Nordhaus–Gaddum Results for Open Irredundance

E.J. Cockayne

Department of Mathematics and Statistics
University of Victoria
BC, Canada

Abstract

It is shown that for $n \geq 16$, the sum of cardinalities of open irredundant sets in an n–vertex graph and its complement is at most $3n/4$.

Keywords: open irredundance, Nordhaus–Gaddum

Subject Classification Number: 05C69

1 Introduction

Let $G = (V, E)$ be a simple n–vertex graph and $v \in X \subseteq V$. The vertex x is called an X*–self private neighbour* (abbreviated X–spn) if x is isolated in the subgraph of $G[X]$ of G induced by X, and $t \in V - X$ is an X*–external private neighbour* (X–epn) of x if $N(t) \cap X = \{x\}$. It is well-known that X is an *irredundant* set if for each $x \in X$, x is an X–spn or x has at least one X–epn. Alternatively, X is irredundant if for each $x \in X$,

$$N[x] - N[X - \{x\}] \neq \phi.$$

Since each neighbourhood here is closed, such an X has also been called CC*–irredundant* [2].

Irredundance is the property which makes a dominating set minimal. This fact, together with applications has stimulated many authors to contribute to an intriguing theory of irredundant sets (see [11]). For a graph parameter $\eta(G)$, results which bound the sum $\eta(G) + \eta(\overline{G})$ or the product $\eta(G)\eta(\overline{G})$ (where $\overline{G}$ is the complement of G) are known as Nordhaus–Gaddum results due to the work of these authors on chromatic numbers

[13]. Such results for CC–irredundance were established in [5] together with a characterisation of the extremal graphs.

It is the purpose of this work to establish analogous results for open irredundance, which was introduced in [7] and applied to broadcast networks. The set X is *open irredundant* if each $x \in X$ has at least one X–epn. Equivalently, X is open irredundant if for each $x \in X$,

$$N(x) - N[X - \{x\}] \neq \phi.$$

Since the neighbourhoods in this characterisation are open and closed, such sets have also been called *OC–irredundant* in [2, 10].

Let $oir(G)$ and $OIR(G)$ denote the smallest and largest cardinalities among the maximal open irredundant set of G. Various inequalities relating these parameters to the domination number and CC–irredundant parameters were established in [7, 8, 9, 12]. In [1] it was shown that any isolate–free graph has an open irredundant minimum dominating set. An algorithm for computing $OIR(T)$ for a tree T was presented in [6]. In [4] a lower bound for $oir(G)$ was obtained when G is an isolate–free n vertex graph with maximum degree Δ. The extremal graphs for this bound were characterised. The concepts of CC– and OC–irredundance have been embedded in larger classes of irredundance models in [2, 3, 10].

In Section 2 we prove that for any n–vertex ($n \geq 16$) graph G, $OIR(G) + OIR(\overline{G}) \leq 3n/4$. The bound is exact for $n \equiv 0 \pmod 4$ and extremal graphs of the inequality are exhibited. A simple corollary shows that for $n \geq 16$, $OIR(G)OIR(\overline{G}) \leq 9n^2/64$. This bound for the product cannot be attained for $n \geq 17$.

2 The bound for $OIR(G) + OIR(\overline{G})$

Let $X(Y)$ be open irredundant sets of $G(\overline{G})$, $|X| = x$ and $|Y| = y$. Note that both x and y are at most $n/2$. Each $u \in X (v \in Y)$ has an X–epn u_r in G (Y–epn v_b in $\overline{G}$). The edges of G (respectively $\overline{G}$) will be coloured red (blue). Occasionally $u_r(v_b)$ will be called a *red epn* of u (*blue epn* of v). Let $X' = \{u_r \mid u \in X\}$. Then each edge of $\{uu_r \mid u \in X\}$ is red while all other edges joining X to X' are blue. Note that the set X' is also an open irredundant set of G and u is an X'–epn of u_r in G. Let $Z = V - (X \cup X')$.

