Unique Independence, Upper Domination and Upper Irredundance in Graphs

Miranca Fischermann and Lutz Volkmann
Lehrstuhl II für Mathematik, RWTH-Aachen, 52056 Aachen, Germany,
e-mail: fischerm, volkm@math2.rwth-aachen.de

Abstract

A set D of vertices in a graph G is irredundant if every vertex v in D has at least one the private neighbour in $N[v,G] \setminus N[D \setminus \{v\}, G]$. A set D of vertices in a graph G is a minimal dominating set of G if D is irredundant and every vertex in $V(G) \setminus D$ has at least one neighbour in D. Further, irredundant sets and minimal dominating sets of maximal cardinality are called IR-sets and Γ-sets, respectively. A set I of the vertex set of a graph G is independent if no two vertices in I are adjacent and independent sets of maximal cardinality are called α-sets.

In this paper we prove that bipartite graphs and chordal graphs have a unique α-set if and only if they have a unique Γ-set if and only if they have a unique IR-set. Some related results are also presented.

Keywords: Independence; Upper domination; Upper irredundance; Uniqueness; Chordal graphs; Bipartite graphs.

1 Terminology and Introduction

For any graph G the vertex set and the edge set of G are denoted by $V(G)$ and $E(G)$, and $n(G) = |V(G)|$ and $m(G) = |E(G)|$ are called the *order* and the *size* of G, respectively. For any subset $A \subseteq V(G)$ we define the induced subgraph $G[A]$ as the graph with vertex set A and edge set $\{ab \in E(G) \mid a, b \in A\}$. For any set $A \subseteq V(G)$ and any vertex $x \in V(G)$ we define $G - A = G[V(G) \setminus A]$ and $G - x = G - \{x\}$. For any positive integers s and t we denote by $K_{s,t}$ the complete bipartite graph where one partite set has order s and the other has order t, and for $t \geq 3$ we denote by C_t the cycle of order t.

For every vertex x in the vertex set of a graph G we denote the set of neighbours of x in G by $N(x,G) = N(x)$ and we define $N[x,G] =$

$N[x] = N(x) \cup \{x\}$. The sets $N(x)$ and $N[x]$ are called the open and the closed neighbourhood of x, respectively. For a subset D of $V(G)$ and a vertex $x \in D$ the set $P(x,D) = N[x] \setminus N[D \setminus \{x\}]$ is called the *private neighbourhood* of x with respect to D. We call a vertex $y \in P(x,D)$ a *private neighbour* of x with respect to D. For a set $A \subseteq D$ we define the set $P(A,D) = \bigcup_{x \in A} P(x,D)$.

A set $D \subseteq V(G)$ is *irredundant* if every vertex in D has at least one private neighbour. An irredundant set D of G is called *maximal irredundant* if $D \cup \{v\}$ is no longer irredundant for every vertex $v \in V(G) \setminus D$. The maximum cardinality of an irredundant set is called the *upper irredundance number* and is denoted by $\mathrm{IR}(G)$. Further, an irredundant set of G of cardinality $\mathrm{IR}(G)$ is called an IR-*set*.

A set $D \subseteq V(G)$ is a *dominating set* of G if $V(G) \subseteq N[D,G]$. A dominating set D of G is called *minimal* if $D \setminus \{x\}$ is no longer a dominating set of G for every vertex $x \in D$. The maximum cardinality of a minimal dominating set is called the *upper domination number* and is denoted by $\Gamma(G)$. We call a minimal dominating set of G of cardinality $\Gamma(G)$ a Γ-*set*.

A subset I of the vertex set of a graph G is called *independent* if no two vertices in I are adjacent and an independent set of a graph G is called a *maximal independent set* of G if for every vertex $u \in V(G) \setminus I$ the set $I \cup \{u\}$ is no longer independent. An independent set of a graph G of maximal cardinality is called a *maximum independent set* of G or an α-*set* and its cardinality is called the *independence number* of G and is denoted by $\alpha(G)$.

