Decomposing Eulerian Graphs

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Abstract

Heinrich et al. [4] characterized those simple eulerian graphs with no Petersen-minor which admit a triangle-free cycle decomposition, a TFCD. If one permits Petersen minors then no such characterization is known even for $E(4,2)$, the set of all the eulerian graphs of maximum degree 4. Let $EM(4,2) \subset E(4,2)$ be the set of all graphs H such that all triangles of H are vertex disjoint, and each triangle contains a degree 2 vertex in H. In the paper it is shown that to each $G \in E(4,2)$ there exists a finite subset $S \subset EM(4,2)$ so that G admits a TFCD if and only if some $H \in S$ admits a TFCD. Further, some sufficient conditions for a graph $G \in E(4,2)$ to possess a TFCD are given.

1 Introduction.

It is very well known that a graph is eulerian iff there exists a decomposition of its edge set into cycles. There are several results along this line where one is looking for a cycle decomposition obeying some additional conditions. For example, Zhang [6] proved that edges of every 2-connected eulerian graph G with no subgraph contractible to K_5 can be decomposed into cycles of even length, for $|E(G)|$ even, and for $|E(G)|$ odd there is a decomposition with exactly one cycle of odd length. Koudier and Sabidussi showed [5] that any 2-connected 4-regular graph can be decomposed into two triangle-free 2 factors. In [1] it is proved that the decision problem: "Given an eulerian graph G of maximum degree 4. Decide whether G can be decomposed into two triangle-free 2-factors" can be solved in a polynomial time with respect to the order of G.

In this area, there are also plenty of interesting conjectures, some of them being around for a long time. Probably the best known of them is a conjecture of Sabidussi, see e.g. [3], saying that to any eulerian trail T of an eulerian graph G without vertices of degree 2 there is a cycle decomposition C of G so that two consecutive edges of T belong to different cycles of C.

In [4] the question when an eulerian graph admits triangle-free cycle decomposition, i.e. a decomposition of its edge set into cycles so that none of them is of length 3, a TFCD, is studied. The authors point out that the question, which is interesting by itself, is motivated also by the famous Double Cycle Cover conjecture. Given a simple graph G. Let H_G be an eulerian graph obtained from G by doubling each edge of G (thus obtaining a multigraph with each edge having the multiplicity 2) and then subdividing all new edges by one vertex. It is not difficult to show that G has a double cycle cover iff H_G possesses a TFCD. Hence, to get a characterization of those eulerian graphs admitting a TFCD is of great interest. The first step has been done in [4], where eulerian graphs with a TFCD and having no Petersen-minor were characterized. Of course, the desire is to find a characterization without the condition no Petersen-minors.

The first non-trivial case are eulerian graphs of maximum degree $\Delta \leq 4$. In fact, this has been asked by Heinrich and Liu (c.f. [7], Problem 10.7.5.), see also [4]. For the sake of simplicity this class of graphs will be denoted by $E(4,2)$ and the members of the class will also be called $(4,2)-$graphs.

As the main result of the paper, we show in two steps that the problem of characterizing those $(4,2)-$graphs possessing a TFCD can be reduced to a relatively small subclass $EM(4,2)$ of $E(4,2)$. A $(4,2)-$graph G belongs to $EM(4,2)$ if all triangles of G are pairwise vertex-disjoint and each triangle of G has a vertex of degree 2 in G.

Reduction A. *To each graph $G \in E(4,2)$ there exists a graph $H \in E(4,2)$ so that the triangles of H are pairwise vertex-disjoint and H possesses a TFCD if and only if G does.*

The above reduction is linear with the number of vertices of G of degree 4. Unfortunately, the reduction presented in the next theorem is not polynomial as in general the number of graphs in the set S is exponential with the number of triangles of G.

Reduction B. *To each graph $H \in E(4,2)$ so that the triangles of H are pairwise vertex-disjoint there exists a finite set S of graphs from $EM(4,2)$ so that H has a TFCD if and only if at least one graph of S has the property.*

Although the class $EM(4,2)$ comprises graphs with more transparent structure than a general $(4,2)$-graph we are not able to characterize those graph of $EM(4,2)$ having a TFCD. Therefore, in Section 3, some sufficient conditions for a graph from $E(4,2)$ to have a TFCD will be presented.