The principal result will follow immediately from three propositions which are broken down into cases depending on the distribution of vertices of Y and blue epns among the three sets X, X', Z. The counting in the various cases is remarkably similar. We repeatedly use the following obvious fact.

Lemma 1 *Let A be an open irredundant set in a graph F and $B \subseteq V(F)$. If each $u \in A \cap B$ has A-epn in B, then $|A \cap B| \leq |B|/2$.*

Proposition 1 *If $n \geq 14$ and $|Y \cap X| \geq 3$, then $x + y \leq 3n/4$.*

Proof Since $|Y \cap X| \geq 3$, for each $u \in Y \cap X$, $u_b \notin X'$. Hence $u_b \in X \cup Z$. Define

$$X_1 = \{u \in Y \cap X \mid u_b \in X\},$$
$$X_2 = \{u \in Y \cap X \mid u_b \in Z\},$$
$$X_3 = X - (X_1 \cup X_2)$$

and for $i = 1, 2, 3$, let $|X_i| = x_i$. For $w \in Y \cap Z$, $w_b \notin X_1 \cup X_2 \cup X'$, hence $w_b \in X_3 \cup Z$.

Case 1 $Y \cap X' = \phi$

Let $t = |\{w \in Y \cap Z \mid w_b \in X_3\}|$. Then by Lemma 1,

$$|\{w \in Y \cap Z \mid w_b \in Z\} \leq (n - 2x - x_2 - t)/2 \tag{1}$$

We will now give more detailed justification for (1). Similar explanations will be omitted in future cases of the propositions. Define

$$B = Z - (\{w \in Y \cap Z \mid w_b \in X_3\} \cup \{w_b \in Z \mid w \in X_2\}) \text{ (disjoint union).}$$

Note that $|B| = (n - 2x - x_2 - t)$ and

$$\{w \in Y \cap Z \mid w_b \in Z\} = \{w \in Y \cap B \mid w_b \in B\}.$$

Then (1) follows by applying Lemma 1 with $A = Y$. Now

$$\begin{aligned} x + y &= x + |Y \cap X| + |Y \cap Z| \\ &\leq x + (x_1 + x_2) + t + \left(\frac{n - 2x - x_2 - t}{2}\right) \\ &= x_1 + \frac{x_2}{2} + \frac{t}{2} + \frac{n}{2}. \end{aligned} \tag{2}$$

The blue epns in X_3 are distinct and so $x_3 \geq t + x_1$, i.e.

$$\frac{t}{2} \leq \frac{x_3}{2} - \frac{x_1}{2}. \tag{3}$$

From (2) and (3) we obtain

$$x + y \le \left(\frac{x_1 + x_2 + x_3}{2}\right) + \frac{n}{2} = \frac{x}{2} + \frac{n}{2} \le \frac{n}{4} + \frac{n}{2} = \frac{3n}{4}.$$

Case 2 $|Y \cap X'| \ge 2$.

In this case $x_1 = 0$, each $w \in Y \cap Z$ has $w_b \in Z$ and for each $w \in Y \cap X'$, $w_b \notin X'$ i.e. $w_b \in X_3 \cup Z$.

Subcase 2(a) $w \in Y \cap X'$ has $w_b \in X_3$.

This implies $|Y \cap X'| = 2$. Let $Y \cap X' = \{w, v\}$. Now

$$\begin{aligned} x + y &= x + |Y \cap X| + |Y \cap X'| + |Y \cap Z| \\ &\le x + x_2 + 2 + \frac{(n - 2x - x_2 - \lambda)}{2} \end{aligned}$$

where $\lambda = 1$ (respectively 0) if $v_b \in Z(X_3)$. Hence

$$x + y \le \frac{n}{2} + \frac{x_2}{2} - \frac{\lambda}{2} + 2. \tag{4}$$

By counting blue epns in X_3 we obtain $x_3 \ge 2 - \lambda$. Hence from (4)

$$x + y \le \frac{n}{2} + \left(\frac{x_2 + x_3}{2}\right) + 1 = \frac{n}{2} + \frac{x}{2} + 1. \tag{5}$$

