Some New Bush-Type Hadamard Matrices of Order 100 and Infinite Classes of Symmetric Designs

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Abstract

There are at least 52432 symmetric $(100, 45, 20)$ designs on which $Frob_{10} \times Z_2$ acts as an automorphism group. All these designs correspond to Bush-type Hadamard matrices of order 100, and each leads to an infinite class of twin designs with parameters

$$v = 100(81^m + 81^{m-1} + \cdots + 81 + 1),\ k = 45(81)^m,\ \lambda = 20(81)^m,$$

and an infinite class of Siamese twin designs with parameters

$$v = 100(121^m + 121^{m-1} + \cdots + 121 + 1),\ k = 55(121)^m,\ \lambda = 30(121)^m,$$

where m is an arbitrary positive integer. One of the constructed designs is isomorphic to that used by Z. Janko, H. Kharaghani and V. D. Tonchev [4].

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1 Introduction and Preliminaries

A symmetric (v, k, λ) design is a finite incidence structure $(\mathcal{P}, \mathcal{B}, I)$, where $\mathcal{P}$ and $\mathcal{B}$ are disjoint sets and $I \subseteq \mathcal{P} \times \mathcal{B}$, with the following properties:

1. $|\mathcal{P}| = |\mathcal{B}| = v$;
2. every element of $\mathcal{B}$ is incident with exactly k elements of $\mathcal{P}$;
3. every pair of distinct elements of $\mathcal{P}$ is incident with exactly λ elements of $\mathcal{B}$.

The elements of the set $\mathcal{P}$ are called points and the elements of the set $\mathcal{B}$ are called blocks.

Given two designs $\mathcal{D}_1 = (\mathcal{P}_1, \mathcal{B}_1, I_1)$ and $\mathcal{D}_2 = (\mathcal{P}_2, \mathcal{B}_2, I_2)$, an isomorphism from $\mathcal{D}_1$ onto $\mathcal{D}_2$ is a bijection which maps points onto points and blocks onto blocks preserving the incidence relation. An isomorphism from a symmetric design $\mathcal{D}$ onto itself is called an automorphism of $\mathcal{D}$. The set of all automorphisms of the design $\mathcal{D}$ forms a group; it is called the full automorphism group of $\mathcal{D}$ and denoted by $Aut\mathcal{D}$.

Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ be a symmetric (v, k, λ) design and $G \leq Aut\mathcal{D}$. The group action of G produces the same number of point and block orbits (see [9, Theorem 3.3, p. 79]). We denote that number by t, the point orbits by $\mathcal{P}_1, \ldots, \mathcal{P}_t$, the block orbits by $\mathcal{B}_1, \ldots, \mathcal{B}_t$, and put $|\mathcal{P}_r| = \omega_r$ and $|\mathcal{B}_i| = \Omega_i$. We shall denote the points of the orbit $\mathcal{P}_r$ by $r_0, \ldots, r_{\omega_r - 1}$, (i.e. $\mathcal{P}_r = \{r_0, \ldots, r_{\omega_r-1}\}$). Further, we denote by γ_{ir} the number of points of $\mathcal{P}_r$ which are incident with a representative of the block orbit $\mathcal{B}_i$. The numbers γ_{ir} are independent of the choice of the representative of the block orbit $\mathcal{B}_i$. For those numbers the following equalities hold (see [5]):

$$\sum_{r=1}^{t} \gamma_{ir} = k\,, \tag{1}$$

$$\sum_{r=1}^{t} \frac{\Omega_j}{\omega_r} \gamma_{ir} \gamma_{jr} = \lambda \Omega_j + \delta_{ij} \cdot (k - \lambda)\,. \tag{2}$$

