Perfect 1-Factorizations of K_{16} with Nontrivial Automorphism Group

Mariusz Meszka, Alexander Rosa
Department of Mathematics and Statistics,
McMaster University, Hamilton, ON, Canada

Abstract

We establish that up to an isomorphism there are exactly 88 perfect 1-factorizations of K_{16} having nontrivial automorphism group. We also present some related results.

1 Introduction

A *factor* of a graph G is a set of edges of G that partition the vertex-set of G. A 1-*factorization* (OF) of G is a partition of the edge-set of G into 1-factors. For a survey of known results on 1-factorizations of the complete graph K_{2n}, see [13], [21].

A 1-factorization of a graph G is *perfect* if the union of any two of its distinct 1-factors is connected, i.e. is a Hamiltonian cycle (cf. [18]). More generally, a 1-factorization of G is *uniform* if the union of any two of its distinct 1-factors is isomorphic to the same (2-regular) graph Q.

Perfect 1-factorizations of the complete graph K_{2n} have been enumerated for $2n \leq 14$. There is a unique perfect OF of K_{2n} for $2n = 4, 6, 8$, and 10 ([13]), there are exactly 5 nonisomorphic 1-factorizations of K_{12} [16], and there are exactly 23 nonisomorphic 1-factorizations of K_{14} [19], [5].

It appears that there are only two nonisomorphic perfect 1-factorizations of K_{16} known in the literature: the unique perfect 1-rotational OF of K_{16} (also called starter induced, see [1]), and the perfect OF discovered by Kotzig in the early 60's [11] (also called even starter induced, see [1]). The full automorphism groups of these perfect OFs are the cyclic groups of order 15 and 14, respectively.

As an initial step towards the complete enumeration of perfect 1-factorizations of K_{16}, we undertook the computational task of determining all

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nonisomorphic perfect 1-factorizations of K_{16} with nontrivial automorphism group. As shown in this paper, they number 88, and they are all listed in the Appendix. In our task we were greatly assisted by theoretical results of Ihrig, as they apply to the case of 16 vertices [9], [10], [7].

As an aside, we also verify that, apart from the perfect 1-factorizations, the only uniform 1-factorizations of $K_{2n}, 2n \leq 16$, are the known ones. We also determine the spectra of maximal perfect sets of 1-factors, as well as maximal uniform sets.

2 Perfect 1-factorizations

Clearly, the only primes that can divide the order of the automorphism group of a perfect OF of K_{16}, are $2, 3, 5$, and 7. Moreover, it follows from [9] and [10] that if α is an automorphism of a perfect OF of K_{16}, then the cycle structure of α must be one of the following:

(1) $1^1 15^1$ (2) $1^2 14^1$ (3) $1^2 7^2$ (4) $1^1 5^3$ (5) $1^1 3^5$ (6) $1^2 2^7$.

It is well known that there is a unique perfect OF of K_{16} admitting an automorphism with cycle structure of type (1), or of type (2) (cf. [1]). Thus in order to generate new perfect 1-factorizations of K_{16}, it suffices to concentrate on the cases (3) to (6). For each of these cases in turn, we generated perfect OFs by using orderly algorithms as outlined in minute detail in [19] (see also [20]), with only minor modifications, and using canonical forms to generate only nonisomorphic perfect 1-factorizations. It is implicit in [3], [4], [19], [20], and can be traced as far back as to the seminal paper by Miller [12] that there is a polynomial time algorithm for deciding isomorphism of perfect 1-factorizations of complete graphs. To our surprise, we did not find an explicit mention of this, other than in the manuscript [15].

We dealt with each of the cases (3) to (6) independently of each other, and only afterwards compared the canonical forms of the perfect 1-factorizations obtained in each of these four cases. The number of perfect OFs obtained is as follows: the number of nonisomorphic perfect OFs of K_{16} in cases (1) to (6) is $1, 1, 4, 6, 20$, and 60, respectively. The nonisomorphic perfect OFs of type (3) and (6) include the unique perfect OF of type (2), and those of type (4) and (5) include the unique perfect OF of type (1); there are no other "cross-over" types. The order of the full automorphism group of perfect OFs of type (3), (4), (5), and (6) is $7, 5, 3$, and 2, respectively. The total number of perfect 1-factorizations of K_{16} with nontrivial group is thus 88.