However $|Z| \ge x_2$ and so $x \le \frac{n - x_2}{2}$. Hence from (5)

$$x + y \le \frac{3n}{4} - \frac{x_2}{4} + 1. \tag{6}$$

By hypothesis $x_2 = |Y \cap X| \ge 3$. If $x_2 \ge 4$, then (6) gives $x + y \le 3n/4$ as required. If $x_2 = 3$, then (6) gives

$$x + y \le \frac{3n}{4} + \frac{1}{4}. \tag{7}$$

However equality in (6) or (7) requires $x_3 = 2 - \lambda$ and $x = (n - x_2)/2$. Therefore

$$\frac{(n-3)}{2} = x = x_2 + x_3 = 3 + (2 - \lambda).$$

Hence $n = 13 - 2\lambda < 14$, a contradiction which shows that

$$x + y < \frac{3n}{4} + \frac{1}{4}.$$

Therefore $x + y \le \frac{3n}{4}$.

Subcase 2(b) Each $w \in Y \cap X'$ has $w_b \in Z$.

Let $s = |Y \cap X'|$. Then

$$\begin{aligned} x + y &= x + |Y \cap X| + |Y \cap X'| + |Y \cap Z| \\ &\leq x + x_2 + s + \left(\frac{n - 2x - x_2 - s}{2}\right) \\ &= \left(\frac{x_2}{2} + \frac{s}{2}\right) + \frac{n}{2} \leq \frac{y}{2} + \frac{n}{2} \leq \frac{n}{4} + \frac{n}{2} = \frac{3n}{4}. \end{aligned}$$

Case 3 $|Y \cap X'| = \{v\}$.

Define λ as in subcase 2(a) and let $\mu(= 0 \text{ or } 1)$ be the number of vertices in $Y \cap Z$ with blue epns in X_3. Hypothesis and the private neighbour property imply

$$x_1 + \mu \leq 1. \tag{8}$$

By counting blue epns in X_3 we obtain

$$x_3 \geq (1 - \lambda) + x_1 + \mu. \tag{9}$$

The set Z contains $\lambda + x_2$ blue epns of vertices in $Y \cap (X \cup X')$ and μ vertices of $Y \cap Z$ have blue epns in X_3. Hence using Lemma 1 we obtain

$$\begin{aligned} x + y &= x + |Y \cap X| + |Y \cap X'| + |Y \cap Z| \\ &\leq x + (x_1 + x_2) + 1 + \mu + \left(\frac{n - 2x - \mu - x_2 - \lambda}{2}\right) \\ &= \frac{n}{2} + x_1 + \frac{x_2}{2} + \frac{(\mu - \lambda)}{2} + 1. \end{aligned}$$

Using (9) we deduce

$$x + y \leq \frac{n}{2} + \left(\frac{x_1 + x_2 + x_3}{2}\right) + \frac{1}{2} = \frac{n}{2} + \frac{x}{2} + \frac{1}{2}. \tag{10}$$

However by (8) $x_1 \leq 1$ and so $x_2 \geq 2$. This implies $x \leq (n-2)/2$ and therefore from (10) we get $x + y \leq \frac{3n}{4}$. ■

Proposition 2 *If* $n \geq 16$ *and* $|Y \cap X\ | \in \{1, 2\}$, *then* $x + y \leq \frac{3n}{4}$.

Proof Let $|Y \cap X| = 1 + \alpha$, where $\alpha \in \{0, 1\}$. If $|Y \cap X'| \geq 3$, then the result is true by the application of Proposition 1 to X' and Y. Hence we may assume that $|Y \cap X'| \leq 2$.

Case 1 $|Y \cap X'| = 1 + \beta$ where $\beta \in \{0, 1\}$.

