Graphs with small upper line-distinguishing and upper harmonious chromatic numbers

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Abstract

A k-line-distinguishing coloring of a graph $G = (V, E)$ is a partition of V into k sets $V_1, \dots, V_k$ such that $q(\langle V_i \rangle) \leq 1$ for $i = 1, \dots, k$ and $q(V_i, V_j) \leq 1$ for $1 \leq i < j \leq k$. If the color classes in a line-distinguishing coloring is also independent, then it is called a harmonious coloring.. A coloring is minimal if, when two color classes are combined, we no longer have a coloring of the given type. The upper harmonious chromatic number, $H(G)$, is defined as the maximum cardinality of a minimal harmonious coloring of a graph G, while the upper line-distinguishing chromatic number, $H'(G)$, is defined as the maximum cardinality of a minimal line-distinguishing coloring of a graph G. For any graph G of maximum degree $\Delta(G)$, $H'(G) \geq \Delta(G)$

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and $H(G) \geq \Delta(G)+1$. We characterize connected graphs G that contain neither a triangle nor a 5-cycle for which $H(G) = \Delta(G)+1$. We show that a triangle-free connected graph G satisfies $H'(G) = \Delta(G)$ if and only if G is a star $K_{1,\Delta(G)}$. A partial characterization of connected graphs G for which $H'(G) = \Delta(G)$ is obtained.

Keywords. Upper harmonious chromatic number, upper line-distinguishing chromatic number

1 Introduction

Graph theory terminology not presented here may be found in [1]. Let $G = (V, E)$ be a graph with n vertices. If $A \subseteq V$ and $B \subseteq V$, we will use $q(A, B)$ to denote the number of edges between the sets A and B. Let $S \subseteq V$. The set S is *independent* if for distinct $u, v \in S$, $uv \notin E$, while S is a *packing* if every two vertices in S are at distance at least 3 apart in G. The subgraph induced by S is denoted by $\langle S \rangle$. The distance $d(v, S)$ from a vertex v to the set S is defined as the minimum distance from v to a vertex of S. If $v \in V$, then the *open neighborhood* of v is $N(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood of* v is $N[v] = \{v\} \cup N(v)$.

A *k-coloring* of G is a partition Π of V into k sets, $V_1, V_2, \ldots, V_k$. A *proper k-coloring* is a k-coloring such that each V_i is independent. A k-coloring is a *complete coloring* if for every i, j, $1 \leq i < j \leq k$, $q(V_i, V_j) \geq 1$.

The *chromatic number* $\chi(G)$ is defined as $\min\{k \mid G$ has a proper k-coloring$\}$, while the *achromatic number* $\psi(G)$ is defined as $\max\{k \mid G$ has a proper complete k-coloring$\}$.

A *k-line-distinguishing coloring* of G is a partition of V into k sets $V_1, \ldots, V_k$ such that $q(\langle V_i \rangle) \leq 1$ for $i = 1, \ldots, k$ and $q(V_i, V_j) \leq 1$ for $1 \leq i < j \leq k$.

If a line-distinguishing coloring is also a proper coloring, then it is called a *harmonious coloring.* In other words, the partition $\{V_1, V_2, \ldots, V_k\}$ is a harmonious coloring of G if and only if $q(\langle V_i \rangle) = 0$ for $i = 1, \ldots, k$ and $q(V_i, V_j) \leq 1$, $1 \leq i < j \leq k$.

The *line-distinguishing coloring number* $h'(G)$ is defined as $\min\{k \mid G$ has a k-line-distinguishing coloring$\}$, while the *harmonious coloring number* $h(G)$ is defined as $\min\{k \mid G$ has a k-harmonious coloring$\}$.

The achromatic number was first introduced and studied by Harary, Hedetniemi and Prins [6]. The line-distinguishing number, $h'(G)$, was introduced independently by Frank, Harary and Plantholt [7] and Hopcroft and Krishnamoorthy [8] even though the latter authors called it the harmonious coloring number. Harmonious colorings were introduced by Miller and Pritikin in [9] and further investigated in [4] and [5].

Consider a partition $\Pi = \{V_1, V_2, \ldots, V_k\}$ of V according to some specified properties P and Q. By definition this means that $\langle V_i \rangle$ has property P for $i = 1, \ldots, k$ and the bipartite graph (V_i, V_j) has property Q for distinct $i, j \in \{1, \ldots, k\}$. The partition is *minimal* with respect to properties P and Q if any partition Π' obtained from Π by combining color classes V_i and V_j no longer satisfies properties P and Q. The smallest and largest cardinality of minimal partitions with respect to properties P and Q give rise to two parameters associated with a graph. For example, the chromatic and achromatic numbers are, respectively, the minimum and maximum cardinality of a minimal partition where the property P specifies that the induced subgraph of each set in the partition contains no edge.

Let P be the property "contains no edges" and Q be the property "contains at most one edge". If $\Pi = \{V_1, \ldots, V_k\}$ is a partition according to the properties P and Q, then Π is a harmonious coloring of G. If we change property P to "contains at most one edge", then Π becomes a line-distinguishing coloring of G. Before proceeding further, we state a characterization of minimal harmonious and minimal line-distinguishing colorings of a graph, as given in [2].

