1 (mod 3).

If $\lambda \equiv 3$ (mod 6), then 2 must divide $n(n-1)/2$. Hence 4 must divide $n(n-1)$ and so $n \equiv 0$ or 1 (mod 4). Since $\lambda(n-1)$ must be even, we must have odd n. Therefore $n \equiv 1$ (mod 4).

Sufficiency: We first observe that by Corollary 3.4 there exists an undirected hinge system of order n and index 2 for each $n \equiv 0$ or 1 (mod 3) where $n \geq 4$. Hence, an undirected hinge system of order $n \equiv 0$ or 1 (mod 3) with $n \geq 4$ and index $\lambda \equiv 0$, 2, or 4 (mod 6) can be constructed by taking $\lambda/2$ copies of an undirected hinge system of order n and index 2.

For even λ, it therefore only remains to consider the case in which $\lambda \equiv 0$ (mod 6), $n \equiv 2$ (mod 3), and $n \geq 4$. By Lemma 4.3 it follows that a hinge decomposition of $6K_n$ exists for each $n \equiv 2$ (mod 3) such that $n \geq 4$. Note now that hinge decompositions for indices $\lambda \equiv 0$ (mod 6), with $n \equiv 2$ (mod 3) and $n \geq 4$, can be obtained by taking $\lambda/6$ copies of a hinge decomposition of $6K_n$.

We now consider the cases in which λ is odd. Necessarily $\lambda \geq 3$ since no hinge decomposition of index 1 can exist (recall that a hinge contains a multiple edge). The situation in which $\lambda = 3$ and $n \equiv 1$, 5, or 9 (mod 12) was handled in Lemma 4.1. So now we must consider $\lambda \geq 5$.

First we consider $\lambda \equiv 3$ (mod 6) with $n \equiv 1$, 5, or 9 (mod 12) and $n \geq 4$. Necessarily λ is then an odd multiple of 3, and hence a hinge decomposition of λK_n can be obtained by taking $\lambda/3$ copies of a hinge decomposition of $3K_n$. And by Lemma 4.1, such a decomposition of $3K_n$ is known to exist.

Now suppose that $\lambda \equiv 1$ or 5 (mod 6), $n \equiv 1$ or 9 (mod 12), and $n \geq 4$. Then $\lambda = \lambda_e + 3$ where $\lambda_e = \lambda - 3$. Thus, to find a hinge decomposition of λK_n, it would be sufficient to find hinge decompositions for each of $\lambda_e K_n$ and $3K_n$ and then combine these two decompositions. Note that $\lambda_e = \lambda - 3 \equiv 4$ or 2 (mod 6) and if $n \equiv 1$ or 9 (mod 12) then $n \equiv 1$ or 0 (mod 3), and so it follows from our earlier statements in this proof that a hinge decomposition of $\lambda_e K_n$ must exist. Note that a hinge decomposition of $3K_n$ also exists, by Lemma 4.1. □

5 Appendix

5.1 A Directed Hinge Decomposition of $2K_6^*$

Consider $2K_6^*$ on the vertex set $V = \{0,1,2,3,4,5\}$. Let $H = \{(1,4,5) - (4,1,3)$, $(1,5,0) - (0,5,2)$, $(1,5,3) - (3,5,2)$, $(1,2,5) - (2,1,4)$, $(2,1,0) - (2,0,3)$, $(1,2,3) - (3,2,4)$, $(1,0,4) - (1,3,0)$, $(2,4,0) - (4,2,5)$, $(3,4,5) - (3,5,0)$, $(3,0,4) - (4,0,5)\}$, where $(a,b,c) - (d,e,f)$ denotes the hinge containing the directed triples (a,b,c) and (d,e,f). Then (V,H) is a directed hinge decomposition of $2K_6^*$.

5.2 A Directed Hinge Decomposition of $3K_5^*$

Consider $3K_5^*$ on the vertex set $V = \{0,1,2,3,4\}$. Let $H_1 = \{(0+i,1+i,2+i) - (0+i,3+i,1+i) \mid 0 \leq i \leq 4\}$, and $H_2 = \{(0+i,2+i,1+i) -$

$(0+i, 1+i, 3+i) \mid 0 \leq i \leq 4\}$, where addition is taken modulo 5. Then $(V, H_1 \cup H_2)$ is a directed hinge decomposition of $3K_5^*$.