We believe that the simplest unsolved case when characterizing graphs with a TFCD is the class $EC(4,2)$, where a $(4,2)$-graph $G \in EC(4,2) \subset EM(4,2)$ if G consists of two vertex disjoint cycles C, C' of the same length n and n vertex-disjoint triangles so that each triangle has one vertex on C, one vertex on C' and the third vertex of the triangle is of degree 2 in G, where $n \geq 4$. We conjecture that the following is true. Given a graph $G \in EC(4,2)$. Then the decision problem whether G admits a TFCD can be solved in a polynomial time with respect to the order of G. The conjecture is discussed in Section 4.

In the last section we point out relation between the TFCD problem and the problem of Compatible Cycle Decompositions. This relation indicates why the TFCD problem is "difficult" even for the class $EM(4,2)$.

2 Proof of Reductions

We start with some more definitions and notation. Let G be a graph. If A is a set of vertices of G then by $[A]$ we will denote a subgraph of G induced by A. Let v be a vertex of G. Then $N(v), and \overline{N(v)}$ stand for the neighborhood and the closed neighborhood of v, respectively. Further, K_n, C_n, and P_n stand for complete graph, cycle, and path on n vertices, respectively.

Proof of Reduction A. To construct H we will create a sequence of $(4,2)-$graphs $F_0 = G, F_1, ..., F_{n-1}, F_n = H$, so that F_{i+1} has a $TFCD$ iff F_i does, $i = 0, 1, ..., n-1$. Suppose that F_i has already been constructed. To get F_{i+1} we choose in F_i a vertex v of degree 4 which is incident with at least two triangles. The vertex v will be replaced in F_{i+1} by two vertices of degree 4, v' and v'', so that either of them is incident in F_{i+1} with at most one triangle, and some new vertices of degree 2 will be created. If $[N(v)]$ has at most one edge, the vertex v is incident with at most one triangle and we do not need to transform the neighborhood of v. There are 9 non-isomorphic subgraphs of K_4 having at least two edges so we will distinguish among 9 cases depending on the structure of $[N(v)]$. In order to avoid indices, in what follows we set $F_i = F$ and $F_{i+1} = F'$. The vertices from $N(v)$ will always be denoted by x, y, z, and w. Further, the vertices of $N(v)$ which are in $[N(v)]$ of degree at most 2 are in F of degree 2 or 4 . In the figures they are always depicted as vertices of degree 4 with the understanding that two of the edges incident with these vertices and not in $[N(v)]$ do not have to belong to F. If these edges are not in F it is for our purpose equivalent to the case when they are in F and in the considered $TFCD$ they form two consecutive edges in some of its cycles.

a) $[N(v)] \simeq K_4$. Then $F = K_5$ and clearly F has a $TFCD$. Hence we can choose as F' any $(4,2)-$ graph having all its triangles pairwise vertex-disjoint and possessing a $TFCD$.

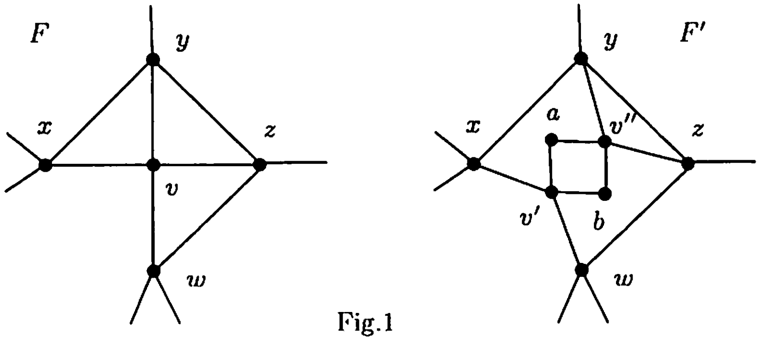

Fig.1

Seven of the remaining eight cases are going to be treated using the same transformation of the neighborhood of v. First the vertex v is removed. Then four new vertices $v', v'', a, b,$ are added together with the edges $v'x$, $v'w$, $v'a$, $v'b$, $v''y$, $v''z$, $v''a$, $v''b$,. We describe the following case (which we believe is a most illustrative for the method) in greater detail.