If $\alpha(H) = \Gamma(H)$ for every induced subgraph H of G, then a graph G is called Γ-*perfect*. Furthermore, a graph G is called IR-*perfect* if $\Gamma(H) = \mathrm{IR}(H)$, for every induced subgraph H of G. For other graph theory terminology we follow the monograph by Haynes, Hedetniemi and Slater [13].

In 1985, Hopkins and Staton [15] have investigated graphs with unique maximum independent sets, and there succeeded a couple of publications on the uniqueness of graph parameters, as e.g. [6], [7], [9], [8], [10], [12], [19] and [20]. There are also several publications on relations between independence, upper domination and upper irredundance as for example [1], [2], [3], [4], [11], [13], [14], [16] and [21].

Cockayne, Favaron, Payan and Thomason [2] found in 1981 that any bipartite graph G satisfies $\alpha(G) = \Gamma(G) = \mathrm{IR}(G)$. About ten years later Jacobson and Peters [16] proved the same for chordal graphs and for a class of graphs defined by three forbidden induced subgraphs, and Topp [21] showed this equality for unicyclic graphs. Using these results, we prove in this paper for any of those graphs the equivalence of the uniqueness of an α-set, the uniqueness of a Γ-set and the uniqueness of an IR-set, and we present some related results.

2 Preliminary results

The following equality chain is well known.

Lemma 2.1 (Cockayne, Hedetniemi and Miller [4]) *For any graph G every maximal independent set is a minimal dominating set and every minimal dominating set is maximal irredundant. Thus,*

$$\alpha(G) \leq \Gamma(G) \leq \mathrm{IR}(G).$$

In 1998, Gutin and Zverovich have found the following relation.

Theorem 2.2 (Gutin and Zverovich [11]) *Any Γ-perfect graph is* IR-*perfect.*

There are several classes of graphs that are Γ-perfect and IR-perfect. Four of them are considered here.

Theorem 2.3 (Cockayne, Favaron, Payan and Thomason [2]) *If G is a bipartite graph, then*

$$\alpha(G) = \Gamma(G) = \mathrm{IR}(G).$$

Theorem 2.4 (Jacobson and Peters [16]) *If G is a chordal graph, then*

$$\alpha(G) = \Gamma(G) = \mathrm{IR}(G).$$

Theorem 2.5 (Jacobson and Peters [16]) *For any graph G that does not contain either $K_{1,3}$, C_4 or the graph H in Figure 1 as an induced subgraph,*

$$\alpha(G) = \Gamma(G) = \mathrm{IR}(G).$$

The complete bipartite graph $K_{1,3}$ is also called the *claw*, and graphs G that do not contain $K_{1,3}$ as an induced subgraph are also called *claw-free*.

H

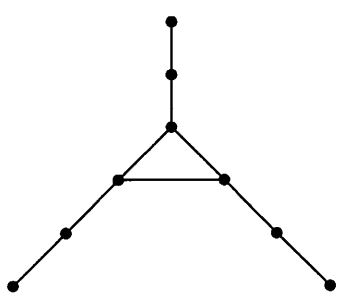


Figure 1

Theorem 2.6 (Topp [21]) *If G is a unicyclic graph, then*

$$\alpha(G) = \Gamma(G) = \mathrm{IR}(G).$$

Since the main properties of the graphs in the last four theorems are hereditary with respect to induced subgraphs, all those graphs are Γ-perfect and IR-perfect.

3 Uniqueness

As a corollary of Lemma 2.1 we obtain the following.

Observation 3.1 Let the graph G be arbitrary.

a) If $\Gamma(G) = \mathrm{IR}(G)$ and G has a unique IR-set D, then D is also the unique Γ-set of G.

b) If $\alpha(G) = \Gamma(G)$ and G has a unique Γ-set D, then D is also the unique α-set of G.