5.3 A Directed Hinge Decomposition of $3K_6^*$

Consider $3K_6^*$ on the vertex set $V = \{0,1,2,3,4,5\}$. Let $H = \{(1,2,3) - (1,3,4),\ (1,4,5) - (1,5,2),\ (1,2,0) - (0,2,3),\ (1,0,5) - (5,0,2),\ (1,3,2) - (2,3,5),\ (1,0,4) - (2,4,0),\ (1,2,5) - (5,2,0),\ (0,1,3) - (0,3,5),\ (1,4,0) - (0,4,2),\ (1,5,3) - (3,5,4),\ (1,0,3) - (3,0,5),\ (1,5,4) - (4,5,3),\ (1,4,2) - (2,4,3),\ (2,5,4) - (4,5,0),\ (2,4,3) - (3,4,0)\}$. Then (V, H) is a directed hinge decomposition of $3K_6^*$.

5.4 A Directed Hinge Decomposition of $3K_8^*$

Consider $3K_8^*$ on the vertex set $V = \{0,1,2,3,4,5,6,7\}$. Let $H = \{(1,2,3) - (7,2,1),\ (1,3,2) - (1,2,5),\ (1,3,5) - (1,5,7),\ (1,4,6) - (1,2,4),\ (1,4,6) - (1,5,4),\ (1,5,2) - (2,5,3),\ (1,6,4) - (1,4,0),\ (1,7,0) - (1,0,6),\ (1,7,0) - (7,1,6),\ (2,4,7) - (3,7,4),\ (2,4,0) - (4,2,6),\ (2,5,6) - (2,6,3),\ (2,7,4) - (2,6,7),\ (2,7,5) - (5,7,6),\ (2,0,4) - (0,2,3),\ (2,0,6) - (0,7,6),\ (3,4,5) - (1,3,5),\ (3,4,7) - (4,3,6),\ (3,5,0) - (5,3,7),\ (3,6,0) - (6,3,1),\ (3,7,6) - (2,3,6),\ (3,0,4) - (4,0,5),\ (4,5,0) - (2,0,5),\ (4,7,3) - (7,4,5),\ (5,6,7) - (6,5,0),\ (6,0,5) - (6,5,4),\ (7,0,2) - (0,7,1),\ (0,3,1) - (0,7,3)\}$. Then (V, H) is a directed hinge decomposition of $3K_8^*$.

5.5 A Directed Hinge Decomposition of $3K_{11}^*$

Consider $3K_{11}^*$ on the vertex set $V = \{0,1,\ldots,10\}$. Let $H_1 = \{(0+i, 5+i, 4+i) - (5+i, 0+i, 2+i) \mid 0 \leq i \leq 10\}$, $H_2 = \{(0+i, 1+i, 2+i) - (0+i, 2+i, 3+i) \mid 0 \leq i \leq 10\}$, $H_3 = \{(0+i, 4+i, 6+i) - (4+i, 7+i, 6+i) \mid 0 \leq i \leq 10\}$, $H_4 = \{(0+i, 5+i, 4+i) - (5+i, 0+i, 7+i) \mid 0 \leq i \leq 10\}$, and $H_5 = \{(0+i, 3+i, 7+i) - (3+i, 0+i, 8+i) \mid 0 \leq i \leq 10\}$, where addition is taken modulo 11. Then $(V, H_1 \cup H_2 \cup H_3 \cup H_4 \cup H_5)$ is a directed hinge decomposition of $3K_{11}^*$.

5.6 A Directed Hinge Decomposition of $3K_{20}^*$

Let $S = \{s_1, s_2, s_3, s_4\}$ and let the vertex set of $3K_{20}^*$ be $V = S \times \{0,1,2,3,4\}$. We first place a directed hinge decomposition of $3K_5^*$ on each of the vertex sets $s_i \times \{0,1,2,3,4\}$ for $0 \leq i \leq 3$.