Definition 1 *Let $(\mathcal{D})$ be a symmetric (v,k,λ) design and $G \leq Aut\ \mathcal{D}$. Further, let $\mathcal{P}_1,\ldots,\mathcal{P}_t$ be the point orbits and $\mathcal{B}_1,\ldots,\mathcal{B}_t$ the block orbits with respect to G, and let $\omega_1,\ldots,\omega_t$ and $\Omega_1,\ldots,\Omega_t$ be the respective orbit lengths. We call $(\mathcal{P}_1,\ldots,\mathcal{P}_t)$ and $(\mathcal{B}_1,\ldots,\mathcal{B}_t)$ the orbit distributions, and $(\omega_1,\ldots,\omega_t)$ and $(\Omega_1,\ldots,\Omega_t)$ the orbit size distributions for the design and the group G. A $(t\times t)$-matrix (γ_{ir}) with entries satisfying conditions (1) and (2) is called an orbit structure for the parameters (v,k,λ) and orbit distributions $(\mathcal{P}_1,\ldots,\mathcal{P}_t)$ and $(\mathcal{B}_1,\ldots,\mathcal{B}_t)$.*

The first step – when constructing designs for given parameters and orbit distributions – is to find all compatible orbit structures (γ_{ir}). The next step, called indexing, consists in determining exactly which points from the point orbit $\mathcal{P}_r$ are incident with a chosen representative of the block orbit $\mathcal{B}_i$ for each number γ_{ir}. Because of the large number of possibilities, it is often necessary to involve a computer in both steps of the construction.

Definition 2 *The set of those indices of points of the orbit $\mathcal{P}_r$ which are incident with a fixed representative of the block orbit $\mathcal{B}_i$ is called the index set for the position (i,r) of the orbit structure and the given representative.*

It is well known that the existence of a symmetric design with parameters $(4u^2, 2u^2-u, u^2-u)$ is equivalent to the existence of a regular Hadamard matrix of order $4u^2$ (see [15, Theorem 1.4 p. 280]). Thus, symmetric designs with parameters (100,45,20) have been known to exist for a long time. However, this fact did not impede combinatorialists from constructing new designs with that parameter triple (see [11], [14]).

A Hadamard matrix of order m is an $(m\times m)$-matrix $H = (h_{i,j})$, $h_{i,j} \in \{-1,1\}$, satisfying $HH^T = H^TH = mI$, where I is the unit matrix. A Bush-type Hadamard matrix of order $4n^2$ is a Hadamard matrix with the additional property of being a block matrix $H = [H_{i,j}]$ with blocks of size $2n\times 2n$, such that $H_{i,i} = J_{2n}$ and $H_{i,j}J_{2n} = J_{2n}H_{i,j} = 0$, $i\neq j$, $1\leq i\leq 2n$, $1\leq j\leq 2n$, where J_{2n} is the all-ones $(2n\times 2n)$-matrix.

H. Kharaghani [8] showed that a Bush-type Hadamard matrix of order $4n^2$ with $2n-1$ or $2n+1$ a prime power can be used to construct infinite classes of symmetric designs. Z. Janko, H. Kharaghani and V. D. Tonchev [4] have constructed a Bush-type Hadamard matrix of order 100 using a symmetric $(100,45,20)$ design admitting an automorphism group isomorphic to $Frob_{10}\times Z_{10}$ acting in 20 orbits of length 5 on points and blocks.

For a $\{0,\pm1\}$-matrix K let $K = K^+ - K^-$, where K^+ and K^- are $\{0,1\}$-matrices and $K = K^+ + K^-$ is a $\{0,1\}$-matrix. A $\{0,\pm1\}$-matrix

D is called a twin design if both D^+ and D^- are incidence matrices of symmetric designs with the same parameters. A $\{0, \pm 1\}$-matrix S is called a Siamese twin design sharing the entries of I, if $S = I + K - L$, where I, K, L are non-zero $\{0,1\}$-matrices and both $I+K$ and $I+L$ are incidence matrices of symmetric designs with the same parameters.