Proof. Lemma 2.1 implies that every α-set of G is a Γ-set of G if $\alpha(G) = \Gamma(G)$, and every Γ-set of G is an IR-set of G if $\Gamma(G) = \mathrm{IR}(G)$. Hence, the required result follows. □

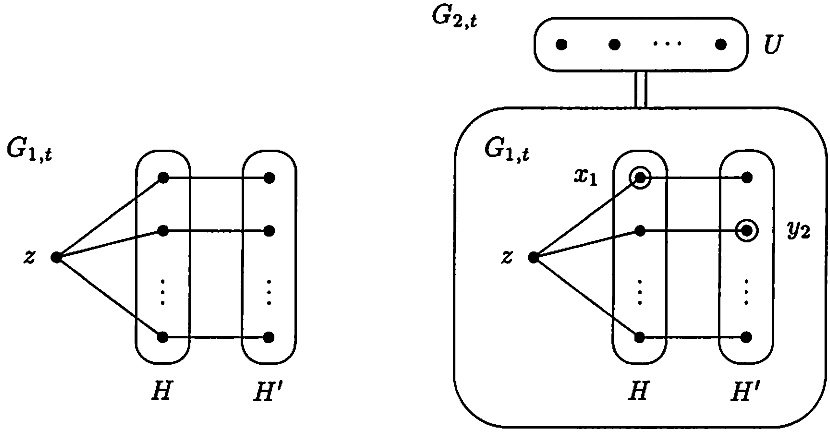
We will now construct graphs which show that the converse of Observation 3.1 does not hold in general.

For every integer $t \geq 3$, let $G_{1,t}$ be the graph consisting of two disjoint complete graphs H and H' with vertex sets $V(H) = \{x_1, x_2, \ldots, x_t\}$ and $V(H') = \{y_1, y_2, \ldots, y_t\}$ and of the additional edges $\{x_i y_i \mid 1 \leq i \leq t\}$ and of a further vertex z that is adjacent to all vertices in H (cf. Figure 2 a)).

Further, let the graph $G_{2,t}$ consist of the graph $G_{1,t}$, of a disjoint independent vertex set U of cardinality $t-2$ and of the additional edges $uv \in E(G_{2,t})$ for every $u \in U$ and $v \in V(G_{1,t}) \setminus \{x_1, y_2\}$ (cf. Figure 2 b)).

Observation 3.2 The graph $G_{1,t}$ has the unique Γ-set $V(H)$ and $\Gamma(G_{1,t}) = \mathrm{IR}(G_{1,t}) = t$ but $G_{1,t}$ has no unique IR-set, since the two sets $V(H)$ and $V(H')$ are IR-sets of $G_{1,t}$.
The graph $G_{2,t}$ has the unique α-set $U \cup \{x_1, y_2\}$ and $\alpha(G_{2,t}) = \Gamma(G_{2,t}) = \mathrm{IR}(G_{2,t}) = t$ but $G_{2,t}$ has the two Γ-sets $U \cup \{x_1, y_2\}$ and $V(H)$.

We will see that for all classes of Γ-perfect graphs G considered in Section 2, a unique α-set I of G also is the unique Γ-set and the unique IR-set of G. In order to prove this, we use the following two lemmas.



$H \cong H' \cong K_t,\ t \geq 3$

U independent, $|U| = t-2$ and $u \in U, v \in V(G_{1,t}), uv \in E(G_{2,t}) \Leftrightarrow v \neq x_1, y_2$

a) b)

Figure 2

Lemma 3.3 *If any graph G has a unique α-set I, then every vertex in $V(G) \setminus I$ has at least two neighbours in I.*

Proof. Let G be an arbitrary graph that has a unique α-set I. Suppose that a vertex v in $V(G) \setminus I$ has at most one neighbour in I. Then, the set $(I \setminus N(v)) \cup \{v\}$ is independent which either contradicts the maximality or the uniqueness of I. □