In the process, we have also verified the results obtained earlier in [16] and [5] on perfect OFs of K_{12} and K_{14}. It is interesting to compare the time needed to generate perfect OFs for complete graphs with 12, 14, and 16 vertices, respectively. It took less than 1 second to obtain the 5 nonisomorphic perfect OFs of K_{12}, and 471 minutes to obtain the 23 nonisomorphic perfect OFs of K_{14}. The time needed to generate and classify the 88 nonisomorphic perfect OFs of K_{16} with nontrivial group was 1714 minutes.

For an easy distinguishing of the obtained perfect OFs, we calculated for each of these also its *train invariant* (cf. [6]) and its *tricolour invariant* (cf. [8]); both of these are $O(n^3)$ invariants. The train invariant distinguishes completely all 88 obtained perfect OFs while the tricolour invariant fails to distinguish just one pair (No.30 and No.56 - see the Appendix).

3 Related results

3.1 Uniform 1-factorizations

We also performed a complete enumeration of uniform 1-factorizations of K_{2n} other than the perfect ones, for $2n \leq 16$. There are no uniform OFs other than the known ones. In other words, we have:

Theorem 1 *If $\mathcal{F}$ is a uniform 1-factorization of $K_{2n}, 2n \leq 16$, then $\mathcal{F}$ is one of the following:*

(a) a perfect 1-factorization

(b) a uniform 1-factorization of K_8 of type 4+4

(c) a uniform 1-factorization of K_{10} of type 4+6

(d) a uniform 1-factorization of K_{12} of type 6+6

(e) a uniform 1-factorization of K_{16} of type 4+4+4+4.

The 1-factorizations in cases (b), (c), (d), (e) are unique up to an isomorphism and are well-known; c.f., e.g., [2].

3.2 Maximal perfect and uniform sets of 1-factors

Let Q be a 2-regular graph with n vertices. A set $\mathcal{F} = \{\mathcal{F}_\infty, \ldots, \mathcal{F}_f\}$ of (edge-disjoint) 1-factors is *uniform* if the union $F_i \cup F_j \simeq Q$ for all $i, j, i \neq j$. A uniform set $\mathcal{F}$ is *maximal* if there is no 1-factor $F \notin \mathcal{F}$ such that $\mathcal{F} \cup \mathcal{F}$ is also a uniform set.

Let $\mathcal{M}_Q(2n) = \{s$: there exists a maximal uniform set of s 1-factors of $K_{2n}\}$. The set $\mathcal{M}_Q$ is the *spectrum* for maximal uniform sets. When Q

is connected, i.e. when it is a Hamiltonian cycle, the spectrum is denoted by $\mathcal{M}_{perf}(2n)$; otherwise, we represent the 2-regular graph Q simply as a partition of $2n$.

The spectrum $\mathcal{M}_{perf}(2n)$ has been determined for $2n = 4, 6, 8$ in [17] and for $2n = 10, 12$ in [14], [15]. We have determined that $\{7, 8, 9, 10, 11, 12, 13\} \subseteq \mathcal{M}_{perf}(14)$ and that $5 \notin \mathcal{M}_{perf}(14)$. Thus it only remains in doubt whether $6 \in \mathcal{M}_{perf}(14)$. That is, we have established the following:

Theorem 2 $\mathcal{M}_{perf}(14) = \{7, 8, 9, 10, 11, 12, 13\} \cup I$ *where either* $I = \emptyset$ *or* $I = \{6\}$.

We have also established the following results on maximal uniform sets:

$\mathcal{M}_{4+4}(8) = \{7\}, \mathcal{M}_{6+4}(10) = \{3, 9\}$,
$\mathcal{M}_{4+4+4}(12) = \{3\}, \mathcal{M}_{6+6}(12) = \{3, 5, 11\}$,
$\mathcal{M}_{8+4}(12) = \{6, 9\}, \mathcal{M}_{6+4+4}(14) = \{3, 5, 7\}$,
$\mathcal{M}_{8+6}(14) = \{5, 6, 7\}, \mathcal{M}_{10+4}(14) = \{5, 6, 7, 8\}$,
$\mathcal{M}_{4+4+4+4}(16) = \{7, 15\}, \mathcal{M}_{6+6+4}(16) = \{3, 4, 5, 7\}$,
$\mathcal{M}_{8+4+4}(16) = \{3, 4, 5, 7\}, \mathcal{M}_{8+8}(16) \supseteq \{5, 6, 7, 8, 9, 10\}$.