Definition 3 *Let G be a group written multiplicatively. A balanced generalized weighing matrix $BGW(v,k,\lambda)$ over G is a $(v \times v)$-matrix $W = (g_{ij})$ with entries from $\overline{G} = G \cup \{0\}$ such that each row of W contains exactly k nonzero entries, and for every $a, b \in \{1, \ldots, v\}$, $a \neq b$, the multiset $\{g_{ai}g_{bi}^{-1} \mid \ 1 \leq i \leq v,\ g_{ai} \neq 0, g_{bi} \neq 0\}$ contains exactly $\frac{\lambda}{|G|}$ copies of each element of G.*

2 Symmetric $(100, 45, 20)$ Designs

Lemma 1 *The following matrix, denoted by OS, is an orbit structure for the parameter triple $(100, 45, 20)$ and orbit distributions resulting in an orbit size distribution $(5, \ldots, 5)$ for blocks and points:*

$$\begin{pmatrix}
0&0&2&3&3&2&3&2&2&3&2&3&3&2&1&4&1&4&3&2\\
0&0&3&2&2&3&2&3&3&2&3&2&2&3&4&1&4&1&2&3\\
2&3&0&0&2&3&3&2&3&2&3&2&2&3&3&2&1&4&1&4\\
3&2&0&0&3&2&2&3&2&3&2&3&3&2&2&3&4&1&4&1\\
3&2&2&3&0&0&2&3&3&2&1&4&3&2&2&3&3&2&1&4\\
2&3&3&2&0&0&3&2&2&3&4&1&2&3&3&2&2&3&4&1\\
3&2&3&2&2&3&0&0&2&3&1&4&1&4&3&2&2&3&3&2\\
2&3&2&3&3&2&0&0&3&2&4&1&4&1&2&3&3&2&2&3\\
2&3&3&2&3&2&2&3&0&0&3&2&1&4&1&4&3&2&2&3\\
3&2&2&3&2&3&3&2&0&0&2&3&4&1&4&1&2&3&3&2\\
2&3&3&2&1&4&1&4&3&2&0&0&3&2&2&3&2&3&3&2\\
3&2&2&3&4&1&4&1&2&3&0&0&2&3&3&2&3&2&2&3\\
3&2&2&3&3&2&1&4&1&4&3&2&0&0&3&2&2&3&2&3\\
2&3&3&2&2&3&4&1&4&1&2&3&0&0&2&3&3&2&3&2\\
1&4&3&2&2&3&3&2&1&4&2&3&3&2&0&0&3&2&2&3\\
4&1&2&3&3&2&2&3&4&1&3&2&2&3&0&0&2&3&3&2\\
1&4&1&4&3&2&2&3&3&2&2&3&2&3&3&2&0&0&3&2\\
4&1&4&1&2&3&3&2&2&3&3&2&3&2&2&3&0&0&2&3\\
3&2&1&4&1&4&3&2&2&3&3&2&2&3&2&3&3&2&0&0\\
2&3&4&1&4&1&2&3&3&2&2&3&3&2&3&2&2&3&0&0
\end{pmatrix}$$

Proof. Use equations (1) and (2). □

Theorem 1 *Up to isomorphism there are exactly four symmetric* $(100,45,20)$ *designs admitting an automorphism group* G *isomorphic to* $Frob_{10} \times Z_5$ *which operates in such a way that* $Frob_{10}$ *acts in* 20 *orbits of length* 5 *on points and blocks and induces the orbit structure* OS *from Lemma 1. Further, a generator of* Z_5 *acts as the permutation*

$$(1,3,5,7,9)(2,4,6,8,10)(11,13,15,17,19)(12,14,16,18,20)$$

on the 20 *point and block orbits of the group* $Frob_{10}$. *The full automorphism groups of these four designs are all isomorphic to* $Frob_{10} \times Z_{10}$. *The designs are not self-dual.*