The remaining arcs of $3K_{20}^*$ induce the multipartite graph $3K_{5,5,5,5}^*$, which we now decompose into directed hinges. Let $H_1 = \{((s_1, 0+i), (s_2, 0+i), (s_4, 0+i)) - ((s_2, 0+i), (s_1, 0+i), (s_3, 0+i)) \mid 0 \leq i \leq 4\}$, $H_2 = \{((s_1, 0+i), (s_2, 1+i), (s_4, 3+i)) - ((s_2, 1+i), (s_1, 0+i), (s_3, 2+i)) \mid 0 \leq i \leq 4\}$,

$H_3 = \{((s_1,0+i),(s_2,2+i),(s_4,1+i))-((s_2,2+i),(s_1,0+i),(s_3,4+i)) \mid 0 \le i \le 4\}$, $H_4 = \{((s_1,0+i),(s_2,3+i),(s_4,4+i))-((s_2,3+i),(s_1,0+i),(s_3,1+i)) \mid 0 \le i \le 4\}$, $H_5 = \{((s_1,0+i),(s_2,4+i),(s_4,2+i))-((s_2,4+i),(s_1,0+i),(s_3,3+i)) \mid 0 \le i \le 4\}$, $H_6 = \{((s_4,0+i),(s_3,0+i),(s_1,0+i))-((s_3,0+i),(s_4,0+i),(s_2,0+i)) \mid 0 \le i \le 4\}$, $H_7 = \{((s_4,0+i),(s_3,1+i),(s_1,3+i)) - ((s_3,1+i),(s_4,0+i),(s_2,2+i)) \mid 0 \le i \le 4\}$, $H_8 = \{((s_4,0+i),(s_3,2+i),(s_1,1+i)) - ((s_3,2+i),(s_4,0+i),(s_2,4+i)) \mid 0 \le i \le 4\}$, $H_9 = \{((s_4,0+i),(s_3,3+i),(s_1,4+i))-((s_3,3+i),(s_4,0+i),(s_2,1+i)) \mid 0 \le i \le 4\}$, and $H_{10} = \{((s_4,0+i),(s_3,4+i),(s_1,2+i)) - ((s_3,4+i),(s_4,0+i),(s_2,3+i)) \mid 0 \le i \le 4\}$, where addition is taken modulo 5. Taking 3 copies of each of the 5 hinges generated by each H_i, we obtain a directed hinge decomposition of $3K^*_{5,5,5,5}$.

5.7 A Directed Hinge Decomposition of $3(K_5^* \setminus K_2^*)$

Consider $3(K_5^* \setminus K_2^*)$ on the vertex set $V = \{0,1,2,a,b\}$. We form a directed hinge decomposition by taking 3 copies of each of the following three directed hinges: $(1,2,b)-(2,1,a)$, $(2,0,b)-(0,2,a)$, $(1,0,a)-(0,1,b)$.

5.8 An Undirected Hinge Decomposition of $2K_4$

Consider $2K_4$ on the vertex set $V = \{0,1,2,3\}$. Let $H = \{(0,1,2)-(0,1,3),(0,2,3)-(1,2,3)\}$. (V,H) is an undirected hinge decomposition of $2K_4$.

5.9 An Undirected Hinge Decomposition of $2K_{3,3,3}$

Consider the multigraph $2K_{3,3,3}$ with vertex tripartition $V = (\{0,1,2\} \cup \{3,4,5\} \cup \{6,7,8\})$. Let $H = \{(0,3,6)-(0,5,6),(0,3,7)-(0,4,7),(0,4,8)-(0,5,8),(1,3,6)-(1,4,6),(1,4,7)-(1,5,7),(1,3,8)-(1,5,8),(2,4,6)-(2,5,6),(2,3,7)-(2,5,7),(2,3,8)-(2,4,8)\}$. Then (V,H) is an undirected hinge decomposition of $2K_{3,3,3}$.

5.10 An Undirected Hinge Decomposition of $2K_{3,3,3,3}$

Consider $2K_{3,3,3,3}$ on the vertex set $\{0,1,\dots,11\}$. The partition is formed by the sets $\{0,1,2\},\{3,4,5\},\{6,7,8\},\{9,10,11\}$. The first step is to obtain a decomposition of $2K_{3,3,3,3}$ into copies of $2K_4$. We do this by obtaining two copies of each K_4 in the set $\{\{0,3,6,11\},\{0,4,7,10\},\{0,5,8,9\},\{1,3,7,9\},\{1,4,8,11\},\{1,5,6,10\},\{2,3,8,10\},\{2,4,6,9\},\{2,5,7,11\}\}$. Now each copy of $2K_4$ can be decomposed into hinges, and this gives us the desired decomposition of $2K_{3,3,3,3}$.