Lemma 3.4 *Let G be a Γ-perfect graph. If G has a unique α-set I and an* IR*-set $D \neq I$, then*

a) for every vertex $x \in I \setminus D$ there exists a unique vertex $w_x \in D \setminus I$ such that $P(w_x, D) = \{x\}$,

b) $\{ab \in E(G) \mid a \in I \setminus D, b \in D \setminus I\} = \{xw_x \mid x \in I \setminus D\}$, and

c) there exists a cycle C in G such that $C = x_1x_2 \ldots x_{4p}x_1$ for some positive integer p and for every $0 \leq i < p$ we have $x_{4i+2} \in D \setminus I$, $x_{4i+3} \in I \cap D$ and $x_{4i+1} \in P(x_{4i+2}, D) \subseteq I \setminus D$, $x_{4i+4} \in P(x_{4i+3}, D) \subseteq P(I \cap D, D) \setminus (I \cup D)$.

Proof. Let G be a Γ-perfect graph that has a unique α-set I and an IR-set $D \neq I$. By Theorem 2.2, we have $\alpha(H) = \Gamma(H) = \mathrm{IR}(H)$, for

every induced subgraph H of G, and especially $|I| = |D|$. Let $x \in I \setminus D$ be arbitrary. We define the induced subgraph $G_x = G - x$ of G and the independent set $I_x = I \setminus \{x\}$. Since I is unique, the set I_x is an α-set of G_x. Thus, we get that $|D| > |I_x| = \alpha(G_x) = \Gamma(G_x) = \mathrm{IR}(G_x)$. This and the fact that D is a subset of $V(G_x)$ imply that D is not irredundant in G_x. Hence, there exists a vertex w_x in D with $P(w_x, D) \cap V(G_x) = \emptyset$. Since the set D is irredundant in G, we obtain that $P(w_x, D) = \{x\}$, $w_x \in D \setminus I$ and $N(x) \cap (D \setminus I) = \{w_x\}$, whereby the proof of a) is complete.
The equality $|D \setminus I| = |I \setminus D|$ implies that every vertex in $D \setminus I$ has exactly one neighbour in $I \setminus D$ and this neighbour is its only private neighbour. This proves b).
It remains to prove c). Let $x_1 \in I \setminus D$ be arbitrary and let x_2 be its unique neighbour in $D \setminus I$. By Lemma 3.3, the vertex x_2 has at least two neighbours in I which yields the existence of a neighbour x_3 of x_2 in $I \cap D$. Since D is irredundant and $x_3 \notin P(x_3, D)$, there exists a fourth vertex $x_4 \in P(x_3, D) \subseteq P(I \cap D, D) \setminus D$. It even yields that $x_4 \in P(I \cap D, D) \setminus (D \cup I)$, since x_3 lies in I. By Lemma 3.3 and since $x_4 \in P(x_3, D)$, the vertex x_4 has a neighbour in $I \setminus D$. If the vertex x_4 is adjacent to x_1, then $x_1x_2x_3x_4x_1$ is a cycle that satisfies the properties in c). Otherwise, the vertex x_4 has a neighbour x_5 in $I \setminus D$ different from x_1, and we can extend the path $x_1x_2x_3x_4$ to a longest path $P = x_1x_2 \ldots x_t$, $t \geq 5$, such that for every $1 \leq i \leq t$ we have

$$
\begin{aligned}
x_i &\in I \setminus D && \text{if } i \equiv 1 \pmod 4,\\
x_i &\in D \setminus I && \text{if } i \equiv 2 \pmod 4,\\
x_i &\in I \cap D && \text{if } i \equiv 3 \pmod 4 \text{ and}\\
x_i &\in P(I \cap D, D) \setminus (I \cup D) && \text{if } i \equiv 0 \pmod 4.
\end{aligned}
$$