Sketch of proof. The designs have been constructed by the method described in [1] and [3]. We denote the points by $1_i, \ldots, 20_i$, $i = 0,1,2,3,4$ and put $G = \langle \rho, \sigma, \tau \rangle$ where the generators for G are permutations defined as follows:

$$\rho = (I_0, I_1, I_2, I_3, I_4), \; I = 1,2,\ldots,20,$$

$$\sigma = (K_0)(K_1, K_4)(K_2, K_3), \; K = 1,2,\ldots,20,$$

$$\tau = (1_i, 3_i, 5_i, 7_i, 9_i)\ (2_i, 4_i, 6_i, 8_i, 10_i)\ (11_i, 13_i, 15_i, 17_i, 19_i)$$
$$(12_i, 14_i, 16_i, 18_i, 20_i), \; i = 0,1,2,3,4.$$

To eliminate isomorphic structures during the indexing process we have used the permutation which – on each $\langle\rho\rangle$-orbit – acts as $x \mapsto 2x$ $(mod\ 5)$, and those automorphisms of our orbit structure OS which commute with τ.

As representatives for the block orbits we chose blocks fixed by $\langle\sigma\rangle$. Therefore, the index sets – numbered from 0 to 6 – which could occur in the designs are among the following:

$0 = \emptyset$, $1 = \{0\}$, $2 = \{1,4\}$, $3 = \{2,3\}$, $4 = \{0,1,4\}$, $5 = \{0,2,3\}$, $6 = \{1,2,3,4\}$.

The indexing process of our orbit structure OS led to four designs, denoted by $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$ and $\mathcal{D}_4$. Using a computer program by V. Krčadinac (see [7], [10]) we get that $(\mathcal{D}_1, \mathcal{D}_3)$ and $(\mathcal{D}_2, \mathcal{D}_4)$ are pairs of mutually dual designs. We write down base blocks for the designs $\mathcal{D}_1$ and $\mathcal{D}_2$ in terms of the index sets defined above:

$$\mathcal{D}_1$$

$B_1 : 0\ 0\ 2\ 4\ 4\ 3\ 5\ 2\ 2\ 4\ 2\ 4\ 4\ 2\ 1\ 6\ 1\ 6\ 5\ 3$
$B_2 : 0\ 0\ 4\ 2\ 3\ 4\ 2\ 5\ 4\ 2\ 4\ 2\ 2\ 4\ 6\ 1\ 6\ 1\ 3\ 5$
$B_{11} : 2\ 4\ 5\ 3\ 1\ 6\ 1\ 6\ 4\ 2\ 0\ 0\ 5\ 3\ 2\ 5\ 3\ 4\ 5\ 3$
$B_{12} : 4\ 2\ 3\ 5\ 6\ 1\ 6\ 1\ 2\ 4\ 0\ 0\ 3\ 5\ 5\ 2\ 4\ 3\ 3\ 5$

$$\mathcal{D}_2$$

$B_1 : 0\ 0\ 2\ 4\ 4\ 3\ 5\ 2\ 2\ 4\ 2\ 4\ 4\ 2\ 1\ 6\ 1\ 6\ 5\ 3$
$B_2 : 0\ 0\ 4\ 2\ 3\ 4\ 2\ 5\ 4\ 2\ 4\ 2\ 2\ 4\ 6\ 1\ 6\ 1\ 3\ 5$
$B_{11} : 3\ 5\ 4\ 2\ 1\ 6\ 1\ 6\ 5\ 3\ 0\ 0\ 4\ 2\ 2\ 5\ 3\ 4\ 4\ 2$
$B_{12} : 5\ 3\ 2\ 4\ 6\ 1\ 6\ 1\ 3\ 5\ 0\ 0\ 2\ 4\ 5\ 2\ 4\ 3\ 2\ 4$

We have determined the automorphism groups of the constructed designs using GAP [2] and a program by V. D. Tonchev [13]. □

Theorem 2 *There are at least* 52432 *pairwise nonisomorphic symmetric* (100, 45, 20) *designs admitting an automorphism group H isomorphic to $Frob_{10} \times Z_2$.*