5.11 An Undirected Hinge Decomposition of $2(K_5 \setminus K_2)$

Consider $2(K_5 \setminus K_2)$ on the vertex set $V = \{0,1,2,3,4\}$. Furthermore, assume that in $2(K_5 \setminus K_2)$, no edges join vertices 0 and 1. Let $H = \{(0,2,3) - (1,2,3), (0,2,4) - (1,2,4), (0,3,4) - (1,3,4)\}$. Then (V, H) is an undirected hinge decomposition of $2(K_5 \setminus K_2)$.

5.12 An Undirected Hinge Decomposition of $3K_{4,4,4}$

Consider $3K_{4,4,4}$ with vertex tripartition $V = (\{1,2,3,4\} \cup \{5,6,7,8\} \cup \{9,10,11,12\})$. Let $H = \{(1,5,11)-(1,5,12), (1,6,10)-(1,6,11), (1,7,9)-(1,7,10), (1,8,9)-(1,8,12), (1,5,10)-(2,5,10), (1,6,9)-(2,6,9), (1,7,12)-(2,7,12), (1,8,11)-(2,8,11), (2,5,9)-(3,5,9), (2,6,12)-(3,6,12), (2,7,11)-(3,7,11), (2,8,10)-(3,8,10), (2,5,11)-(4,5,11), (2,6,10)-(4,6,10), (2,7,9)-(4,7,9), (2,8,12)-(4,8,12), (3,5,10)-(3,5,12), (3,6,9)-(3,6,11), (3,7,10)-(3,7,12), (3,8,9)-(3,8,11), (4,5,9)-(4,5,12), (4,6,11)-(4,6,12), (4,7,10)-(4,7,11), (4,8,9) - (4,8,10)\}$. Then (V, H) is an undirected hinge decomposition of $3K_{4,4,4}$.

5.13 An Undirected Hinge Decomposition of $3K_5$

Consider $3K_5$ defined on the vertex set $V = \{0,1,2,3,4\}$. Let $H = \{(0+i, 1+i, 2+i) - (0+i, 1+i, 3+i) \mid 0 \le i \le 4\}$, where all sums are reduced modulo 5. Notice that the differences 1 and 2 have been covered three times each by the base hinge $(0,1,2)-(0,1,3)$. Therefore, (V, H) is an undirected hinge decomposition of $3K_5$.

5.14 An Undirected Hinge Decomposition of $3K_9$

Consider $3K_9$ defined on the vertex set $V = \{0,1,\ldots,8\}$. Let $H_1 = \{(0+i, 1+i, 4+i) - (0+i, 1+i, 6+i) \mid 0 \le i \le 8\}$, and $H_2 = \{(0+i, 2+i, 3+i) - (0+i, 2+i, 4+i) \mid 0 \le i \le 8\}$ where all sums are reduced modulo 9. Notice that the differences 1, 2, 3, and 4 have each been covered three times by the base hinges $(0,1,4)-(0,1,6)$ and $(0,2,3)-(0,2,4)$. Therefore, $(V, H_1 \cup H_2)$ is an undirected hinge decomposition of $3K_9$.

5.15 An Undirected Hinge Decomposition of $3K_{17}$

Consider $3K_{17}$ defined on the vertex set $V = \{0,1,\ldots,16\}$. Let $H_1 = \{(0+i, 1+i, 8+i) - (0+i, 1+i, 10+i) \mid 0 \le i \le 16\}$, $H_2 = \{(0+i, 2+i, 6+i) - (0+i, 2+i, 7+i) \mid 0 \le i \le 16\}$, $H_3 = \{(0+i, 3+i, 6+i) - (0+i, 3+i, 8+i) \mid 0 \le i \le 16\}$, and $H_4 = \{(0+i, 4+i, 5+i) - (0+$

$i, 4+i, 6+i) \mid 0 \leq i \leq 16\}$, where all sums are reduced modulo 17. Notice that the differences 1 through 8 have each been covered three times by the base hinges $(0,1,8)-(0,1,10)$, $(0,2,6)-(0,2,7)$, $(0,3,6)-(0,3,8)$ and $(0,4,5)-(0,4,6)$. Therefore, $(V, H_1 \cup H_2 \cup H_3 \cup H_4)$ is an undirected hinge decomposition of $3K_{17}$.