3.3 Cubic graphs and 4-regular graphs in perfect 1-factorizations

The union of any three [four] 1-factors in a perfect 1-factorization of K_{2n} yields a perfect OF of a cubic [4-regular] graph. Conversely, one may ask whether a perfect OF of a cubic [4-regular] graph on $2n$ vertices can be extended, by adding further 1-factors, to a perfect 1-factorization of K_{2n}. This is not always possible. The following table shows the number of completable (C3, C4) and non-completable (NC3, NC4) perfect 1-factorizations of $2n$-vertex cubic [4-regular] graphs.

Table.

$2n$	C3	NC3	C4	NC4
8	2	0	2	0
10	5	5	9	40
12	30	2	380	1735
14	247	2	4691	208817
16	≥ 2209	≤ 7	≥ 51266	≤ 30093734

3 Conclusion

From the complete enumeration of perfect 1-factorizations of K_{12} and of K_{14} in [16] and [5], the total number of perfect 1-factorizations of K_{12} and of K_{14} with trivial automorphism group is 1 and 2, respectively. In spite of our considerable effort, we were so far unable to obtain an example of a perfect OF of K_{16} with the trivial group. Thus the question of the existence of such a perfect OF remains open.

Acknowledgements

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References

[1] B. A. Anderson, Some perfect 1-factorizations, *Proc. 7th S.-E. Conf. Combinatorics, Graph Theory and Computing, Congr. Numer.* 17 (1976), 79–91.

[2] P. J. Cameron, *Parallelisms of Complete Designs, London Math. Soc. Lect. Note Ser.* **23** (Cambr. Univ. Press, 1976).

[3] C. J. Colbourn, M. J. Colbourn, Combinatorial isomorphism problems involving 1-factorizations, *Ars Combinat.* **9** (1980), 191–200.

[4] M. J. Colbourn, C. J. Colbourn, 1-factorizations of simple graphs, *manuscript.*

[5] J. H. Dinitz, D. K. Garnick, There are 23 nonisomorphic perfect one-factorizations of K_{14}, *J. Combin. Designs* **4** (1996), 1–4.

[6] J. H. Dinitz, W. D. Wallis, Trains: an invariant for one-factorizations, *Ars Combin.* **32** (1991), 161–180.

[7] D. Duncan, E. Ihrig, Automorphism groups of 1-factorizations, *Congr. Numer.* **94** (1993), 89–97.

[8] T. S. Griggs, A. Rosa, An invariant for 1-factorizations of the complete graph, *Ars Combin.* **42** (1996), 77–88.

[9] E. Ihrig, Symmetry groups related to the construction of perfect one-factorizations of K_{2n}, *J. Combin. Theory* (B) **40** (1986), 121–151.

[10] E. C. Ihrig, The structure of the symmetry groups of perfect 1-factorizations of K_{2n}, *J. Combin. Theory* (B) **47** (1989), 307–329.

[11] A. Kotzig, Hamilton graphs and Hamilton circuits, in *Theory of Graphs and its Applications, Proc. Sympos. Smolenice* 1963 (Naklad. CSAV, 1964), 63–82.

[12] G. Miller, On the n^{logn} isomorphism technique, in *Proc. Tenth Annual ACM Sympos. Theory Comput., San Diego, CA, May* 1-3, 1978 (1978), 51–58.

[13] E. Mendelsohn, A. Rosa, One-factorizations of the complete graph – A survey, *J. Graph Theory* **9** (1985), 43–65.

[14] A. J. Petrenjuk, O spektre maksimal'nykh soveršennykh semeistv 1-faktorov v polnykh grafakh malykh poriadkov, in *Optimizacia i yeyo priloženia* (Kiev, 1977), 60–68.

[15] A. J. Petrenjuk, Perečeň soveršennykh 1-faktorizacii poradka 12 ranga 4, in *Gos. Letnaya Akad. Ukrainy, Kirovograd* 1997, 21 pp.

[16] A. P. Petrenjuk, A. J. Petrenjuk, Intersection of perfect 1-factorizations of complete graphs, *Kibernetika* (Kiev) **16** (1980), 6–9.

[17] A. Rosa, Maximal partial designs and configurations *Le Matematiche* (Catania) **45** (1990), 149–162.

[18] E. Seah, Perfect one-factorizations of the complete graph – a survey, *Bull. Inst. Combin. Appl.* **1** (1991), 59–70.

[19] E. Seah, D. R. Stinson, Some perfect one-factorizations of K_{14}, *Ann. Discrete Math.* **34** (1987), 419-436.

[20] E. Seah, D. R. Stinson, On the enumeration of one-factorizations of complete graphs containing prescribed automorphism groups, *Math. Comput.* **50** (1988), 607–618.

