A note on the complexity of graph parameters and the uniqueness of their realizations

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Abstract: Let ν be some graph parameter and let $\mathcal{G}$ be a class of graphs for which ν can be computed in polynomial time. In this situation it is often possible to devise a strategy to decide in polynomial time whether ν has a unique realization for some graph in $\mathcal{G}$. We first give an informal description of the conditions that allow one to devise such a strategy, and then we demonstrate our approach for three well-known graph parameters: the domination number, the independence number, and the chromatic number.

Keywords: complexity; unique realization; domination number; independence number; chromatic number

1 Introduction

Let ν be some graph parameter. As typical choices for ν we will consider well-known graph parameters such as the *domination number* γ, the *independence number* α, or the *chromatic number* χ (for notation and definitions see e.g. [14]). In these examples, ν corresponds to the minimum or maximum cardinality of special subsets of the vertex set or to a certain partition of the vertex set, i.e. ν measures some property either of a set of vertices or of a partition of the vertices of a graph.

We say that ν has a *unique realization* in $G = (V, E)$ if G has only one set, or partition, which satisfies the property measured by ν. In such a case, we say that this unique set is a unique ν-set of G. For example, γ has a unique realization in some graph G if G has a unique minimum dominating set, that is, a unique γ-set. Similarly, α has a unique realization in G if G has a unique maximum independent set (α-set), and χ has a unique realization in G if G has a unique partition into χ independent sets.

Usually, the problem of characterizing graphs for which some parameter ν has a unique realization is considered separately from the algorithmic

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problem of determining the value of ν. There are several characterizations in the literature of classes of graphs for which some parameter ν has a unique realization (e.g. [1, 6, 7, 8, 9, 10, 12, 13, 16, 21, 22] and [23]). Sometimes these characterizations lead to polynomial time algorithms for deciding if a graph in this class has a unique ν-realization. For example, in [10], Gunther, Hartnell, Markus and Rall characterize trees having a unique γ-set. A corollary of this characterization is that there is a linear time algorithm for deciding if an arbitrary tree has a unique γ-realization. All one needs to do is to use any existing $O(n)$-time algorithm for finding a γ-set (cf. [3]) D in a tree T, and then, check to see if every vertex in D has at least two private neighbours. If the answer is 'yes', then the tree T has a unique γ-set; otherwise the tree T has at least one other γ-set.

Often, these characterizations do not immediately lead to polynomial time algorithms for deciding if an arbitrary graph in one of these classes has a unique ν-realization. In many of these cases, however, it is possible to determine the value of ν in polynomial time, and, as we shall show, it is also possible to decide whether a graph has a unique ν-realization.

Let us assume now that there is an (easy-to-prove) characterization of the graphs for which ν has a unique realization that uses a property which can be checked by evaluating ν for different graphs that arise from a given graph by some local changes. Let $\mathcal{G}$ be a class of graphs. If it is possible to determine ν in polynomial time for all graphs that arise from a graph in $\mathcal{G}$ by the above-mentioned local changes and if we can check the above-mentioned property by looking only at a polynomial number of graphs, then we can decide - again in polynomial time - using the characterization whether ν has a unique realization for some graph in $\mathcal{G}$. We will now demonstrate this informally described strategy in the next section for γ, α and χ. It is clear that our approach works for several other graph parameters.

2 Examples of the strategy

We begin with characterizations of graphs with unique minimum dominating sets, unique maximum independent sets, and unique χ-colorings that are as described in the introduction.

Lemma 2.1 *(i) A graph G has a unique minimum dominating set if and only if the set of vertices that belong to every minimum dominating set of G is a dominating set of G.*

(ii) A graph G has a unique maximum independent set if and only if the set of vertices that belong to every maximum independent set is maximal independent.

(iii) Let G be a graph and let $E' = \{uv \notin E(G) | u, v \in V(G),\ \chi(G) = \chi(G + uv)\}$, i.e. E' is the set of non-edges of G whose addition to G does not increase the chromatic number.

The graph G is uniquely colorable if and only if the graph $G' = (V(G), E(G) \cup E')$ is a complete $\chi(G)$-partite graph.

Proof: (i): '⇒' (trivial). '⇐' Let D_1 and D_2 be two different minimum dominating sets of G, then the set $D_1 \cap D_2$ dominates G and $|D_1 \cap D_2| < \gamma(G)$ which is a contradiction.
(ii) '⇒' (trivial). '⇐' Let I_1 and I_2 be two different maximum independent sets of G, then the set $I_1 \cap I_2 \neq I_1$ is maximal independent which is a contradiction.
(iii) Let $V_1 \cup V_2 \cup ... \cup V_{\chi(G)} = V(G)$ be a $\chi(G)$-coloring of G. Clearly, all non-edges of G with endpoints in different sets V_i belong to E'. If G is uniquely colorable, then clearly $(V(G), E(G) \cup E')$ is a complete $\chi(G)$-partite graph. If G is not uniquely colorable, then there is a pair of vertices x, y and a second $\chi(G)$-coloring $V_1' \cup V_2' \cup ... \cup V_{\chi(G)}' = V(G)$ of G such that without loss of generality $x, y \in V_1$, $x \in V_1'$, and $y \in V_2'$. This implies that $xy \in E'$ and $(V(G), E(G) \cup E')$ is no complete $\chi(G)$-partite graph. □

Now, we describe the local changes that allow to check the properties used in the above characterizations. For χ the local change consists just of adding a specific edge to the graph.