5.16 An Undirected Hinge Decomposition of $3K_{21}$

Consider $3K_{21}$ defined on the vertex set $V = \{0,1,\ldots,20\}$. Let $H_1 = \{(0+i, 6+i, 5+i)-(0+i, 6+i, 8+i) \mid 0 \leq i \leq 20\}$, $H_2 = \{(0+i, 7+i, 3+i)-(0+i, 7+i, 4+i) \mid 0 \leq i \leq 20\}$, $H_3 = \{(0+i, 8+i, 3+i)-(0+i, 8+i, 6+i) \mid 0 \leq i \leq 20\}$, $H_4 = \{(0+i, 9+i, 4+i)-(0+i, 9+i, 7+i) \mid 0 \leq i \leq 20\}$, and $H_5 = \{(0+i, 10+i, 9+i)-(0+i, 10+i, 20+i) \mid 0 \leq i \leq 20\}$, where all sums are reduced modulo 21. Notice that the differences 1 through 10 have each been covered three times by the base hinges $(0,6,5)-(0,6,8)$, $(0,7,3)-(0,7,4)$, $(0,8,3)-(0,8,6)$, $(0,9,4)-(0,9,7)$ and $(0,10,9)-(0,10,20)$. Therefore, $(V, H_1 \cup H_2 \cup \cdots \cup H_5)$ is an undirected hinge decomposition of $3K_{21}$.

5.17 An Undirected Hinge Decomposition of $3K_{41}$

Let $S = \{s_1, s_2, \ldots, s_{10}\}$, and consider $3K_{41}$ defined on the vertex set $(S \times \{1,2,3,4\}) \cup \{\infty\}$. We define the hinge decomposition of $3K_{41}$ in the following manner.

(1) Place a hinge decomposition of $3K_{17}$ on the vertex set $(\{s_1, s_2, s_3, s_4\} \times \{1,2,3,4\}) \cup \{\infty\}$.

(2) For $3 \leq i \leq 5$, place a hinge decomposition of $3K_9$ on each of the vertex sets $(\{s_{2i-1}, s_{2i}\} \times \{1,2,3,4\}) \cup \{\infty\}$.

(3) We first consider $G = K_{10} \setminus K_4$ defined on the vertex set S in which the edges between pairs of vertices in $\{s_1, s_2, s_3, s_4\}$ have been removed. Our first goal is to partition the edges of $G - \{\{s_5, s_6\}, \{s_7, s_8\}, \{s_9, s_{10}\}\}$ into triples. we consider K_6 defined on the vertex set $\{s_5, s_6, \ldots, s_{10}\}$. Now K_6 contains a 1-factorisation $F_0, F_1, \ldots, F_4$, in which we can assume, without loss of generality, F_0 contains the edges $\{s_5, s_6\}, \{s_7, s_8\}, \{s_9, s_{10}\}$ which correspond to the sets $(\{s_{2i-1}, s_{2i}\} \times \{1,2,3,4\})$, for $3 \leq i \leq 5$. For $1 \leq i \leq 4$ and for each edge $\{s_a, s_b\} \in F_i$, we can form a triple $\{s_a, s_b, s_i\}$. Notice that these triples partition the edges of $G - \{\{s_5, s_6\}, \{s_7, s_8\}, \{s_9, s_{10}\}\}$. Each of these triples $\{s_a, s_b, s_c\}$ will correspond to a copy of $3K_{4,4,4}$ defined on the vertex set $\{s_a, s_b, s_c\} \times \{1,2,3,4\}$, which can be decomposed into hinges. Since each of these triples corresponds to a copy of $3K_{4,4,4}$, we have a partition of the remaining edges of $3K_{41}$ into hinges.

5.18 An Undirected Hinge Decomposition of $3K_{45}$

Let $S = \{s_1, s_2, \ldots, s_{11}\}$, and consider $3K_{45}$ defined on the vertex set $(S \times \{1,2,3,4\}) \cup \{\infty\}$. We define the hinge decomposition of $3K_{45}$ in the following manner.