This implies that $x_i \in P(x_{i+1}, D)$ for every $1 \leq i < t$ with $i \equiv 1 \pmod 4$ and $x_i \in P(x_{i-1}, D)$ for every $1 \leq i \leq t$ with $i \equiv 0 \pmod 4$.
If $t \equiv 1 \pmod 4$, then the unique neighbour w of x_t in $D \setminus I$ does not belong to P, since it has no other neighbour in $I \setminus D$, and we can extend P with w, which is a contradiction.
Analogously, if $t \equiv 3 \pmod 4$, then x_t has a private neighbour u outside I and D and, since $u \in P(I \cap D, D) \setminus (I \cup D)$, this vertex u does not belong to P and we can extend P with u, which is a contradiction.
If $t \equiv 0 \pmod 4$, then $x_t \in P(x_{t-1}, D)$ and by Lemma 3.3, the vertex x_t has a second neighbour in I besides x_{t-1} that has to lie outside of D. Since P is a longest path, this neighbour has to lie in the set $(I \setminus D) \cap V(P)$. Let $x_\nu \in N(x_t) \cap (I \setminus D) \cap V(P)$ such that ν is maximal. Then, we know that $\nu \equiv 1 \pmod 4$. If we define $y_{i-(\nu-1)} = x_i$ for every $\nu \leq i \leq t$, then $y_1y_2 \cdots y_{t-(\nu-1)}y_1$ is a cycle that satisfies the properties in c).

If $t \equiv 2 \pmod 4$, then x_{t-1} is the only neighbour of x_t in $I \setminus D$, and by Lemma 3.3, x_t has at least one neighbour in $I \cap D$. Since the path P is a longest path, there exists a greatest index $1 \leq \nu < t-1$ such that the vertex $x_\nu \in N(x_t) \cap (I \cap D)$, and we know that $\nu \equiv 3 \pmod 4$. In this case, we define $y_1 = x_{t-1}$, $y_2 = x_t$ and $y_{3+(i-\nu)} = x_i$ for every $\nu \leq i \leq t-2$. Hence, the cycle $y_1 y_2 \ldots y_{t-(\nu-1)} y_1$ satisfies the properties in c). □

With this lemma we are able to prove the following results.

Theorem 3.5 *Let G be a bipartite graph and let D be a subset of $V(G)$. Then the following conditions are equivalent:*

(i) D is the unique IR*-set of G.*

(ii) D is the unique Γ-set of G.

(iii) D is the unique α-set of G.

Proof.
(i) ⇒ (ii) ⇒ (iii) Follows immediately from Theorem 2.3 and Observation 3.1.

(iii) ⇒ (i) Let G be a bipartite graph with partite sets A and B, and let I be the unique α-set of G. Suppose that G has an IR-set $D \neq I$. Let $D_0 = \{x \in D \mid P(x, D) = \{x\}\}$. Since G is Γ-perfect, Lemma 3.4 yields that every vertex $w \in D \setminus I$ has its unique private neighbour in $I \setminus D$ which implies that the set D_0 is a subset of $I \cap D$. We define the four subsets

$$\begin{aligned} A_1 &= (D \setminus I) \cap A, \\ A_2 &= (P(I \cap D, D) \setminus (D \cup I)) \cap A, \\ B_1 &= (I \setminus D) \cap B, \\ B_2 &= ((I \cap D) \setminus D_0) \cap B. \end{aligned}$$

Figure 3 illustrates these sets where $P = P(I \cap D, D) \setminus (D \cup I)$. Note, that the sets A_1 and A_2 are disjoint subsets of $A \setminus I$ and the sets B_1 and B_2 are disjoint subsets of $B \cap I$. Next, we define the set

$$I' = (I \setminus (B_1 \cup B_2)) \cup (A_1 \cup A_2) = (I \setminus B) \cup (D_0 \cap B) \cup (A_1 \cup A_2).$$

Suppose that there exist two adjacent vertices a and b in I'. The union $(I \setminus B) \cup (D_0 \cap B)$ is independent as a subset of I, and the union $(I \setminus B) \cup (A_1 \cup A_2)$ is independent as a subset of the partite set A. Thus, without loss of generality, we deduce that $a \in A_1 \cup A_2$ and $b \in D_0 \cap B$.
If $a \in A_1$, then we obtain the contradiction that $b \in D_0$ is its own private neighbour with regard to D but b is adjacent to $a \in D$.