Let G be a graph, let $v \in V(G)$, and let $u \in N(v, G)$. The graph $G_{v,u}$ has vertex set $V(G_{v,u}) = (V(G) \setminus \{v\}) \cup \{u'\}$ and edge set $E(G_{v,u}) = (E(G) \setminus \{vw | w \in N(v, G)\}) \cup \{uu'\}$.

Lemma 2.2 *Let G be a graph and let $v \in V(G)$.*

(i) The vertex v belongs to every minimum dominating set of G if and only if $\gamma(G) < \gamma(G_{v,u})$ for every $u \in N(v, G)$.

(ii) The vertex v belongs to every maximum independent set of G if and only if $\alpha(G) > \alpha(G[V(G) \setminus N[u, G]]) + 1$ for every $u \in N(v, G)$.

Proof: (i) Let D be a minimum dominating set of $G_{v,u}$. Since in $G_{v,u}$ the vertex u has a neighbour of degree one, we can assume without loss of generality that $u \in D \subseteq V(G)$. Hence D is also a dominating set of G and we obtain that $|D| = \gamma(G_{v,u})$ is the minimum cardinality of a dominating set of G that contains u. Therefore, $\min\{\gamma(G_{v,u}) | u \in N(v, G)\}$ is the minimum cardinality of a dominating set of G that does not contain v and the result follows.
(ii) As above, $\alpha(G[V(G) \setminus N[u, G]]) + 1$ is the maximum cardinality of an independent set of G that contains u. Therefore $\max\{\alpha(G[V(G) \setminus N[u, G]]) +$

$1|u \in N(v,G)\}$ is the maximum cardinality of an independent set of G that does not contain v and the result follows. □

We will now complete our exposition by considering the properties of graph classes that allow to decide efficiently if γ, α, or χ have unique realizations.

Let $\mathcal{G}_\gamma$ be a class of graphs such that for every $G \in \mathcal{G}_\gamma$ and every $v \in V(G)$ and $u \in N(v,G)$, it is possible to determine γ for the graphs G and $G_{v,u}$ in polynomial time.

Proposition 2.3 *For a graph $G \in \mathcal{G}_\gamma$ it can be decided in polynomial time whether G has a unique minimum dominating set.*

Proof: We may assume that we can determine γ for G and $G_{v,u}$ in time $p_\gamma(|V(G)|,|E(G)|)$ for every $G \in \mathcal{G}$, $v \in V(G)$ and $u \in N(v,G)$ where p_γ is some polynomial. By Lemma 2.2, we can decide in time $|V(G)| \cdot p_\gamma(|V(G)|,|E(G)|)$ for a specific vertex $v \in V(G)$, whether v is contained in every minimum dominating set of G. We can therefore find the set of all vertices of G that are in every minimum dominating set of G in time $|V(G)|^2 \cdot p_\gamma(|V(G)|,|E(G)|)$. By Lemma 2.1(i), it is now trivial to decide whether G has a unique minimum dominating set. □

The property of $\mathcal{G}_\gamma$ is not very restrictive and many of the classes of graphs for which γ can be computed efficiently have this property. As an example we cite the ***strongly chordal graphs*** [5], [4] which contain several other well-known classes of graphs (see [17]) and for which γ can be computed in polynomial time. If G is a strongly chordal graph, then $G_{v,u}$ is also a strongly chordal graph for every $v \in V(G)$ and $u \in N(v,G)$ (note that if $v_1v_2...v_n$ is a strong elimination ordering of the vertices of G and $v = v_i$ and $u = v_j$, then $v_1v_2...v_{i-1}v_{i+1}...v_{j-1}u'v_j...v_n$ is a strong elimination ordering of $G_{v,u}$). Exactly as Proposition 2.3 we can now prove the following two results for α and χ.

Let $\mathcal{G}_\alpha$ be a class of graphs such that for every $G \in \mathcal{G}_\alpha$ and every $v \in V(G)$, it is possible to determine α for the graphs G and $G[V(G) \setminus N[v,G]]$ in polynomial time.

Proposition 2.4 *For a graph $G \in \mathcal{G}_\alpha$ it can be decided in polynomial time whether G has a unique maximum independent set.*

Again, the property of $\mathcal{G}_\alpha$ is not very restrictive and there are several classes of graphs for which α can be computed in polynomial time that have this property because they are closed under taking induced subgraphs (see e.g. [2, 15, 18, 19] and [20]).

Let $\mathcal{G}_\chi$ be a class of graphs such that for every $G \in \mathcal{G}_\chi$ and every $uv \notin E(G)$, it is possible to determine χ for the graphs G and $(V(G), E(G) \cup \{uv\})$ in polynomial time.

Proposition 2.5 *For a graph $G \in \mathcal{G}_\chi$ it can be decided in polynomial time whether G is uniquely colorable.*

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