b) $[N(v)] \simeq P_4$, see Fig.1. Let D be a $TFCD$ of F. The four edges incident to v belong to two different cycles of D. Suppose first that the path xvw is a part of a cycle C of D, and the path yvz is on a cycle K of D. To obtain a $TFCD$ of F', we form a cycle C' by substituting in C the path xvw by the path $xv'w$, and a cycle K' by substituting in K the path yvz by the path $yv''z$. Then $D' = (D - \{C, K\}) \cup \{C', K', L = v'av''bv'\}$ is a $TFCD$ of F' because the cycles C' and K' have the same length as the cycles C and K, respectively, and the cycle L is of length 4. Assume now that the path $xvy(xvz)$ is on a cycle C of D, and the path $zvw(yvw)$ is on a cycle K of D. We form a cycle C' by substituting in C the path $xvy(xvz)$ by the path $xv'av''y(xv'av''z)$, and a cycle K' by substituting in K the path $zvw(yvw)$ by the path $zv''bv'w(yv''bv'w)$. Then $D' = (D - \{C, K\}) \cup \{C', K'\}$ is a $TFCD$ of F' as cycles C' and K' are even longer then the cycles C and K, respectively.

Now let D' be a $TFCD$ of F'. We will show how to construct a $TFCD$, D, of F.

If the cycle $L = v'av''bv'$ belongs to D' , to obtain D, we first remove L from D' and in the cycles of D' containing v' and v'' we write v instead of v' and v''. The only problem can arise if the vertices v' and v'' are on the same cycle $C' \neq L$. Then we first modify D' in such a way that the path $yv''z$ is in C' replaced by the edge yz, and the edge yz which belonged in D' to a cycle C'' is replaced in C'' by the path $yv''z$. Clearly, the modified D' retains the property to be a TFCD. To be able to perform the modification, in all following seven cases we choose notation so that the edge yz belongs to G.

Suppose now that the edges of the cycle L belong to two cycles of D'. Let the path $\alpha v''av'\beta$ be a part of a cycle C', the path $\gamma v''bv'\delta$ be a part of a cycle K' of D', where $\{\alpha,\beta,\gamma,\delta\} = \{x,y,z,w\}$. Further, let cycles C and K be obtained from C' and K' by substituting the paths $\alpha v''av'\beta$ and $\gamma v''bv'\delta$ by the path $\alpha v\beta$ and $\gamma v\delta$, respectively. If both cycles C and K are of length at least 4, then $D = (D' - \{C', K'\}) \cup \{C, K\}$ is a $TFCD$ of F. The only non-trivial case arises when C' and/or K' is of length 5, i.e., when at least one of the four cycles $xyv''bv'x, xyv''av'x, zv'av''wz$ and $zv'bv''wz$ are in D'.

The edges of the subgraph of F' induced by the set $\{x,y,z,w,v',v'',a,b\}$ form up in D' a set S' of cycles and/or paths. To construct D one can proceed as follows. Decompose the edges of the subgraph $\overline{[N(v)]}$ of F into a set S of cycles and/or paths so that

(i) each cycle of S is of length at least 4;

(ii) there is a bijection of paths in S' and S where the two corresponding paths P' and P have the same pair of endvertices (from $N(v)$) so that substituting of P' in a cycle C' of D' by a path P results in a cycle C of length at least 4.

This is guarantied, if the length of P is at least 2 or if both P and P' are of length 1. Note that if a path P contains $\alpha \in \{x,y,z,w\}$ as an inner vertex, where $\deg_{\overline{[N(v)]}}(\alpha) = 2$ and $\deg_F(\alpha) = 4$, then substituting P for P' into C' might lead to a trail passing through α twice. This would occur in the case $\alpha \in C'$. We will alsways choose paths P to avoid this possibility.

To obtain D we remove from D' the cycles of S' and the cycles C' containing paths from S' and add the cycles of S and cycles C formed in the above mentioned manner. We need to distinguish among the following cases.

(i) $S' = \{xv'av''yx, wv'bv''zw, yz\}$. Then $S = \{xvwzyx, yvz\}$ has the required property. Note that if we set $S = \{xvzyx, yvwz\}$ then, in the case that the vertex w would be of degree 4 in F and the cycle C' of D' containing the path yz would pass through the vertex w, replacing the path

yz by the path $yvwz$ would not result in a cycle of F but in a trail consisting of two cycles.