Lemma 1 (Chen et al. [2]) *A harmonious coloring $\{V_1, \ldots, V_k\}$ is minimal if and only for distinct $i, j \in \{1, \ldots, k\}$*

(1) $q(V_i, V_j) = 1$, *or*

(2) *if $V_i \cup V_j$ is independent, there is an $r \in \{1, \ldots, k\} - \{i, j\}$ such that $q(V_i, V_r) = 1$ and $q(V_r, V_j) = 1$.*

Lemma 2 (Chen et al. [2]) *A line-distinguishing coloring $\{V_1, \ldots, V_k\}$ is minimal if and only for distinct $i, j \in \{1, \ldots, k\}$*

(1) $q(\langle V_i \cup V_j \rangle) > 1$, *or*

(2) *if $q(\langle V_i \cup V_j \rangle) \leq 1$, there is an $r \in \{1, \ldots, k\} - \{i, j\}$ such that $q(V_i, V_r) = 1$ and $q(V_r, V_j) = 1$.*

The *upper harmonious chromatic number*, $H(G)$, is defined as the maximum cardinality of a minimal harmonious coloring of a graph G, while the *upper line-distinguishing chromatic number*, $H'(G)$, is defined as the maximum cardinality of a minimal line-distinguishing coloring of a graph G. These parameters were first introduced and studied in [2]. In particular, it was shown that the decision problems corresponding to the computation of $H(G)$ and $H'(G)$ for a general graph G are NP-complete, that the two parameters are incomparable, even for trees, and, lastly, $H(P_n)$ and $H'(P_n)$ were determined for the path P_n of order n.

For any graph G of maximum degree $\Delta(G)$, $H'(G) \geq \Delta(G)$ and $H(G) \geq \Delta(G) + 1$. In this paper, we characterize connected graphs G that contain neither a triangle nor a 5-cycle for which $H(G) = \Delta(G) + 1$. We show that a triangle-free connected graph G satisfies $H'(G) = \Delta(G)$ if and only if G is a star $K_{1,\Delta(G)}$. A partial characterization of connected graphs G for which $H'(G) = \Delta(G)$ is obtained.

2 Graphs G satisfying $H(G) = \Delta(G) + 1$

For any graph G, $H(G) \geq \Delta(G) + 1$. Our aim in this section is to characterize graphs G that have neither a triangle nor a 5-cycle for which $H(G) = \Delta(G) + 1$. We begin with two lemmas.

Lemma 3 *If F is an induced subgraph of a graph G, then $H(G) \geq H(F)$.*

Proof. Let C be a minimal harmonious coloring of F that uses $H(F)$ colors. Then, C satisfies conditions (1) and (2) of Lemma 1. We show that C can be extended to a minimal harmonious coloring of G. We consider each vertex v of $V(G) - V(F)$ in turn. If v can be added to any one of the existing color classes so that the resulting coloring remains harmonious, then we add v to that color class; otherwise, we leave v uncolored. Once all vertices of $V(G) - V(F)$ have been considered in turn, we give to every remaining uncolored vertex, if any, a different color. No new color class can be combined with any original color class by construction. If no new colors are used, we have a minimal harmonious coloring of G that uses all the original $H(F)$ colors, and so $H(G) \geq H(F)$. On the other hand, if new colors are used, then we simply combine pairs of the new color classes as long as the resulting coloring remains harmonious. When we reach a point where no further pairs of new color classes can be combined (without destroying the harmonious coloring), then the coloring must be minimal.

Hence we have a minimal harmonious coloring of G that uses the original colors and at least one new color, and so $H(G) \geq H(F) + 1$. □

Lemma 4 *Let G be a connected graph with maximum degree Δ and let x be a vertex of degree Δ. If $H(G) = \Delta + 1$, then every vertex of G is within distance 2 from x.*

Proof. Suppose there exists a vertex z at distance 3 from x in G. Let x, u, v, z be a x–z path and let F be the subgraph of G induced by $N[x] \cup \{v, z\}$. We now construct a minimal harmonious coloring of F that uses at least $\Delta + 2$ colors. The vertices in $N[x]$ require $\Delta + 1$ colors in any harmonious coloring of F. Let x and z be colored with color 1, the Δ vertices in $N(x)$ with the colors $2, \ldots, \Delta + 1$, and let v be colored with the color $\Delta + 2$. Let $V_1, V_2, \ldots, V_{\Delta+2}$ denote the color classes containing the vertices of colors $1, 2, \ldots, \Delta + 2$, respectively. Then, no two of the color classes $V_1, V_2, \ldots, V_{\Delta+2}$ can be combined since the vertex in each of $V_2, \ldots, V_{\Delta+2}$ is adjacent to a vertex in V_1. Hence, by Lemma 1, there exists a minimal harmonious coloring of F that uses at least $\Delta + 2$ colors, and so $H(F) \geq \Delta + 2$. Hence, by Lemma 3, $H(G) \geq \Delta + 2$, a contradiction. Hence, every vertex of G is within distance 2 from x. □

Recall that a packing in G is a set of vertices that are pairwise at distance at least 3 apart in G. For disjoint nonempty subsets $A = \{a_1, \ldots, a_\ell\}$ and B of $V(G)$, if there exists a partition $B_1, \ldots, B_\ell$ of B such that for all $i = 1, \ldots, \ell$, $\{a_i\} \cup B_i$ is a packing in G and for all i, j, $1 \leq i < j \leq \ell$, there is at most one edge between $\{a_i\} \cup B_i$ and $\{a_j\} \cup B_j$, then we say that A is *specially matched* to B. Moreover, if $|B_i| = 1$ for $i = 1, \ldots, \ell$, then we say that A is *specially 1-matched* to B.

Theorem 5 *Let $G = (V, E)$ be a connected graph that has neither a triangle nor a 5-cycle and let x be a vertex of maximum degree Δ. For a packing $D \subseteq V - N[x]$ in G, let $R_D = \{v \in V - N[x] - D \mid d(v, D) = 2\}$, $S_D = N(x) \cap N(D)$ and $T_D = N(x) - S_D$. Then $H(G) = \Delta + 1$ if and only if*

(1) *every vertex of G is within distance 2 from x, and*

(2) *for every packing $D \subseteq V - N[x]$ in G, $T_D \neq \emptyset$ and if $R_D \neq \emptyset$, then*

   (a) *T_D cannot be specially 1-matched to a subset of R_D, and*

   (b) *for all nonempty subsets S' of S_D and all subsets R' of $V - N[x] - D$ such that $|S'| \leq |R'|$ either S' cannot be specially matched to*

R' or $T_D - N(R')$ cannot be specially 1-matched to a subset of $R_D - R'$.