Sketch of proof. The designs are obtained from the orbit structure OS together with the assumed group H. We put $H = \langle \rho, \sigma, \tau \rangle$, where the generators of H are permutations defined as follows:

$$\rho = (I_0, I_1, I_2, I_3, I_4),\ I = 1, 2, \ldots, 20,$$

$$\sigma = (K_0)(K_1, K_4)(K_2, K_3),\ K = 1, 2, \ldots, 20,$$

$$\tau = (1_i, 2_i)(3_i, 4_i)(5_i, 6_i)(7_i, 8_i)(9_i, 10_i)(11_i, 12_i)(13_i, 14_i)(15_i, 16_i)\ (17_i, 18_i)\ (19_i, 20_i),\ i = 0, 1, 2, 3, 4.$$

We have constructed 104864 designs. Among them there are 52432 designs with mutually different statistics of intersections of any three blocks. All constructed designs except the designs described in Theorem 1 have $Frob_{10} \times Z_2$ as full automorphism group. □

3 Bush-Type Hadamard Matrices of Order 100 and Classes of Symmetric Designs

Replacing each zero by 1 and each one by −1 in the incidence matrix of each symmetric (100, 45, 20) design from Theorem 2 results in a Bush-type

Hadamard matrix of order 100. Some of these Hadamard matrices might be equivalent.

Let $U = circ(0,1,0,\ldots,0)$ be the circulant matrix of order 10 and $N = diag(-1,1,1,\ldots,1)$ be the diagonal matrix of order 10. Let $E = UN$ and $G_{20} = \{E^i \otimes I_{10} | i = 1,2,\ldots,20\}$, where $E^i \otimes I_{10}$ is a Kronecker product, and I_{10} is the unit matrix of order 10. Then E is a signed permutation matrix of order 10, G_{20} is a cyclic group of order 20, and $\Sigma_{g \in G_{20}} \, g = 0$.

The q-ary simplex code $S_d(q)$ of length $(q^d - 1)/(q-1)$, where $d \geq 2$ and q is a prime power, is a linear code over $GF(q)$ with a generator matrix having as columns representatives of all distinct one-dimensional subspaces of the d-dimensional vector space $GF(q)^d$. The code $S_d(q)$ is the dual of the unique linear perfect single-error-correcting code of length $(q^d - 1)/(q-1)$ over $GF(q)$. We use the following theorem of Jungnickel and Tonchev [6]:

Theorem 3 *Any $(q^d-1)/(q-1) \times (q^d-1)/(q-1)$ matrix M with rows a set of representatives of the $(q^d-1)/(q-1)$ distinct one-dimensional subspaces of $S_d(q)$ is a balanced generalized weighing matrix with parameters*

$$v = \frac{q^d - 1}{q - 1}, \quad k = q^d - 1, \quad \lambda = q^{d-1} - q^{d-2}$$

over the multiplicative group $GF(q)^*$.

Corollary 1 *There exists a balanced generalized weighing matrix $BGW(q^m + q^{m-1} + \ldots + q + 1, q^m, q^m - q^{m-1})$ over the group G_{20} for each positive integer m and $q \in \{81, 121\}$.*

Proof. This is a direct consequence of Theorem 3, because 81 and 121 are prime powers and 20 divides 80 and 120. □

Each Bush-type Hadamard matrix of order 100 leads to infinite classes of twin and Siamese twin designs.