(ii) $S' = \{xv'av''yx, wv'bv''z, wzy\}$ or $S' = \{xv'av''yx, wv'bv''zy, wz\}$. Set $S = \{xvzyx, wz, wvy\}$. A problem can occur only in the case when the cycle $C' = wv'bv''ztw \in D'$, $t \notin N(v)$, as we would have a triangle $wztw$ in D. We set $S = \{wvxyz, wzy\}$ as we know that in this case the cycle of C' does not pass through the vertex x. Thus replacing the $w - z$ path in C' does not result in a trail consisting of two cycles.

(iii) The other cases are symmetric to the previous ones.

The first part of the proof in the following six cases is identical to the first part of the proof of b) (since the structure of $[N(v)]$ does not play any role there), and therefore will be omitted. As to the non-trivial part of the proof (i.e., if at least one of the cycles $xyv''bv'x, xyv''av'x, zv'av''wz$ and $zv'bv''wz$ is in S'), we only list sets S' and the corresponding set S.

c) $[N(v)] \simeq K_4 - P_2$, yw being the removed edge.
S' comprises two cycles and a $y - w$ path, $S = \{wxzyvw, wzvxy\}$.

d) $[N(v)] \simeq C_4$, $xyzwx \in [N(v)]$.
S' always contains exactly two paths, an $\alpha - \beta$ path and a $\gamma - \delta$ path, $\{\alpha, \beta, \gamma, \delta\} = \{x, y, z, w\}$. We can take $S = \{xyzwx, \alpha v\beta, \gamma v\delta\}$.

e) $[N(v)] \simeq K_4 - P_3, xwz$ being the removed path.
$S' = \{xv'av''yx, wv'bv''zyw, xz\}, S = \{wvzxyw, xvyz\}$.
$S' = \{xv'av''yx$, a $y - z$ path, a $y - x$ path $\}$, $S = \{wvyzw, yxvz, yzx\}$.
$S' = \{xv'av''yx$, a $w - z$ path, a $w - x$ path$\}, S = \{wyzvx, wvyxz\}$.

f) $[N(v)] \simeq C_3$, $xyzx$ being the cycle.
$S' = \{xv'av''yx, wv'bv''z, xzy\}, S = \{xvzy, wvyxz\}$.
$S' = \{xv'av''yx, wv'bv''zy, xz\}, S = \{wvxzy, xyvz\}$.
$S' = \{xv'av''yx, wv'bv''zx, yz\}, S = \{wvzyx, yvxz\}$.

g) $[N(v)] \simeq P_3, xyz$ being the path.
$S' = \{xv'av''yx, wv'bv''zy\}, S = \{xvzyx, wvy\}$.
If $S' = \{xv'av''yx, wv'bv''z, yz\}$, then $S = \{wvyz, yxvz\}$ for the case when x is not on the cycle of D' containing yz, otherwise $S = \{wvxyz, yvz\}$.

h) $[N(v)] \simeq 2K_2, xw$ and yz being the two edges.
There is not any non-trivial case as the edges xy and zw are not in $[N(v)]$, i.e., none of the cycles $xyv''bv'x, xyv''av'x, zv'av''wz$ and $zv'bv''wz$ is in D'. Note that the transformation was made with the assumption that the edge $yz \in G$.