Proof. Suppose, first, that $H(G) = \Delta + 1$. Then Condition (1) holds by Lemma 4. Let $D \subseteq V - N[x]$ be a packing in G.

Claim 1 *If $T_D = \emptyset$, then $H(G) \geq \Delta + 2$.*

Proof. Since $T_D = \emptyset$, $S_D = N(x)$. Let F be the subgraph of G induced by $N[x] \cup D$. We now construct a minimal harmonious coloring of F that uses at least $\Delta + 2$ colors. The vertices in $N[x]$ require $\Delta + 1$ colors in any harmonious coloring of F. Let x be colored with color 1, the Δ vertices in $N(x)$ with the colors $2, \dots, \Delta + 1$, and each vertex in D with the color $\Delta + 2$. Let $V_1, V_2, \dots, V_{\Delta+2}$ denote the color classes containing the vertices of colors $1, 2, \dots, \Delta + 2$, respectively. Since V_2 contains a vertex which is adjacent to a vertex in V_1 and to a vertex in $V_{\Delta+2}$, the color classes V_1 and $V_{\Delta+2}$ cannot be combined. Furthermore, by Lemma 1, $V_{\Delta+2}$ cannot be combined with any of the color classes $V_2, \dots, V_{\Delta+1}$ since, by assumption, the vertex in each such color class is a vertex of $N(x)$ which is adjacent to some vertex of D. It follows that no two of the color classes $V_1, V_2, \dots, V_{\Delta+2}$ can be combined. Hence, by Lemma 1, there exists a minimal harmonious coloring of F that uses at least $\Delta + 2$ colors, and so $H(F) \geq \Delta + 2$. Hence, by Lemma 3, $H(G) \geq \Delta + 2$. □

By Claim 1, $T_D \neq \emptyset$. Assume then that $R_D \neq \emptyset$.

Claim 2 *If T_D can be specially 1-matched to a subset of R_D, then $H(G) \geq \Delta + 2$.*

Proof. Suppose T_D can be specially 1-matched to a subset R'_D of R_D. Let F be the subgraph of G induced by $N[x] \cup D \cup R'_D$. We now construct a minimal harmonious coloring of F that uses at least $\Delta + 2$ colors. Let the vertices in $N[x] \cup D$ be colored as in the proof of Claim 1. We now extend the colors of T_D to R'_D by coloring each vertex in R'_D with the same color used to color the corresponding vertex of T_D with which it is specially 1-matched. Let $V_1, V_2, \dots, V_{\Delta+2}$ denote the color classes containing the vertices of colors $1, 2, \dots, \Delta + 2$, respectively. The color class $V_{\Delta+2}$ cannot be combined with any of the color classes that contain a vertex of S_D by Property 1 of Lemma 1. Since each vertex in R'_D is at distance 2 from a vertex of D, the color class $V_{\Delta+2}$ cannot be combined with any of the color classes that contain a vertex of T_D by Property 2 of Lemma 1 (since each such color class also contains a vertex of R_D). Hence, since G is triangle-free, there is a minimal harmonious coloring of F with at least $\Delta+2$ colors, and so by Lemma 3, $H(G) \geq H(F) \geq \Delta+2$. □

By Claim 2, T_D cannot be specially 1-matched to a subset of R_D, i.e., Condition 2(a) holds.

Claim 3 *If there is a nonempty subset S' of S_D and a subset R' of $V - N[x] - D$ such that $|S'| \leq |R'|$, S' can be specially matched to R' and $T_D - N(R')$ can be specially* 1*-matched to some subset of $R_D - R'$, then $H(G) \geq \Delta+2$.*

Proof. For $i = 1, \ldots, |S'|$, let $\{s_i\} \cup R_i$ be the special matching of S' to R', while for $j = 1, \ldots, |T_D - N(R')|$, let $\{t_j, r_j\}$ be the special 1-matching of $T_D - N(R')$ to some subset R'_D of $R_D - R'$. Let F be the subgraph of G induced by $N[x] \cup D \cup R' \cup R'_D$. We now construct a minimal harmonious coloring of F that uses at least $\Delta+2$ colors. Let the vertices in $N[x] \cup D$ be colored as in the proof of Claim 1. Color each vertex of R_i with the same color that s_i ($i = 1, \ldots, |S'|$) received and color r_j with the same color that t_j ($j = 1, \ldots, |T_D - N(R')|$) received. Let $V_1, V_2, \ldots, V_{\Delta+2}$ denote the color classes containing the vertices of colors $1, 2, \ldots, \Delta+2$, respectively. Then by the definition of a special matching and since $t_j \notin N(R')$ for $j = 1, \ldots, |T_D - N(R')|$ and since G has neither a triangle nor a 5-cycle, it follows that there is at most one edge between any two of the color classes V_i and V_j, $1 \leq i < j \leq \Delta+1$. (Note that if we relax the condition that G has neither a triangle nor a 5-cycle, then it is possible, for example, for s_1 to be adjacent to t_1 and for a vertex of R_1 to be adjacent to r_1, thereby producing at least two edges between the color class $\{s_1\} \cup R_1$ and the color class $\{t_1, r_1\}$.)