(1) Place a hinge decomposition of $3K_{21}$ on the vertex set $(\{s_1, s_2, s_3, s_4, s_5\} \times\{1,2,3,4\}) \cup \{\infty\}$.

(2) For $6 \leq i \leq 11$, place a hinge decomposition of $3K_5$ on each of the vertex sets $(\{s_i\} \times \{1,2,3,4\}) \cup \{\infty\}$.

(3) We first consider $G = K_{11} \setminus K_5$ defined on the vertex set S in which the edges between pairs of vertices in $\{s_1, s_2, s_3, s_4, s_5\}$ have been removed. Our first goal is to partition the edges of G into triples. Next, we consider K_6 defined on the vertex set $\{s_6, s_7, \ldots, s_{11}\}$. Clearly, K_6 contains a 1-factorisation $F_1, \ldots, F_5$. For $1 \leq i \leq 5$ and for each edge $\{s_a, s_b\} \in F_i$, we can form a triple $\{s_a, s_b, s_i\}$. Notice that these triples partition the edges of G. As in the previous example, each of the triples $\{s_a, s_b, s_c\}$ will correspond to a copy of $3K_{4,4,4}$ defined on the vertex set $\{s_a, s_b, s_c\} \times \{1,2,3,4\}$, which can be decomposed into hinges. Therefore, we have a partition of the remaining edges of $3K_{45}$ into hinges.

5.19 An Undirected Hinge Decomposition of $3K_{89}$

Let $S = \{s_1, s_2, \ldots, s_{11}\}$, and consider $3K_{89}$ defined on the vertex set $(S \times \{1,2,\ldots,8\}) \cup \{\infty\}$. We define the hinge decomposition of $3K_{89}$ in the following manner.

(1) Place a hinge decomposition of $3K_{41}$ on the vertex set $(\{s_1, s_2, s_3, s_4, s_5\} \times\{1,2,\ldots,8\}) \cup \{\infty\}$.

(2) For $6 \leq i \leq 11$, place a hinge decomposition of $3K_9$ on each of the vertex sets $(\{s_i\} \times \{1,2,\ldots,8\}) \cup \{\infty\}$.

(3) We first consider $G = K_{11} \setminus K_5$ defined on the vertex set S in which the edges between all pairs of vertices in $\{s_1, s_2, s_3, s_4, s_5\}$ have been removed. Our first goal is to partition the edges of G into triples. We consider K_6 defined on the vertex set $\{s_6, s_7, \ldots, s_{11}\}$. Clearly, K_6 contains a 1-factorisation $F_1, F_2, \ldots, F_5$. For $1 \leq i \leq 5$ and for each edge $\{s_a, s_b\} \in F_i$, we can form a triple $\{s_a, s_b, s_i\}$. Notice that these triples partition the edges of G. Each of these triples $\{s_a, s_b, s_c\}$ will correspond to a copy of $3K_{8,8,8}$ defined on the vertex set $\{s_a, s_b, s_c\} \times \{1,2,\ldots,8\}$, which can be decomposed into hinges.

Again since each of these triples corresponds to a copy of $3K_{8,8,8}$, we have a partition of the remaining edges of $3K_{89}$ into hinges.

5.20 An Undirected Hinge Decomposition of $3K_{93}$

Let $S = \{s_1, s_2, \ldots, s_{23}\}$, and consider $3K_{93}$ defined on the vertex set $(S \times \{1,2,3,4\}) \cup \{\infty\}$. We define the hinge decomposition of $3K_{93}$ in the following manner.

(1) Place a hinge decomposition of $3K_{45}$ on the vertex set $(\{s_1, s_2, \ldots, s_{11}\} \times \{1,2,3,4\}) \cup \{\infty\}$.

(2) For $12 \leq i \leq 23$, place a hinge decomposition of $3K_5$ on each of the vertex sets $(\{s_i\} \times \{1,2,3,4\}) \cup \{\infty\}$.

(3) We first consider $G = K_{23} \setminus K_{11}$ defined on the vertex set S in which the edges between pairs of vertices in $\{s_1, s_2, \ldots, s_{11}\}$ have been removed. Our first goal is to partition the edges of G into triples. As in the previous example, each of these triples $\{s_a, s_b, s_c\}$ will correspond to a copy of $3K_{4,4,4}$ defined on the vertex set $\{s_a, s_b, s_c\} \times \{1,2,3,4\}$, which can be decomposed into hinges.