Theorem 4 *Let m be a positive integer, H any Bush-type Hadamard matrix of order 100, and $M = H - I_{10} \otimes J_{10}$. Further, let $W = [w_{ij}]$ be the balanced generalized weighing matrix $BGW(81^m + 81^{m-1} + \ldots + 81 + 1, 81^m, 81^m - 81^{m-1})$ from Corollary 1. Then, the matrix $D = [Mw_{ij}]$ is a twin design with parameters*

$$(1) \quad v = 100(81^m + 81^{m-1} + \cdots + 81 + 1), \; k = 45(81)^m, \; \lambda = 20(81)^m.$$

Proof. We follow the proof by H. Kharaghani ([8, Theorem 8, p. 8]). Let $P = J_{100} - I_{10} \otimes J_{10}$. For a matrix $w_{ij} = [a_{kl}]$ denote by $|w_{ij}|$ the matrix $[|a_{kl}|]$. The matrices

$$D^{+} = \frac{1}{2}[P|w_{ij}| + Mw_{ij}] \quad and \quad D^{-} = \frac{1}{2}[P|w_{ij}| - Mw_{ij}]$$

are symmetric designs with parameters (1).

To prove that D^+ is a symmetric (v, k, λ) design we have to prove that $D^+ D^{+t} = \lambda J + (k - \lambda)I$.

For $k \neq l$,
$\Sigma_{j=1}^{(81^m+\ldots+81+1)} Mw_{kj}(Mw_{jl})^t = \Sigma_{j=1}^{(81^m+\ldots+81+1)} M(w_{kj}w_{lj}^t)M^t =$
$M(\frac{81^m - 81^{m-1}}{20}\Sigma_{g\in G_{20}}\, g)M^t = 0.$

For $k = l$,
$\Sigma_{j=1}^{(81^m+\ldots+81+1)} Mw_{kj}(Mw_{jk})^t = 81^m MM^t.$

Therefore, $[Mw_{ij}][Mw_{ij}]^t = 81^m(I_{81^m+\ldots+81+1} \otimes MM^t)$. Also, for every i, j, k, l, $(P|w_{ij}|)(Mw_{kl})^t = (Mw_{kl})(P|w_{ij}|)^t = 0$.

Thus, we have,
$4D^+D^{+t} = [P|w_{ij}|][P|w_{ij}|]^t + [Mw_{ij}][Mw_{ij}]^t =$
$[P|w_{ij}|][P|w_{ij}|]^t + 81^m(I_{81^m+\ldots+81+1} \otimes MM^t)$.

For $k \neq l$,
$\Sigma_{j=1}^{(81^m+\ldots+81+1)} P|w_{kj}|(P|w_{jl}|)^t = \Sigma_{j=1}^{(81^m+\ldots+81+1)} P|w_{kj}||w_{lj}|^t P^t =$
$\frac{81^m - 81^{m-1}}{10} P(J_{10} \otimes I_{10})P^t = 80(81)^m J_{100}$.

For $k = l$,
$\Sigma_{j=1}^{(81^m+\ldots+81+1)} P|w_{kj}|(P|w_{jk}|)^t + \Sigma_{j=1}^{(81^m+\ldots+81+1)} Mw_{kj}(Mw_{jk})^t =$
$81^m(PP^t + MM^t) = 81^m(100I_{100} + 80J_{100})$.

Therefore, D^+ is a symmetric design with parameters (1). In a similar way one can prove that D^- is also a symmetric design with parameters (1). □

Theorem 5 *Let m be a positive integer, H any Bush-type Hadamard matrix of order 100, and $M = H - I_{10} \otimes J_{10}$. Further, let $W = [w_{ij}]$ be the balanced generalized weighing matrix $BGW(121^m + 121^{m-1} + \ldots + 121 + 1, 121^m, 121^m - 121^{m-1})$ from the above corollary. Then, the matrices*

$$\frac{1}{2}[((J_{10}+I_{10})\otimes J_{10})|w_{ij}| + Mw_{ij}] \quad and \quad \frac{1}{2}[((J_{10}+I_{10})\otimes J_{10})|w_{ij}| - Mw_{ij}]$$

are symmetric designs with parameters

$$v = 100(121^m + 121^{m-1} + \cdots + 121 + 1), \ k = 55(121)^m, \ \lambda = 30(121)^m.$$

sharing the entries of $I = [(I_{10} \otimes J_{10})|w_{ij}|]$.

Proof. Similar to the proof of Theorem 4. □

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