The color class $V_{\Delta+2}$ cannot be combined with any of the color classes that contain a vertex of S_D by Property 1 of Lemma 1. Furthermore, $V_{\Delta+2}$ cannot be combined with any of the color classes in $N(R') \cap N(x)$, since every vertex in $N(R') \cap N(x)$ is adjacent to a vertex in a color class in R' which also contains a vertex adjacent to some vertex in $V_{\Delta+2}$. Since each vertex in $R_D - R'$ is at distance 2 from a vertex of D, the color class $V_{\Delta+2}$ cannot be combined with any of the color classes that contain a vertex of $T_D - N(R')$ by Property 2 of Lemma 1 (since each such color class also contains a vertex of $R_D - R'$). Hence, there is a minimal harmonious coloring of F with at least $\Delta+2$ colors, and so by Lemma 3, $H(G) \geq H(F) \geq \Delta+2$. □

By Claim 3, Condition 2(b) holds. This proves the necessity.

To prove the sufficiency, suppose that Conditions (1) and (2) hold but $H(G) \geq \Delta + 2$. Then there is a minimal harmonious coloring C of G using the colors $1, 2, \ldots, c$, where $c \geq \Delta + 2$. Let $V_1, V_2, \ldots, V_c$ denote the color classes containing the vertices of colors $1, 2, \ldots, c$, respectively. We may assume that the Δ vertices in $N[x]$ are colored with the colors $2, \ldots, \Delta + 1$ and that x is colored with color 1. By Condition (1), each vertex of $V - N[x]$ is adjacent to some vertex of $N(x)$. Hence since G has neither a triangle nor a 5-cycle, the sets $N(x)$ and $V - N[x]$ are both independent. We now consider the set $V_{\Delta+2}$ which is contained in $V - N[x]$. Let $D = V_{\Delta+2}$, and note that D is a packing of G.

Since C is a minimal harmonious coloring of G, the color class $V_{\Delta+2}$ cannot be combined with any color class that contains a vertex of T_D. Let $u \in T_D$ and let C_u be the color class of C that contains u. Since there is no edge between $V_{\Delta+2}$ and T_D, and since the set $V - N[x]$ is independent, the set $C_u \cup V_{\Delta+2}$ is independent. Hence, by Property 2 of Lemma 1, we must have a color class, B_u say, which contains a vertex adjacent to a vertex of C_u and a vertex adjacent to a vertex of D. Either $B_u \subseteq N(x)$, in which case $|B_u| = 1$ and C_u also contains a vertex from R_D (since G is triangle-free), or $B_u \not\subseteq N(x)$, in which case $|B_u \cap N(x)| = 1$ and $B_u \cap (V - N[x] - D) \neq \emptyset$. If $B_u \not\subseteq N(x)$, we say that u is a vertex of T_D of type-1, while if $B_u \subseteq N(x)$, we say that u is of type-2. If T_D has at least one vertex of type-1, then let $\{u_1, \ldots, u_\ell\}$ denote the vertices of T_D of type-1. Let $S' = (\cup_{i=1}^{\ell} B_{u_i}) \cap S_D$ and let $R' = (\cup_{i=1}^{\ell} B_{u_i}) - S'$. Then, $S' \subseteq S_D$ and $R' \subseteq V - N[x] - D$ such that $|S'| \leq |R'|$ and S' can be specially matched to R'. Suppose $T_D - N(R') \neq \emptyset$. Then each vertex of $T_D - N(R')$ is of type-2. For each vertex $u \in T_D - N(R')$, let $r_u \in C_u \cap (V - N[x])$ and let $R = \cup\{r_u\}$ over all vertices $u \in T_D - N(R')$. Then $R \subseteq R_D - R'$, and $T_D - N(R')$ can be specially 1-matched to R. Thus we have established the existence of a subset S' of S_D and a subset R' of $V - N[x] - D$ such that $|S'| \leq |R'|$, S' can be specially matched to R' and $T_D - N(R')$ can be specially 1-matched to a subset of $R_D - R'$. This, however, contradicts Condition (2). We deduce, therefore, that $H(G) = \Delta + 1$. □

Note that if G is a graph in the statement of Theorem 5 that satisfies $H(G) = \Delta + 1$, then G is in fact bipartite. Following the notation introduced in the statement of Theorem 5, we have the following immediate consequence of Theorem 5.

Corollary 6 *Let $G = (V, E)$ be a connected graph that has neither a triangle nor a 5-cycle and let x be a vertex of maximum degree Δ. Suppose* *every vertex of G is within distance 2 from x. Then,* $H(G) > \Delta + 1$ *if and only if there exists a packing* $D \subseteq V - N[x]$ *in G such that either* $T_D = \emptyset$ *or* $R_D \neq \emptyset$ *and*

(1) T_D *can be specially* 1*-matched to a subset of* R_D *or*

(2) *there exists a nonempty subset* S' *of* S_D *and a subset* R' *of* $V - N[x] - D$ *such that* $|S'| \leq |R'|$, S' *can be specially matched to* R', *and* $T_D - N(R')$ *can be specially* 1*-matched to a subset of* $R_D - R'$.

As a consequence of Theorem 5 when the graph G is a tree, we have the following characterization of trees T for which $H(T) = \Delta(T) + 1$ due to Domke and Jonck [3].

Theorem 7 (Domke and Jonck [3]) *Let* $T = (V, E)$ *be a tree of maximum degree* Δ *and let* x *be a vertex of degree* Δ *where* $L \subseteq N(x)$ *is the set of all leaves adjacent to* x *and* $|L| = \ell$. *Then,* $H(T) = \Delta + 1$ *if and only if the following three conditions hold:*

(1) $V - N[x]$ *is an independent set,*
(2) $\ell \geq 1$, *and*
(3) *every set* $S \subseteq N(x) - L$ *where* $|S| = k$ *has* $|N(S)| \leq k + \ell$.