    Next, we consider K_{12} defined on the vertex set $\{s_{12}, s_{13}, \ldots, s_{23}\}$. Clearly, K_{12} contains a 1-factorisation $F_1, F_2, \ldots, F_{11}$. For $1 \leq i \leq 11$ and for each edge $\{s_a, s_b\} \in F_i$, we can form a triple $\{s_a, s_b, s_i\}$. Notice that these triples partition the edges of G. Again since each of these triples corresponds to a copy of $3K_{4,4,4}$, we have a partition of the remaining edges of $3K_{93}$ into hinges.

5.21 An Undirected Hinge Decomposition of $3K_{8,8,8}$

Let $S = \{s_1, s_2, \ldots, s_6\}$ and consider $3K_{8,8,8}$ on the vertex set $S \times \{1,2,3,4\}$. The tripartition of the vertices is formed by the sets $\{s_{2i-1}, s_{2i}\} \times \{1,2,3,4\}$, for $1 \leq i \leq 3$.

Now consider $K_{2,2,2}$ defined on S with vertex tripartition $V = (\{s_1, s_2\}, \{s_3, s_4\}, \{s_5, s_6\})$. If we can decompose $K_{2,2,2}$ into copies of K_3, then we can make a one-to-one correspondence between each K_3 and a copy of $3K_{4,4,4}$. Furthermore, these copies of $3K_{4,4,4}$ will form a decomposition of $3K_{8,8,8}$. The set $\{\{s_1, s_3, s_5\}, \{s_1, s_4, s_6\}, \{s_2, s_3, s_6\}, \{s_2, s_4, s_5\}\}$ forms a decomposition of $K_{2,2,2}$ into triples. Each of these triples $\{s_i, s_j, s_k\}$ corresponds to a copy of $3K_{4,4,4}$ on the vertex set $\{s_i, s_j, s_k\} \times \{1,2,3,4\}$, and each copy of $3K_{4,4,4}$ can be decomposed into hinges. Therefore, we have the desired hinge decomposition of $3K_{8,8,8}$.

5.22 An Undirected Hinge Decomposition of $6K_8$

Consider $6K_8$ defined on the vertex set $V = \{0,1,\ldots,7\}$. We let $H = \{(0,1,2)-(0,2,7),(2,4,5)-(3,4,5),(0,1,2)-(1,2,4),(0,3,4)-(3,4,5),(1,2,3)-(1,3,7),(0,4,6)-(4,6,7),(3,5,7)-(5,6,7),(1,2,5)-(1,5,6),(2,3,6)-(3,5,6),(0,3,4)-(0,3,6),(1,3,7)-(1,4,7),(0,3,6)-(3,6,7),(1,4,7)-(4,5,7),(0,2,5)-(0,2,6),(0,1,7)-(1,4,7),(0,1,5)-(1,5,6),(0,3,5)-(0,5,7),(2,3,7)-(2,6,7),(0,2,7)-(2,3,7),(1,4,6)-(1,5,6),(0,4,5)-(0,5,7),(0,1,6)-(1,3,6),(2,3,4)-(2,4,6),(2,4,5)-(2,4,6),(0,6,7)-(4,6,7),(2,5,6)-(2,5,7),(1,2,3)-(1,3,5),(0,1,4)-(0,3,4)\}$. Then (V,H) is an undirected hinge decomposition of $6K_8$.

5.23 An Undirected Hinge Decomposition of $6K_{11}$

Let $S = \{s_1,s_2,s_3\}$, and define $6K_{11}$ on the vertex set $(S \times \{1,2,3\}) \cup \{\infty_1,\infty_2\}$. We define a hinge decomposition as follows:

(1) place two copies of a hinge decomposition of $3K_5$ on the vertex set $(\{s_1\} \times \{1,2,3\}) \cup \{\infty_1,\infty_2\}$;

(2) place three copies of a hinge decomposition of $2(K_5 \setminus K_2)$ on each of the vertex sets $(\{s_2\} \times \{1,2,3\}) \cup \{\infty_1,\infty_2\}$ and $(\{s_3\} \times \{1,2,3\}) \cup \{\infty_1,\infty_2\}$ (in this decomposition, the pair $\{\infty_1,\infty_2\}$ does not occur in any hinge since this pair has already occurred six times in (1)); and

(3) place three copies of a hinge decomposition of $2K_{3,3,3}$ on the vertex set $S \times \{1,2,3\}$ such that the tripartition is formed by the three sets $(\{s_i\} \times \{1,2,3\})$, for $1 \le i \le 3$.