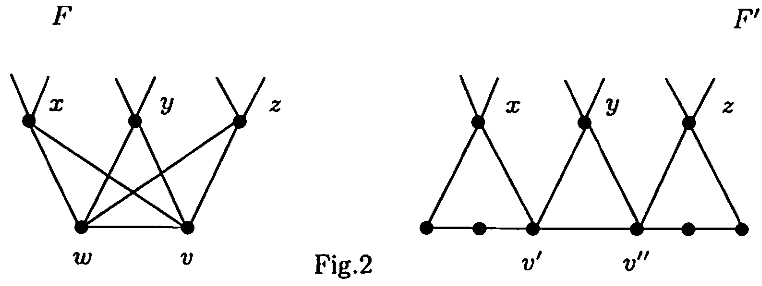

Fig.2

i) $[N(v)] \simeq K_{1,3}$. In this case we use a transformation depicted in Fig.2. The vertices of F' of degree 2 are new vertices, the vertex w has been removed. Let $D(D')$ be a $TFCD$ of $F(F')$. Then $S(S')$ (S and S' have the same meaning as before) contain either two paths both having the same pair of endvertices from $\{x, y, z\}$ or an $x - y, y - z$, and an $z - x$ paths, or S' contains a cycle and two paths both having the same pair of endvertices. On the other hand, both $\overline{[N(v)]}$ and the subgraph of F' induced by x, y, z, v', v'' and the new vertices of degree 2 can be decomposed into two (three) paths with above described endvertices such that each path has length at least two, and possibly a cycle. Substituting corresponding paths in the cycles of $D(D')$ provides a $TFCD$,$D'(D)$, of $F'(F)$. The proof is complete.

Proof of Reduction B.

Let $T = xyzx$ be a triangle of H, x being a vertex of degree 4. Denote by $H(T, x)$ the graph obtained from H by first removing the vertex x, then adding two new vertices x' and x'' of degree 2, where x' is adjacent to y and z, x'' to the other two neighbors of x. Let $T = xyzx$ be a triangle of H so that all vertices of T are in H of degree 4. We assign to H a set H_T of four $(4, 2)$-graphs whose triangles are vertex disjoint. Three of them are graphs $H(T, x), H(T, y), H(T, z)$, the fourth one, $H(T, 3)$ is depicted in Fig.3. The vertices $n', n'', n''', m, x', y', z'$ are new vertices. If the triangles of H were vertex disjoint then clearly also the graphs of H_T have the property.

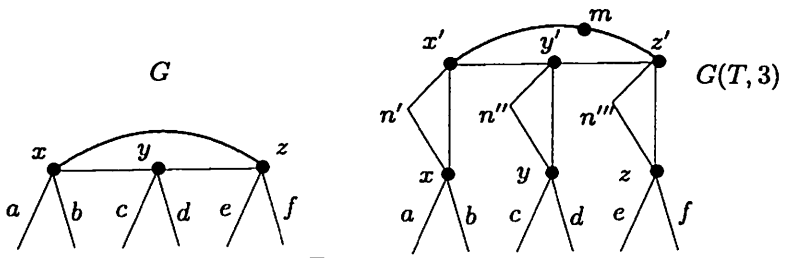

Fig.3

Claim 1. H has a $TFCD$ iff at least one of the graphs in H_T does.

Suppose first that H has a $TFCD$, D. Then the edges of T belong to either two or three cycles of D. In the former case, for exactly one vertex of T, say x, the two edges incident with x and not in T are on the same cycle C of D (and are consecutive edges of the cycle). The edges incident with $y(z)$ and not from T are on distinct cycles of C. Clearly, then $H(T,x)$ has a $TFCD$. So suppose that the edges of T belong to three distinct cycles of D. Denote by M the subgraph of $H(T,3)$ induced by the set $\{x,x',y,y',z,z',n',n'',n''',m\}$. The edges of M can be decomposed into $x-y, y-z$ and $x-z$ paths. Let $e \in T$. To construct a $TFCD$ of $H(T,3)$ all one needs to do is to replace in the cycle C_e of $D, e \in C_e$, the edge e by a path of M having the same endvertices as e does.

Assume now that one of the graphs in H_T has a $TFCD$, D'. If it is the graph $H(T,x)$ then to obtain a $TFCD$ of H we formally write x instead x' and x'' in the cycles of D' containing the vertices x' and x''. In the case x', x'' belong to the same cycle of D' what we get is a trail consisting of two cycles sharing the vertex x. However, both cycles of the trail are of length at least 4 as all triangles of H are vertex-disjoint. Suppose that $H(T,3)$ possesses a $TFCD$ D'. Then the edges of the induced subgraph M form in D' three paths with endvertices from $\{x,y,z\}$. To obtain a $TFCD$ of H we replace the paths by the edges of T with the same pair of endvertices. The resulting cycles of H are shorter than the cycles of $H(T,3)$ but are of length at least 4 as the triangles of H are vertex-disjoint. The proof of the claim follows.