Proof. Suppose $H(T) = \Delta + 1$. Then Conditions (1) and (2) of Theorem 5 hold. Since T is a tree, Condition (1) of Theorem 5 implies that no two vertices of $V - N[x]$ are adjacent, i.e., Condition (1) of Theorem 7 holds. Let D be a set of vertices formed by taking for each vertex in $N(x) - L$, a vertex in $V - N[x]$ adjacent to it. Then D is a packing for which $T_D = L$, and by Condition (2) of Theorem 5, $T_D \neq \emptyset$, i.e., Condition (2) of Theorem 7 holds. To show that Condition (3) of Theorem 7 holds, suppose to the contrary that there is a subset $S \subseteq N(x) - L$ with $|S| = k$ and $|N(S)| \geq k + \ell + 1$. Let D be the packing formed by taking for each vertex in $N(x) - L$, a vertex in $V - N[x]$ adjacent to it. Then, $R_D \supseteq N(S) - D - \{x\}$, $|R_D| \geq \ell$ and $T_D = L$. Thus T_D can be specially 1-matched to a subset of R_D, and so, by Corollary 6, $H(T) > \Delta + 1$, a contradiction. Thus, Condition (3) of Theorem 7 holds.

Conversely, suppose Conditions (1), (2) and (3) of Theorem 7 hold. Then Condition (1) of Theorem 5 holds. It remains to show that Condition (2) of Theorem 5 holds. Let $D \subseteq V - N[x]$ be a packing in G for which $R_D \neq \emptyset$. Note that $L \subseteq T_D$, and so, by Condition (2) of Theorem 7, $|T_D| \geq \ell \geq 1$.

Let $S = N(x) \cap N(D)$. Then $N(S) - D - \{x\} = R_D$, and so, since $|D| \geq |S|$, $|N(S)| \geq |S| + |R_D| + 1$. If T_D can be specially 1-matched to a subset of R_D, then $|R_D| \geq |T_D| \geq \ell$, and so $|N(S)| \geq |S| + \ell + 1$, which contradicts Condition (2) of Theorem 7. Hence, T_D cannot be specially 1-matched to a subset of R_D. Suppose that there are nonempty subsets S' of S_D and R' of $V - N[x] - D$ with $|S'| \leq |R'|$ such that S' can be specially matched to R' and $T_D - N(R')$ can be specially 1-matched to a subset of $R_D - R'$. Note that $L \subseteq T_D - N(R')$. Since L is specially 1-matched to some subset of $R_D - R'$, $|R_D| \geq \ell$ and so once again $|N(S)| \geq |S| + \ell + 1$, a contradiction. Thus, Condition (2) of Theorem 5 holds, and $H(T) = \Delta + 1$. □

Our next result provides a simpler characterization of trees T for which $H(T) = \Delta(T) + 1$ than that presented in Theorem 7.

Theorem 8 *Let $T = (V, E)$ be a tree of order n and maximum degree Δ and let x be a vertex of degree Δ. Then, $H(T) = \Delta + 1$ if and only if the following two conditions hold:*
(1) *$V - N[x]$ is an independent set, and*
(2) *$n \leq 2\Delta$.*

Proof. Suppose $H(T) = \Delta + 1$. Then Conditions (1) and (2) of Theorem 5 hold. Since T is a tree, Condition (1) of Theorem 5 implies that no two vertices of $V - N[x]$ are adjacent, i.e., Condition (1) of Theorem 8 holds. Suppose that $n \geq 2\Delta + 1$. Then, $|V - N[x]| \geq \Delta$. Let D be a maximum packing in G consisting of vertices from $V - N[x]$. Using the notation introduced in Theorem 5, the set T_D is nonempty by Condition (2) of Theorem 5. Since T is a tree, it follows from our choice of the set D that $|S_D| = |D|$ and that no vertex of T_D is adjacent to any vertex of $V - N[x]$. In particular, each vertex of T_D is a leaf in T and each vertex of $V - N[x] - D$ is adjacent to a vertex of S_D. Thus, since $|D| + |R_D| = |V - N[x]| \geq \Delta$, $|R_D| \geq \Delta - |D| = \Delta - |S_D| = |T_D|$. The set T_D can be specially 1-matched to a subset of R_D, contradicting Condition (2) of Theorem 5. Hence, $n \leq 2\Delta$, i.e., Condition (2) of Theorem 8 holds.

To prove the sufficiency, suppose the two conditions of Theorem 8 hold. We use the notation of Theorem 7. Clearly, Condition (1) of Theorem 7 holds. If $\ell = 0$, then $|V - N[x]| \geq \Delta$, so that $n \geq 2\Delta + 1$, a contradiction. Thus, $\ell \geq 1$ and Condition (2) of Theorem 7 holds. Suppose there exists a set $S \subseteq N(x) - L$ with $|N(S)| \geq |S| + \ell + 1$. Note that $|N(S) \cap (V - N[x])| \geq |S| + \ell$ and $|V - N[x] - N(S)| \geq |N(x) - S - L| = \Delta - |S| - \ell$. Thus, $|V - N[x]| \geq (|S| + \ell) + (\Delta - |S| - \ell) = \Delta$, and so $n \geq 2\Delta + 1$, a contradiction. Hence Condition (3) of Theorem 7 holds. By Theorem 7, $H(T) = \Delta + 1$. □

3 Graphs G satisfying $H'(G) = \Delta(G)$

In any line-distinguishing coloring of a graph, no vertex is adjacent to two vertices in the same color class, and so the neighbors of each vertex in a graph require distinct colors. Hence for any graph G, $H'(G) \geq \Delta(G)$. Our aim in this section is twofold: First to characterize triangle-free connected graphs G for which $H'(G) = \Delta(G)$ and, secondly, to show that a characterization of connected graphs G for which $H'(G) = \Delta(G)$ appears difficult to obtain. Using an identical argument to that used in the proof of Lemma 3 (but with "harmonious" replaced by "line-distinguishing" and with "Lemma 1" replaced by "Lemma 2"), we have the following lemma.