At most $1-\alpha$ (respectively $1-\beta$) vertices of $Y \cap Z$ have blue epns in X' (respectively X). Hence

$$\begin{aligned}
x+y &\le x+|Y\cap X|+|Y\cap X'|+|Y\cap Z| \\
&\le x+(1+\alpha)+(1+\beta)+(1-\alpha)+(1-\beta)+\left(\frac{n-2x-2+(\alpha+\beta)}{2}\right) \\
&= \frac{n}{2}+3+\frac{\alpha+\beta}{2} \le \frac{n}{2}+4 \\
&\le \frac{3n}{4} \quad \text{(since } n \ge 16\text{)}.
\end{aligned}$$

Case 2 $|Y \cap X'| = 0$.

In this case at most $(1-\alpha)$ vertices of $Y \cap Z$ have blue epns in X'. Suppose that t vertices of $Y \cap Z$ have blue epns in X. Then estimating as above yields

$$\begin{aligned}
x+y &\le x+(1+\alpha)+(1-\alpha)+t+\left(\frac{n-2x-1+\alpha-t}{2}\right) \\
&= \frac{n}{2}+\frac{\alpha}{2}+\frac{t}{2}+\frac{3}{2}. \qquad (11)
\end{aligned}$$

Now $t+(1+\alpha) \le x$. Therefore from (11)

$$x+y \le \frac{n}{2}+\frac{x}{2}+1. \qquad (12)$$

If $x \le \frac{n-4}{2}$, then (12) gives $x+y \le \frac{3n}{4}$. Suppose that $x = \frac{n-m}{2}$, where $m \le 3$. Then $|Z| = m$ and

$$\begin{aligned}
x+y &= x+|Y\cap X|+|Y\cap Z| \\
&\le \left(\frac{n-m}{2}\right)+2+m \\
&= \frac{n}{2}+\frac{m}{2}+2 \\
&\le \frac{n}{2}+\frac{7}{2} < \frac{3n}{4} \quad \text{(for } n \ge 16\text{)}.
\end{aligned}$$

■

Proposition 3 *If $n \ge 16$ and $|Y \cap X| = 0$, then $x+y \le \frac{3n}{4}$.*

Proof By Propositions 1 and 2 applied to X' and Y we may assume that $|Y \cap X'| = 0$. Let s (respectively t) vertices of $Y \cap Z$ have blue epns in

$X(X')$. By estimating as usual we obtain

$$x+y \le x+s+t+\left(\frac{n-s-t-2x}{2}\right)$$
$$=\frac{n}{2}+\frac{s+t}{2}=\frac{n}{2}+\frac{y}{2}.$$

Since $y \le \frac{n}{2}$, we obtain $x+y \le \frac{3n}{4}$ as required. ■

The principal result of the paper now follows immediately from Propositions 1, 2 and 3.

Theorem 1 *For any n–vertex graph G where $n \ge 16$,*

$$OIR(G)+OIR(\overline{G}) \le \frac{3n}{4}.$$

In order to describe the following example and the extremal graphs in Section 3, we let $X=\{u_1,\ldots,u_x\}$ and $X'=\{v_1,\ldots,v_x\}$, where for each $i=1,\ldots,x$ vertex v_i is a red epn of u_i. Note that each edge joining X to X' which is not in $\{u_iv_i \mid i=1,\ldots,x\}$, is blue. Let $Z=\{w_1,\ldots,w_k\}$ (where $2x+k=n \ge 16$). Examples will be described by specifying x and disjoint sets Y, Y' of cardinality y. The order of vertices within the set parentheses describing Y, Y', will be such that a vertex in Y' will be a blue epn of the corresponding vertex in Y, so that all other edges joining Y to Y' are red. Any edge which joins neither X to X' nor Y to Y' may be arbitrarily red or blue.

We now present an example to show that $OIR(G)+OIR(\overline{G})$ can exceed $3n/4$ for $n<16$. Let $n=13$, $x=5$, $Y=\{u_1,u_2,w_1,w_2,w_3\}$ and $Y'=\{v_2,v_1,u_3,u_4,u_5\}$. Then

$$x+y=10>\frac{3\times 13}{4}.$$

3 Extremal graphs: the product $OIR(G)OIR(\overline{G})$

In order to find the extremal graphs of the bound of Theorem 4, one substitutes equality for each inequality used in the proof of the bound in the various cases in Propositions 1, 2 and 3. This analysis will be carried out in (a) – (g) below. Note that the bound can only be attained when $n \equiv 0 \pmod 4$.