Now we are ready to construct the set S_H. We do it in an iterative way. At the beginning we set $S_H = \{H\}$. If $H \in EM(4,2)$ we are done. If not suppose that a graph $K \in S_H$ has a triangle T whose all vertices are of degree 4 in H. Then the graph K will be replaced by the four graphs from

H_T. After a final number of steps all graphs in S_H are from $EM(4,2)$. The claim stated above finishes the proof of Reduction B.

3 Sufficient conditions.

In this section we state some sufficient conditions for a $(4,2)-$graph to have a TFCD. We start with one which is a reduction of the main result in [4] to graphs of maximum degree 4. Let T_3 be a graph on 5 vertices with 7 edges consisting of three triangles sharing an edge. Then:

Theorem 1 *If G is a $(4,2)-$graph with no Petersen-minor and is not isomorphic to either a triangle C_3 or a graph T_3 then G has a TFCD.*

Corollary 1 *Every planar $(4,2)-$graph not isomorphic to C_3 or T_3 has a TFCD.*

Thus, our interest is concentrated on graphs with Petersen-minors. As already pointed in [4], graphs with Petersen-minor might but do not have to have a TFCD and there are infinitely many examples of either sort. The same is valid if we restrict ourselves to the class $E(4,2)$, and even the class $EM(4,2)$. The graph P obtained from Petersen graph by first doubling the 5 spokes and then subdividing each new edge by one vertex does not have a TFCD, see [4]. On the other hand, take two copies of the graph P and remove an edge from both outer and inner 5 cycle of each copy. Then add 2 edges joining vertices of degree 3 on outer cycles and two edges joining vertices of degree 3 on inner cycles. The obtained graph possesses a TFCD.

As we mentioned before we are not able to characterize those graphs of $EM(4,2)$ which contain a Petersen-minor and admit a TFCD. Instead, we present here some sufficient conditions. Let $G \in EM(4,2)$ be a 2-connected graph. Then by G_{-T} we denote a (multi)graph obtained from G by first removing all edges of G belonging to any triangle of G, then removing isolated vertices, and finally by suppressing vertices of degree 2 in G (an operation "inverse" to subdividing an edge). Note, that G_{-T} may contain parallel edges and/or triangles. However, G is 2-connected, hence there are no loops in G_{-T}.

Theorem 2 *Let G be a 2-connected graph from $EM(4,2)$. If each component of G_{-T} has an even number of edges then G has a TFCD.*

Proof. Each component of G_{-T} is an eulerian graph. For any component S of G_{-T} take an eulerian trail of S and color the edges of the trail alternately by two colors, red and blue. Then we subdivide edges by vertices of degree

2 which were removed in the process of constructing G_{-T}. Both edges obtained by a subdivision will retain the color of the original edge. Further, we color edges of triangles of G so that the edges incident with the vertex of degree 2 get the same color, the third edge will get the other color. Then the monochromatic cycles of G are of length at least 4 and thus make up a TFCD of G.

Corollary 2 *If G_{-T} is connected, then G has a TFCD.*

Proof. The number of edges of G_{-T} is even as $G_{-T} \in E(4,2)$ and its number of vertices of degree 2 equals twice the number of triangles of G.

We note that Theorem 2 is a slight generalization of Lemma 10.2.6 [7] which is formulated in terms of Compatible Cycle Decompositions.

4 Conjecture

As graphs G with G_{-T} being connected have a TFCD, the next step are graphs $G \in EM(4,2)$ so that G_{-T} has two components. However, this case is already a complicated one and it is believed that understanding this case would be a crucial step toward solving Sabidussi's conjecture, see [3, Conjecture 12 and below]. To simplify the problem even further we consider the case where $G \in EC(4,2)$ (for the definition of $EC(4,2)$ see the end of the introduction). We believe that:

Conjecture 1 *Let $G \in EC(4,2)$. Then the decision problem whether G has a TFCD can be solved in a polynomial time with respect to the order of G.*

As an immediate consequence of Theorem 2 it follows that Conjecture is true in the case G_{-T} is formed by two cycles of an even length. We point out that the validity of Fleischner-Jackson conjecture [3, page 21] about 4-regular graphs would imply our conjecture.