Lemma 9 *If F is an induced subgraph of a graph G, then $H'(G) \geq H'(F)$.*

We begin with the following lemma.

Lemma 10 *Let G be a connected graph with maximum degree Δ and let x be a vertex of degree Δ. If $H'(G) = \Delta$, then*

(1) *$\langle N(x)\rangle$ contains at least one isolated vertex;*
(2) *each isolated vertex in $\langle N(x)\rangle$ is an end-vertex of G;*
(3) *every vertex of G is within distance 2 from x.*

Proof. We may assume that the Δ vertices in $N(x)$ are colored with the colors $1, 2, \ldots, \Delta$. Since $H'(G) = \Delta$, we may assume x is colored with color 1. But then the vertex in $N(x)$ that is colored 1 must be isolated in $\langle N(x)\rangle$. This proves (1).

Let y be a vertex in $N(x)$ that is isolated in $\langle N(x)\rangle$. Suppose $\deg y \geq 2$. Then, y is adjacent to a vertex $w \in V(G) - N[x]$. Let F be the subgraph of G induced by $N[x] \cup \{w\}$. We now construct a minimal line-distinguishing coloring of F that uses at least $\Delta + 1$ colors. Let x and y be colored with color 1, the $\Delta - 1$ vertices in $N(x) - \{y\}$ with colors $2, \ldots, \Delta$, and w with color $\Delta + 1$. Let $V_1, V_2, \ldots, V_{\Delta+1}$ denote the color classes containing the vertices of colors $1, 2, \ldots, \Delta + 1$, respectively. Then, no two of the color classes $V_1, V_2, \ldots, V_{\Delta+1}$ can be combined since the vertex in each of $V_2, \ldots, V_{\Delta+1}$ is adjacent to a vertex in V_1 and since $q(\langle V_1\rangle) = 1$. Hence, by Lemma 2, there exists a minimal line-distinguishing coloring of F that uses $\Delta + 1$ colors, and so $H'(F) \geq \Delta + 1$. Hence, by Lemma 9, $H'(G) \geq \Delta + 1$, a contradiction. Hence, $\deg y = 1$. This proves (2).

Suppose there exists a vertex z at distance 3 from x. Let x, u, v, z be a x–z path. Let L be the subgraph of G induced by $N[x] \cup \{v, z\}$. We now construct a minimal line-distinguishing coloring of L that uses at least $\Delta+1$ colors. Let x be colored with color 1, the Δ vertices in $N(x)$ with the colors $1, 2, \ldots, \Delta$ where u is colored with color Δ, the vertex v with color $\Delta+1$ and z with color 1. Let $V_1, V_2, \ldots, V_{\Delta+1}$ denote the color classes containing the vertices of colors $1, 2, \ldots, \Delta+1$, respectively. Then, no two of the color classes $V_1, V_2, \ldots, V_{\Delta+1}$ can be combined. Hence, by Lemma 2, there exists a minimal line-distinguishing coloring of L that uses at least $\Delta+1$ colors, and so $H'(L) \geq \Delta+1$. Hence, by Lemma 9, $H'(G) \geq \Delta+1$, a contradiction. Hence, every vertex of G is within distance 2 from x. This proves (3). □

As an immediate consequence of Lemma 10 (cf. Properties (1) and (2)), we have the following results.

Corollary 11 *Let G be a connected graph that contains a vertex of maximum degree Δ that belongs to no triangle. Then, $H'(G) = \Delta$ if and only if G is a star $K_{1,\Delta}$.*

The following result provides a characterization of triangle-free connected graphs G for which $H'(G) = \Delta(G)$.

Corollary 12 *Let G be a triangle-free connected graph with maximum degree Δ. Then, $H'(G) = \Delta$ if and only if G is a star $K_{1,\Delta}$.*

If v is a vertex of a graph G and $W \subseteq V(G)$, then we let $\deg_W v$ denote the number of vertices in W that are adjacent to v. In particular, if $W = V(G)$, then $\deg_W v = \deg v$.

Theorem 13 *Let $G = (V, E)$ be a connected graph of order n with maximum degree Δ and let x be a vertex of degree Δ. Let $W = V - N[x] \neq \emptyset$ and let $S = \{v \in N(x) \mid$ there exists a path of length 2 from v to a vertex of $W\}$. Let $A = \{v \in N(x) - S \mid \deg_W v = 1\}$, $B = \{v \in N(x) - S \mid \deg_W v \geq 2\}$, $C = \{v \in S \mid \deg_W v \geq 1\}$, $E = \{v \in S - C \mid \deg_{B \cup C} v \geq 1\}$, and $D = N(x) - (A \cup B \cup C \cup E)$. Let $|A| = a$, $|B| = b$, $|C| = c$ and $|D| = d$. Suppose $\deg w = 1$ for every $w \in W$. Then, $H'(G) = \Delta$ if and only if the following three conditions hold:*

(1) $\langle N(x) \rangle$ contains at least one isolated vertex;
(2) each isolated vertex in $\langle N(x) \rangle$ is an end-vertex of G;
(3) $b + c = 0$ or $n \leq \Delta - 1 + a + \lceil 3b/2 \rceil + c + d$.

Proof. We first note that the set $A \cup B$ is an independent set and there is no edge between $A \cup B$ and C.