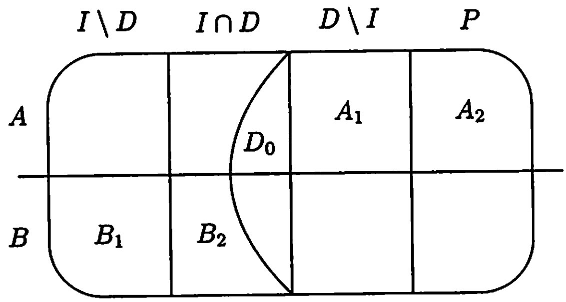


Figure 3

If $a \in A_2$, then a has to be a private neighbour of b, which contradicts that $b \in D_0$.
Hence, the set I' is independent, and by the uniqueness of I, we obtain that $|I'| < |I|$, which is equivalent to $|B_1 \cup B_2| > |A_1 \cup A_2|$.
By Lemma 3.4 a), we get that $|B_1| \leq |A_1|$, since a vertex in B_1 only has neighbours in the partite set A.
Furthermore, every vertex $x \in B_2 = ((I \cap D) \setminus D_0) \cap B$ has a private neighbour p_x in $P(I \cap D, D) \setminus (D \cup I)$. Since x lies in the partite set B, we deduce that $p_x \in A_2$, which implies that $|B_2| \leq |A_2|$. This results in the contradiction $|B_1 \cup B_2| \leq |A_1 \cup A_2|$. □

Corollary 3.6 *Let G be a unicyclic graph and let D be a subset of $V(G)$. Then the following conditions are equivalent:*

(i) D is the unique IR*-set of G.*

(ii) D is the unique Γ-set of G.

(iii) D is the unique α-set of G.

Proof.
(i) ⇒ (ii) ⇒ (iii) Follows immediately from Theorem 2.6 and Observation 3.1.

(iii) ⇒ (i) Let G be a unicyclic graph. If the only cycle in G is even, then G is bipartite and the required result follows from Theorem 3.5. Now, let the only cycle in G be odd and let I be the unique α-set of G. Suppose that G has an IR-set $D \neq I$. Since G - as a unicyclic graph - is Γ-perfect, we obtain by Lemma 3.4 c) the existence of a cycle C in G of even length. This contradiction completes the proof. □

Theorem 3.7 *Let G be a Γ-perfect, claw-free graph and let D be a subset of $V(G)$. Then the following conditions are equivalent:*

(i) D is the unique IR*-set of G.*

(ii) D is the unique Γ-set of G.

(iii) D is the unique α-set of G.

Proof.
(i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) Follows immediately from Theorem 2.2 and Observation 3.1.

(iii) $\Rightarrow$ (i) Let G be a Γ-perfect, claw-free graph and let I be the unique α-set of G. By Lemma 3.3, every vertex in $V(G) \setminus I$ has at least two neighbours in I. Since G is claw-free, every vertex in $V(G) \setminus I$ has exactly two neighbours in I. Suppose that G has an IR-set $D \neq I$. Then, there exists a cycle C in G as in Lemma 3.4 c). Let $C = x_1 x_2 \ldots x_{4p} x_1$ for some positive integer p such that $x_1 \in I \setminus D$ and $x_2 \in D \setminus I$. Furthermore, let

$$\begin{aligned} (I \setminus D)_C &= V(C) \cap (I \setminus D) \\ (D \setminus I)_C &= V(C) \cap (D \setminus I) \\ (I \cap D)_C &= V(C) \cap (I \cap D) \\ P_C &= V(C) \cap (P(I \cap D, D) \setminus (I \cup D)) \end{aligned}$$