(a) **Proposition 1, Case 1**

Substitution of equality for inequalities used to establish the bound (in future cases these words will be omitted) gives $x = n/2$ and $x_3 = t + x_1$. Therefore, if the bound is attained, then $|Z| = t = x_2 = 0$ and $x_3 = x_1$. Thus

$$2x_1 = x_1 + x_3 = x = \frac{n}{2}$$

so that $y = n/4$. without loss of generality $Y = \{u_1, u_3, \ldots, u_{\frac{n}{2}-1}\}$ and $Y' = \{u_2, u_4, \ldots, u_{\frac{n}{2}}\}$.

(b) **Proposition 1, Subcase 2(a)**

In this situation we have $x = (n - x_2)/2$. This implies $\lambda = 0$ and since $x_3 = 2 - \lambda$, we deduce $x_3 = 2$. Also $x_1 = 0$ and $x_2 = 4$ which implies $k = 4$. Hence $x = (n-4)/2$ and $y = x_2 + |Y \cap X'| = 6$. Therefore $6 + (n-4)/2 = 3n/4$ which gives $n = 16$ and $x = y = 6$. Without loss of generality

$$Y = \{v_1, v_2, u_3, u_4, u_5, u_6\} \text{ and } Y' = \{u_2, u_1, w_1, w_2, w_3, w_4\}.$$

(c) **Proposition 1, Subcase 2(b)**

Here we have $y = n/2$, hence $x = n/4$. The structure is that of (a) with red and blue edges interchanged.

(d) **Proposition 1, Case 3**

In this case $x = (n-2)/2$ which implies $k = 2$ and $x_2 \leq 2$. By hypothesis $x_1 + x_2 \geq 3$ and (8) gives $x_1 \leq 1$. We conclude $x_2 = 2$, $x_1 = 1$ and $\lambda = \mu = 0$. Equality in (9) gives $x_3 = 2$. Hence $n = 12$, a contradiction.

(e) **Proposition 2, Case 1**

We have $(\alpha + \beta)/2 = 1$ i.e. $\alpha = \beta = 1$ and $4 + n/2 = 3n/4$ so that $n = 16$. Hence

$$(x, y) \in \{(4,8), (8,4), (5,7), (7,5), (6,6)\}.$$

If $(x, y) = (8, 4)$ then without loss in generality $Y = \{u_1, u_2, v_3, v_4\}$ and $Y' = \{v_2, v_1, u_4, u_3\}$. The situation with $(x, y) = (4, 8)$ is a blue/red interchange of this.

If $(x, y) = (7, 5)$ then $Y = \{u_1, u_2, v_3, v_4, w_1\}$ and $Y' = \{v_2, v_1, u_4, u_3, w_2\}$ and $(x, y) = (5, 7)$ gives a blue/red interchange of this.

Finally $(x, y) = (6, 6)$ gives

$$Y = \{u_1, u_2, v_3, v_4, w_1, w_2\} \text{ and } Y' = \{v_2, v_1, u_4, u_3, w_3, w_4\},$$

which is a different situation from that in (b).

(f) **Proposition 2, Case 2**

In this case $t+1+\alpha = x$. Equality is possible if $x = (n-4)/2$ so that $k = 4$, $t \leq 4$ and $x \leq 6$. We deduce $n \leq 16$ i.e. $n = 16$, $\alpha = 1$ and $x = 6$. Without loss of generality $Y = \{u_1, u_2, w_1, w_2, w_3, w_4\}$ and $Y' = \{v_2, v_1, u_3, u_4, u_5, u_6\}$.

(g) **Proposition 3**

We have $y = k = n/2$ and $x = n/4$. The situation is the same as (a) with Y', Y, X playing the roles of X, X', Y and blue and red edges interchanged.

The above analysis shows that there are several different structures of extremal graphs for $n = 16$. For $n \geq 20$ and $n \equiv 0 \pmod 4$ every extremal graph is one of those specified in (a) (we emphasize that we are ignoring interchanges of X, X' of Y, Y', and of red/blue edges).