Suppose first that $H'(G) = \Delta$. Then, by Lemma 10, conditions (1) and (2) are both satisfied. In particular, Condition (2) implies that each vertex of $A \cup B$ is adjacent to some vertex of S. It remains only to verify that condition (3) is satisfied. Suppose, to the contrary, that $b + c \geq 1$ and $n \geq \Delta + a + \lceil 3b/2 \rceil + c + d$. By Condition (1), x is adjacent to at least one end-vertex y, say, and so $d \geq 1$. Consider now a line-distinguishing coloring of G in which x and y are both colored with color 1 and the $\Delta - 1$ vertices in $N(x) - \{y\}$ with colors $2, \ldots, \Delta$. We may assume the vertices in $D - \{y\}$, if any, are colored with colors $2, \ldots, d$.

Suppose $b \geq 1$. Let $B = \{v_1, \ldots, v_b\}$. For each $i = 1, \ldots, b$, suppose v_i is colored with color i'. For each vertex v_i of B, let $w_i \in N(v_i) \cap W$ and color w_i with the color $\Delta + 1$. For each $i = 1, \ldots, \lfloor b/2 \rfloor$, color one vertex of $(W - \{w_{2i}\}) \cap N(v_{2i})$ with the color $(2i - 1)'$. If b is odd, then color one vertex of $(W - \{w_b\}) \cap N(v_b)$ with the color b'. If $c \geq 1$, then for each $v \in C$, color exactly one vertex of $N(v) \cap W$ with the color $\Delta + 1$. If $a \geq 1$, then let k be any color used to color a vertex in $B \cup C$ and color each of the a vertices in $W \cap N(A)$ with the color k.

Let $W_A = N(A) \cap W$, $W_B = N(B) \cap W$ and $W_C = N(C) \cap W$. Then, W_A, W_B and W_C are disjoint sets and $W = W_A \cup W_B \cup W_C$. Note that $|W_A| = a$. We have now colored all a vertices of W_A, $\lceil 3b/2 \rceil$ vertices of W_B and c vertices of W_C. By assumption, at least $d - 1$ vertices of W have yet to be colored and these vertices are contained in $W_B \cup W_C$. Color $d - 1$ of these vertices of W (that have not yet been colored) with the colors $2, \ldots, d$. Let $V_1, V_2, \ldots, V_{\Delta+1}$ denote the color classes containing the vertices of colors $1, 2, \ldots, \Delta + 1$, respectively.

We claim that no two of the color classes $V_1, V_2, \ldots, V_{\Delta+1}$ can be combined. No two of the color classes $V_1, V_2, \ldots, V_\Delta$ can be combined since there is at least one vertex in each of $V_2, \ldots, V_\Delta$ which is adjacent to a vertex in V_1 and since $q(\langle V_1 \rangle) = 1$. Hence we need only show that $V_{\Delta+1}$ cannot be combined with any of the color classes $V_1, V_2, \ldots, V_\Delta$.

Since each vertex in $B \cup C$ is adjacent to a vertex in V_1 and a vertex in $V_{\Delta+1}$, the classes V_1 and $V_{\Delta+1}$ cannot be combined.

If $d \geq 2$, then for each j with $2 \leq j \leq d$, there is a vertex in W that is colored with color j. But such a vertex is adjacent to a vertex of $B \cup C$. Since each vertex of $B \cup C$ is adjacent to a vertex colored $\Delta + 1$, there is therefore a vertex of $B \cup C$ which is adjacent to a vertex colored j and a vertex colored $\Delta + 1$. Thus, $V_{\Delta+1}$ cannot be combined with any color class containing a vertex of D.

Suppose $c \geq 1$. Since each vertex of C is adjacent to at least one other vertex of C, and since each vertex of C is adjacent to a vertex colored $\Delta+1$, it follows that $V_{\Delta+1}$ cannot be combined with any color class containing a vertex of C.

Suppose $|E| \geq 1$. Since each vertex of E is adjacent to at least one vertex of $B \cup C$, and since each vertex of $B \cup C$ is adjacent to a vertex colored $\Delta+1$, it follows that $V_{\Delta+1}$ cannot be combined with any color class containing a vertex of E.

Since there is an edge between the color class V_k (where recall that k is the color used to color the vertices of W_A) and each color class containing a vertex of A, and between the color class V_k and $V_{\Delta+1}$, the color class $V_{\Delta+1}$ cannot be combined with any color class containing a vertex of A.

Suppose $i \in \{1, \ldots, \lfloor b/2 \rfloor\}$. Since v_{2i} is adjacent to a vertex in $V_{(2i-1)'}$ and a vertex in $V_{\Delta+1}$, we cannot combine the classes $V_{\Delta+1}$ and $V_{(2i-1)'}$. Moreover, since there is an edge between the color classes $V_{(2i-1)'}$ and $V_{(2i)'}$ and an edge between the color classes $V_{(2i-1)'}$ and $V_{\Delta+1}$, we cannot combine the classes $V_{\Delta+1}$ and $V_{(2i)'}$. Finally, if b is odd, then since $q(\langle V_{(b)'} \rangle) = 1$ and since there is an edge between the color classes $V_{(b)'}$ and $V_{\Delta+1}$, we cannot combine the classes $V_{(b)'}$ and $V_{\Delta+1}$. It follows that $V_{\Delta+1}$ cannot be combined with any color class containing a vertex of B.

Hence, no two of the color classes $V_1, V_2, \ldots, V_{\Delta+1}$ can be combined. Thus, by Lemma 2, there exists a minimal line-distinguishing coloring of G that uses at least $\Delta + 1$ colors, a contradiction. Hence condition (3) must be satisfied. This proves the necessity.

To prove the the sufficiency, suppose that the three conditions (1), (2) and (3) are all satisfied but $H'(G) \geq \Delta + 1$. Then there is a minimal line-distinguishing coloring of G using the colors $1, 2, \ldots, r$, where $r \geq \Delta + 1$. Let $V_1, V_2, \ldots, V_r$ denote the color classes containing the vertices of colors $1, 2, \ldots, r$, respectively. We may assume that the Δ vertices in $N(x)$ are colored with the colors $1, 2, \ldots, \Delta$. By Condition (1), x is adjacent to at least one end-vertex y, say. We may assume that y is colored with color 1.