Suppose that there is an edge ab in the induced subgraph $G[(D \setminus I)_C \cup P_C]$. Since every vertex in P_C is a private neighbour of a vertex in $I \cap D$, there is no edge between any vertex in P_C and any vertex in $(D \setminus I)_C$. Thus, it yields either $a, b \in (D \setminus I)_C$ or $a, b \in P_C$. If $a, b \in (D \setminus I)_C$, then let, without loss of generality, $a = x_2$ and $b = x_j$ for some $j \equiv 2 \pmod 4$, $2 < j \leq 4p$. Since $N(x_2) \cap I = \{x_1, x_3\}$ and $N(x_j) \cap I = \{x_{j-1}, x_{j+1}\}$, the induced subgraph $G[\{x_1, x_2, x_3, x_j\}]$ is a claw, which is a contradiction. Analogously, if $a, b \in P_C$ and, without loss of generality, $a = x_4$ and $b = x_j$ for some $j \equiv 0 \pmod 4$, $4 < j \leq 4p$, then $N(x_4) \cap I = \{x_3, x_5\}$, $N(x_j) \cap I = \{x_{j-1}, x_{j+1}\}$, and the induced subgraph $G[\{x_3, x_4, x_5, x_j\}]$ is a claw, which is a contradiction. Hence, the set $(D \setminus I)_C \cup P_C$ is independent. Every vertex x in $(D \setminus I)_C \cup P_C$ has two neighbours in $I \cap V(C)$ and hence it has no neighbour in $I \setminus V(C)$. This implies that the set $I' = (I \setminus V(C)) \cup (D \setminus I)_C \cup P_C$ is independent. Since the cardinalities $|I \cap V(C)|$ and $|(D \setminus I)_C \cup P_C|$ are both equal $2p$, we obtain the contradiction that $|I'| = |I|$ and I' is a second α-set of G different from I. □

By Theorem 2.5 and Theorem 3.7, we obtain the following.

Corollary 3.8 *Any graph G that does not contain either $K_{1,3}$, C_4 or the graph H in Figure 1 as an induced subgraph has a unique* IR*-set if and only if it has a unique Γ-set if and only if it has a unique α-set.*

At least we consider chordal graphs.

Theorem 3.9 *Let G be a chordal graph and let D be a subset of $V(G)$. Then the following conditions are equivalent:*

(i) D is the unique IR*-set of G.*

(ii) D is the unique Γ-set of G.

(iii) D is the unique α-set of G.

Proof.
(i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) Follows immediately from Theorem 2.4 and Observation 3.1.

(iii) $\Rightarrow$ (i) Let G be a chordal graph and let I be the unique α-set of G. As mentioned at the end of Section 2, the graph G is Γ-perfect. Suppose that G has an IR-set $D \neq I$. Then, there exists a cycle C in G as described in Lemma 3.4 c). Let $C = x_1x_2 \ldots x_{4p}x_1$ for some positive integer p such that $x_1 \in I \setminus D$ and $x_2 \in D \setminus I$. Furthermore, let C' be a cycle of minimal length in the induced subgraph $G[V(C)]$ that contains the edge x_1x_2. Since the induced subgraph $G[V(C)]$ is chordal, we obtain that $C' = x_1x_2yx_1$ for some vertex $y \in V(C) \setminus \{x_1, x_2\}$. The fact that $x_1 \in I$ leads to $y \in V(C) \setminus I$. Note that $V(C) \setminus I \subseteq (D \setminus I) \cup (P(I \cap D, D) \setminus (I \cup D))$. If $y \in P(I \cap D, D) \setminus (I \cup D)$, then we obtain the contradiction that y is adjacent to the vertex x_2 in $D \setminus I$. Hence, it remains that $y \in D \setminus I$. But in this case the vertex y lies in D and is adjacent to $x_1 \in P(x_2, D)$ which is a contradiction. □

Siemes, Topp, and Volkmann [19] have investigated so called k-independent sets - a generalization of unique α-sets - and they have found characterizations of k-independent sets for several classes of graphs. For $k = 1$ their results contain a further characterization of unique α-sets in chordal graphs (cf. Theorem 4 in [19]).