This gives the desired hinge decomposition of $6K_{11}$.

5.24 An Undirected Hinge Decomposition of $6K_{14}$

Let $S = \{s_1,s_2,s_3,s_4\}$, and consider $6K_{14}$ on the vertex set $(S \times \{1,2,3\}) \cup \{\infty_1,\infty_2\}$. Before we describe the hinge decomposition, we note that since a hinge decomposition of $2K_{3,3,3,3}$ exists, a hinge decomposition of $6K_{3,3,3,3}$ exists, as well. We define the hinge decomposition of $6K_{14}$ in the following manner.

(1) Place a hinge decomposition of $6K_5$ on the vertex set $(\{s_1\} \times \{1,2,3\}) \cup \{\infty_1,\infty_2\}$.

(2) For $2 \le i \le 4$, place a hinge decomposition of $6(K_5 \setminus K_2)$ on each of the vertex sets $(\{s_i\} \times \{1,2,3\}) \cup \{\infty_1,\infty_2\}$. In each of these

decompositions, it is important that the pair $\{\infty_1, \infty_2\}$ appears in no hinges, since that pair occurs six times in hinges defined in (1).

(3) Place a hinge decomposition of $6K_{3,3,3,3}$ on the vertex set $S \times \{1, 2, 3\}$, where the partition of the vertices is formed by the vertex sets $\{s_i\} \times \{1, 2, 3\}$ for $1 \leq i \leq 4$.

This forms the desired hinge decomposition of $6K_{14}$.

5.25 An Undirected Hinge Decomposition of $6(K_8 \setminus K_2)$

Consider $6(K_8 \setminus K_2)$ defined on the vertex set $V = \{0, 1, \ldots, 7\}$. Furthermore, assume that in $6(K_8 \setminus K_2)$, no edges join vertices 0 and 1. We let $H = \{(2,3,6) - (2,4,6), (3,5,7) - (4,5,7), (0,3,4) - (0,4,7), (1,3,5) - (1,5,6), (0,3,6) - (0,6,7), (1,2,4) - (1,2,7), (1,3,4) - (1,3,5), (0,2,5) - (0,4,5), (1,4,6) - (1,5,6), (1,2,7) - (1,3,7), (0,2,5) - (0,2,7), (0,2,5) - (0,5,6), (1,4,6) - (1,6,7), (0,3,7) - (0,6,7), (1,2,4) - (2,3,4), (2,3,5) - (2,3,6), (4,5,7) - (4,6,7), (0,4,5) - (0,4,6), (0,3,4) - (3,4,7), (1,5,7) - (5,6,7), (1,2,4) - (1,2,7), (1,3,5) - (1,3,6), (0,2,3) - (0,3,7), (0,2,6) - (2,5,6), (2,4,5) - (4,5,7), (3,4,6) - (3,5,6), (2,3,7) - (2,6,7)\}$. (V, H) is an undirected hinge decomposition of $6(K_8 \setminus K_2)$.

5.26 An Undirected Hinge Decomposition of $6(K_{11} \setminus K_5)$

Consider $6(K_{11} \setminus K_5)$ on the vertex set $\{0, 1, \ldots, 10\}$ in which there are no edges between pairs of vertices in the set $\{6, 7, 8, 9, 10\}$. We define the hinge decomposition in the following manner.

(1) Place three copies of a hinge decomposition of $2(K_5 \setminus K_2)$ on each of the vertex sets $\{0, 1, 2, 9, 10\}$ and $\{3, 4, 5, 9, 10\}$. We should be sure that the edge $\{9, 10\}$ occurs in no hinges.

(2) Place three copies of a hinge decomposition of $2K_{3,3,3}$ on the vertex set $\{1, 2, \ldots, 9\}$ such that the sets $\{0, 1, 2\}$, $\{3, 4, 5\}$, and $\{6, 7, 8\}$ form the tripartition of $2K_{3,3,3}$.

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