Suppose x is colored with color d where $d \notin \{1, 2, \ldots, \Delta\}$. Since no vertex at distance 2 from x can be colored with the same color as x, x is therefore the only vertex colored d. But then we can recolor y with the color d. Hence, we may assume that x and y are colored with the same color, namely color 1.

Consider now the color class V_r. By assumption, $r \geq \Delta + 1$, and so $V_r \subseteq W$. Since W is an independent set, it follows that V_r is a packing in G. Suppose $b + c = 0$. Then, $|W| = a$ and at least one of the colors, say color t, used to color the vertices of A is not used to color any vertex of W. But then we can combine the color classes V_t and V_r, a contradiction. Hence, $b + c \geq 1$. Thus, by condition (3), $n \leq \Delta - 1 + a + \lceil 3b/2 \rceil + c + d$.

We proceed further by introducing some additional notation. Let $L_B = \{v \in B \mid v \text{ has a neighbor colored } r\}$ and let $M_B = B - L_B$. Hence for each vertex $v \in L_B$, V_r contains a vertex in $N(v) \cap W$. Let $|L_B| = \ell_b$ and let $|M_B| = m_b$. Then, $b = \ell_b + m_b$. Define L_C, M_C, ℓ_c and m_c in a similar way. Then, $c = \ell_c + m_c$. Let $L = L_B \cup L_C$ and $M = M_B \cup M_C$.

Let β denote the number of vertices in $N(L) \cap W$ that belong to a color class that contains a vertex of $D - \{y\}$.

Claim 4 $\beta \geq d - 1$.

Proof. Let t be a color used to color a vertex of $D - \{y\}$. Since we cannot combine the color classes V_r and V_t, it follows that there must exist a vertex $v \in B \cup C$ that has a neighbor in W colored r and a neighbor in W colored t. In particular, $v \in L$. Hence each of the $d - 1$ colors used to color the vertices in $D - \{y\}$ is used to color a vertex in $N(L) \cap W$. The result follows. □

Let $L_1 = \{v \in L_B \mid \text{no vertex in } N(v) \cap W \text{ belongs to a color class that contains a vertex of } L_B\}$, and let $L_2 = L - L_1$. Let $|L_1| = \ell_1$, and so $|L_2| = \ell_b + \ell_c - \ell_1$.

Claim 5 *There are at least ℓ_1 vertices in $N(L_2) \cap W$ that are colored with the same color as a vertex in L_1.*

Proof. Let $v \in L_1$ and suppose v is colored with color k. Since we cannot combine the color classes V_k and V_r, it follows that there must exist a vertex $u \in B \cup C$ that has a neighbor in W colored r and a neighbor u' in W colored k. Then, $u \in L_2$ and so $u' \in N(L_2) \cap W$. Hence for each vertex $v \in L_1$, there exists a vertex in $N(L_2) \cap W$ that belongs to the color class that contains v. The result follows. □

Let γ denote the number of vertices in $N(L) \cap W$ that belong to V_r or to a color class that contains a vertex of L_B.

Claim 6 $\gamma \geq \left\lceil \frac{3\ell_b}{2} \right\rceil + \ell_c$.

Proof. By definition, each vertex $v \in L$ has a neighbor (in W) colored r, and so there are exactly $\ell_b+\ell_c$ vertices in $N(L) \cap W$ that belong to V_r. Hence if $\ell_1 \geq \lceil \ell_b/2 \rceil$, then by Claim 5, $\gamma \geq (\ell_b+\ell_c)+\ell_1 \geq \lceil 3\ell_b/2 \rceil + \ell_c$. Therefore we may assume that $\ell_1 \leq \lceil \ell_b/2 \rceil - 1$, for otherwise the desired result follows. By definition, each vertex $u \in L_B - L_1$ has a neighbor (in W) colored r and a neighbor (in W) colored with a color used to color a vertex of L_B. Hence there are at least $\ell_b - \ell_1$ vertices in $N(L_B - L_1) \cap W$ that belong to a color class that contains a vertex of L_B. Thus, $\gamma \geq (\ell_b + \ell_c) + (\ell_b - \ell_1) \geq 2\ell_b + \ell_c - \lceil \ell_b/2 \rceil + 1 = \ell_b + \ell_c + \lfloor \ell_b/2 \rfloor + 1 \geq \lceil 3\ell_b/2 \rceil + \ell_c$. □

By Claims 4 and 6, we have $|N(L) \cap W| \geq \beta + \gamma \geq d - 1 + \lceil 3\ell_b/2 \rceil + \ell_c$. Since each vertex of M_B has at least two neighbors in W, $|N(M_B) \cap W| \geq 2m_b \geq \lceil 3m_b/2 \rceil$. Since each vertex of M_C has at least one neighbor in W, $|N(M_C) \cap W| \geq m_c$. Hence, $|N(M) \cap W| = |N(M_B) \cap W| + |N(M_C) \cap W| \geq \lceil 3m_b/2 \rceil + m_c$. Thus, $|W| = |W_A| + |N(L) \cap W| + |N(M) \cap W| \geq a + (d - 1 + \lceil 3\ell_b/2 \rceil + \ell_c) + (\lceil 3m_b/2 \rceil + m_c) \geq a + \lceil 3b/2 \rceil + c + d - 1$. Hence, $n \geq \Delta + a + \lceil 3b/2 \rceil + c + d$, contradicting condition (3). We deduce, therefore, that $H'(G) = \Delta$. □

We have yet to characterize all connected graphs G for which $H'(G) = \Delta(G)$. Theorem 13 illustrates that such a characterization appears difficult to obtain.

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