5 Concluding Remark.

In this section we point out the relation between a TFCD problem and the Compatible Cycle Decompositions. First we recall some definitions.

Let G be an eulerian graph. For a vertex v in G a forbidden set of v, F_v, is a partition of the set of edges incident with v. A forbidden system P of G is the union of F_v taken over all vertices v of G. We say that (G,P) has a CCD if G has a cycle decomposition F so that any cycle of F has at most one edge in common with any element of P. A trivial necessary condition for the existence of CCD is that $|p \cap T| \leq \frac{1}{2}|T|$ for any element p of P and any edge cut T of G. The pair (G,P) satisfying the necessary condition is called admissible. The problem of the existence of CCD generalizes several problems in the area, among others also the above mentioned Sabidussi's conjecture. To study the problem Fan and Zhang [2] introduced the notion of a minimal contra-pair. Define a partial order $\preceq$ on the set $\mathbf{B}$ of all admissible pairs (G,P) as follows: $(G_1,P_1) \preceq (G_2,P_2)$ if G_1 is a subgraph of G_2 and each member of P_1 is a subset of some member of P_2. A pair $(G,P) \in \mathbf{B}$ is called a contra-pair if it has no CCD. A minimal contra-pair is a contra-pair which is minimal with respect to the partial order $\preceq$. In [2] it is proved:

Theorem 3 *([2], Theorem 10.5.1) If (G,P) is a minimal contra-pair then every member of P has cardinality at most 2 and $\Delta(G) \leq 4$.*

Thus, the graphs from $E(4,2)$ play a crucial role in the CCD problem. At the end of the paper we will prove the following theorem which states the relation between the CCD and the TFCD problems.

Theorem 4 *Let (G,P) be an admissible pair, $G \in E(4,2)$. Then there exists a graph $H \in EM(4,2)$ so that (G,P) has a CCD iff H has a TFCD. Furthermore, to each graph $H \in EM(4,2)$ there is an admissible pair (G,P) so that H has a TFCD iff (G,P) has a CCD.*

The following proof uses well-known ideas in the theory of eulerian graphs and we present it here for the readers convenience.

Proof. First of all we note that if S is the forbidden set of a vertex v of degree 4 then without loss of generality we may assume that S consists either of two parts of cardinality 2 or four parts of cardinality 1. Let us start with a $(4,2)-$graph G endowed with a forbidden system P. By subdividing each edge of G by one vertex we form up a new triangle-free graph G'. Forbidden set of each new vertex (of degree 2) of G' consists of two parts of cardinality 1, a forbidden system P' of G' is the union of the forbidden system P with the forbidden sets of all new vertices. It is obvious, that the graph G has a CCD iff and only iff G' does. To obtain a desired graph $H \in EM(4,2)$ we split some vertices of G' of degree 4. A vertex v will be split if v is of degree 4 and the forbidden set of v comprises two parts of cardinality 2. We split v into two new vertices v' and v'' so that the edges belonging to the same part of the forbidden system of v are incident after the splitting with the same new vertex. Further, we add the edges of a triangle $v'v''nv'$, where n is another new vertex of degree 2. Now it is a matter of routine to check that G' has a CCD (note that G' is triangle-free) iff H admits a TFCD.

Assume now that $H \in EM(4,2)$. We construct an admissible pair (G, P) as follows. Let $T = xyzx$ be a triangle of H. Suppose that $d(z) = 2, d(x) = d(y) = 4$. By shrinking the triangle T into a vertex we understand removing the edges of T and the vertex z, and then identifying the vertices x and y into a new vertex. To obtain the graph G we shrink simultaneously all triangles of H. Note that after carrying out the operation the resulting graph G might have a triangle but we will not apply the operation to the triangles of G. Further, G might have multiple edges. In such a case, only for the formal reason in order to confine ourselves to graphs, we subdivide each parallel edge by one vertex. Now we define a forbidden system of G. Two edges incident with a vertex of G of degree 4 made up by shrinking a triangle T will form up a forbidden part iff they were incident with the same vertex of T. For the other vertices of G we define all the forbidden parts to be of cardinality 1. Clearly, H has a $TFCD$ iff G has a cycle decomposition compatible with P. The proof follows.

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