Finally we consider the product of $OIR(G)$ and $OIR(\overline{G})$.

Corollary 1 *If G has n vertices, then*

(*i*) $OIR(G)OIR(\overline{G}) \leq 36$ *for* $n = 16$.

(*ii*) $OIR(G)OIR(\overline{G}) < \dfrac{9n^2}{64}$ *for* $n \geq 17$.

Proof Let $OIR(G) = x$ and $OIR(\overline{G}) = y$. By Theorem 4 for $n \geq 16$,

$$xy \leq x\left(\frac{3n}{4} - x\right). \tag{13}$$

By elementary calculus the maximum of the right hand side of (13) occurs when $x = y = 3n/8$ and the maximum is $9n^2/64$.

(*i*) If $n = 16$, then the analysis of (b), (e) and (f) above give the extremal graphs of the bound $9n^2/64 = 36$.

(*ii*) If $n \geq 17$, then (a) – (g) above show that $x = y = 3n/8$ is impossible and so $OIR(G)OIR(\overline{G}) < 9n^2/64$. ■

Note that (a) above implies

$$\max_G OIR(G)OIR(\overline{G}) \geq \frac{n^2}{8} \text{ for } n \equiv 0 \pmod 4.$$

We can obtain a lower bound very close to $\dfrac{n^2}{8}$ for $n \not\equiv 0 \pmod 4$ with $x = \lfloor \frac{n}{2} \rfloor$, $Y = \{u_1, u_3, \ldots\}$ and $Y' = \{u_2, u_4, \ldots\}$.

Acknowledgements

The paper was written while the author was enjoying the hospitality of the Department of Mathematics, Applied Mathematics and Astronomy, University of South Africa, during 2001.

The research Support of the Canadian Natural Sciences and Engineering Research Council (NSERC) is gratefully acknowledged.

References

[1] B. Bollobás and E.J. Cockayne, The irredundance number and maximum degree of a graph, *Discrete Math.* **69** (1984), 197–199.

[2] E.J. Cockayne, Generalized irredundance in graphs: hereditary properties and Ramsey numbers, *J. Combin. Math. Combin. Comput.* **31** (1999), 15–31.

[3] E.J. Cockayne, O. Favaron, P.J.P. Grobler, C.M. Mynhardt and J. Puech, Ramsey properties of generalised irredundant sets in graphs, *Discrete Math.* **231**(2001), 123–134.

[4] E.J. Cockayne, O. Favaron, C.M. Mynhardt, Open irredundance and maximum degree in graphs (submitted).

[5] E.J. Cockayne, C.M. Mynhardt, On the product of upper irredundance numbers of a graphs and its complement. *Discrete Math.* **76** (1988), 117–121.

[6] A.M. Farley and A. Proskurowski, Computing the maximum order of an open irredundant set in a tree, *Congr. Numer.* **41** (1984), 219–228.

[7] A.M. Farley and N. Shacham, Senders in broadcast networks: open irredundancy in graphs, *Congr. Numer.* **38** (1983), 47–57.

[8] O. Favaron, A note on the open irredundance in a graph, *Congr. Numer.* **66** (1988), 316–318.

[9] O. Favaron, A note on the irredundance number after vertex deletion, *Discrete Math.* **121** (1993), 51–54.

[10] M.R. Fellows, G.H. Fricke, S.T. Hedetniemi and D. Jacobs, The private neighbor cube, SIAM J. *Discrete Math.* **7** (1994), 41–47.

[11] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, *Fundamentals of Domination in Graphs.* Marcel Dekker, New York, 1998.

[12] S.T. Hedetniemi, D.P. Jacobs and R.C. Laskar, Inequalities involving the rank of a graph, *J. Combin. Math. Combin. Comput.* **6** (1989), 173–176.

[13] E.A. Nordhaus and J.W. Gaddum, On complementary graphs, *Amer. Math. Monthly* **63** (1956), 175–177.