Theorem 3.10 (Siemes, Topp, and Volkmann [19]) *Let G be a graph in which every even cycle possesses a chord. Then the following statements are equivalent.*

a) D is the unique α-set of G.

b) D is an independent dominating set of G such that every vertex in $V(G) \setminus D$ has at least two neighbours in D.

The following lemma contains a simple characterization of dominating sets as in Condition b) of Theorem 3.10. The proof of this result is trivial.

Lemma 3.11 *Let G be an arbitrary graph. Then the following two conditions are equivalent.*

a) D is a (minimal) dominating set of G such that $P(x, D) = \{x\}$ for every vertex $x \in D$.

b) D is an independent dominating set of G such that every vertex in $V(G) \setminus D$ has at least two neighbours in D.

Theorem 3.9 together with Theorem 3.10 and Lemma 3.11 yields the following.

Corollary 3.12 *Let G be a chordal graph of order at least 3 and let D be a subset of $V(G)$. Then the following conditions are equivalent:*

(i) D is the unique IR*-set of G.*

(ii) D is the unique Γ-set of G.

(iii) D is the unique α-set of G.

(iv) D is a independent dominating set of G such that every vertex in $V(G) \setminus D$ has at least two neighbours in D.

(v) D is a (minimal) dominating set of G such that $P(x, D) = \{x\}$ for every vertex $x \in D$.

Remark 3.13 By Lemma 3.3, Condition (iv) and (v) in Corollary 3.12 are necessary for the uniqueness of α-sets in arbitrary graphs. But they are not necessary for the uniqueness of Γ-sets in arbitrary graphs (cf. the graph $G_{1,t}$ in Figure 2 a)).
Furthermore, in arbitrary graphs Condition (iv) and (v) are not sufficient for the uniqueness of α-sets, Γ-sets or IR-sets, even not if the graph G satisfies $\alpha(G) = \Gamma(G) = \mathrm{IR}(G)$ and an α-set I of G fulfils Condition (iv) and (v). For example consider for some integer $s \geq 3$ the complete bipartite graph $K_{s,s}$ satisfying $\alpha(K_{s,s}) = \Gamma(K_{s,s}) = \mathrm{IR}(K_{s,s}) = s$ and both partite sets are α-, Γ- and IR-sets that fulfil Condition (iv) and (v). Thus, Corollary 3.12 does not even hold for bipartite graphs.

Remark 3.14 There exist polynomial time algorithms to compute α for bipartite graphs and for chordal graphs (cf. Chapter 12.3.4 in [13]).

If we consider an arbitrary unicyclic graph G and an edge xy on its cycle, then it is straightforward to see that $\alpha(G) = \max\{\alpha(G-x), \alpha(G-y)\}$. Since $G - x$ and $G - y$ are trees, we can determine $\alpha(G)$ for unicyclic graphs G in linear time, by using the algorithm of Daykin and Ng [5].

Also for claw-free graphs it is possible to find an α-set in polynomial time (cf. [17], [18]).

In [7] the authors have proved that, if $\mathcal{G}_\alpha$ is a class of graphs such that for every graph $G \in \mathcal{G}_\alpha$ and every vertex $v \in V(G)$ it is possible to determine α for the graphs G and $G - N[v, G]$ in polynomial time, then it can be decided in polynomial time whether a graph in $\mathcal{G}_\alpha$ has a unique maximum independent set.

Since every one of the four graph classes considered in this paper fulfils this condition, we can decide in polynomial time whether any graph in one of these classes has a unique α-set. By our results in Theorem 3.9, Theorem 3.7, Theorem 3.5 and Corollary 3.6, this also yields an efficient algorithm for the decision problem whether a graph in these classes has a unique Γ-set or a unique IR-set.

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