[21] W. D. Wallis, One-factorizations of the complete graphs, in *Contemporary Design Theory: A Collection of Surveys*, ed. J. H. Dinitz, D. R. Stinson (J. Wiley & Sons, 1992), 593–639.

Appendix 1

We list here the 88 nonisomorphic perfect 1-factorizations of K_{16} with nontrivial automorphism group. The first two 1-factors form in each case the hamiltonian cycle 0123456789ABCDEF, and are therefore omitted. The next six pairs of 1-factors are represented by corresponding hamiltonian cycles; the sequence for the last 1-factor is a list of neighbors of consecutive vertices.

1. 027C41DF9EB6A583, 04E31628ACFB7D95, 063F1A49B8D52CE7, 08C1EA3B5F2D6479, 0A7F684D3C915E2B, 0C53718E692AF4BD, EB4927C5F3D16A08.
2. 02C9DFA71485BE63, 04AE829FB16C73D5, 06A193E4B8CF52D7, 08F6D47B3C1E2A59, 0A38D15EC4697F2B, 0CA81F357E9426BD, E371FC826BD95A04.
3. 025AD941EB7CF863, 04B192CEA83D6F75, 06951C4AFB8D2E37, 082AC3B5E64F1D79, 0A74E9C8135DF26B, 0C6A17E85F39B24D, E67A8C124F3D5B09.
4. 02CA7B5841FD9E63, 0462BFC18D793EA5, 06A153F9BD2E8C47, 086F24AD173BEC59, 0AF4D5E13827C96B, 0C3A294E75F8B16D, E95DB2CFA1846307.
5. 0295EA8D7BFC1463, 04FA2837C61BE9D5, 068F3AD1EC9B5247, 084DB3E751AC2F69, 0A6E85C49317FD2B, 0C35A79F18B4E26D, E97DAFB2C1468305.
6. 0285BF9D7ACE4163, 04F7E1938DA6B2C5, 06DF5942EB3A18C7, 0846EA2FC35DB179, 0A475E3D269C1F8B, 0C6F315A8E927B4D, ED57C2836BF9410A.
7. 02F8EB147C59DA63, 04A869EC271FD3B5, 06E48D25A1C39BF7, 0853EACF4B7D1629, 0A28C49731E5F6DB, 0C9FA38157E2B64D, E94F2DCAB1786503.
8. 029E58AF14C7DB63, 04A6D8EB91CF2735, 06CE3AD952B184F7, 083C2A7B4E615DF9, 0AC57E1D49628F3B, 0C9317468BF5AE2D, EA4D2BF9C7158306.
9. 02B758DFAE9C4163, 047C1E68A3BF92D5, 06B935A2FC81D4E7, 08B5172EC3DA6F49, 0A48E31F7952C6DB, 0C5EB1A73F82469D, E968BF2D31C4A705.
10. 02C7DBE958FA1463, 04FB3EC1796A8D25, 06E42B5AD19C83F7, 084C53D62F1EA7B9, 0ACF5D492E73186B, 0C61B475E82A39FD, E57CA1F2BD483906.
11. 02BDFA85E9417C63, 04F6A2E1C83D79B5, 069153FCA4BE8D27, 08B7E316DA5C42F9, 0A3C926E481D57FB, 0CEA739528F1B64D, EACB7F846D132905.
12. 028FAD4159BE7C63, 047DF96B2C38E1A5, 06AE94FC531B8D27, 084AC162F5E3DB79, 0A7193F6D58CE24B, 0C4E681FB573A29D, ED5B624FAC839107.
13. 02CE58147AFBD963, 049CF16B8EAD2735, 06F95D1B4E382AC7, 08FD46C5A31E7B29, 0A86E251C4F7D39B, 0C3F5BE971A6248D, E9FBA7D5C1438602.

14. 02859BD7CE41AF63, 04B2C6E8DA3971F5, 0691B38AE25DFC47, 08BF7E164A2D35C9, 0A726D49F81C3E5B, 0CA6842F3157BE9D, ED97FAB3C2568104.
15. 0259DF7B8E41AC63, 04FC182973D6BEA5, 06E1DA85FB39C247, 084D2F6A317CE5B9, 0A4627E91F83C5DB, 0C861B49FA2E357D, E5BFC19AD6724803.
16. 0241E8597BDFAC63, 047A28D13CE69BF5, 062B4F1839C5EAD7, 08A4D2F6B351C7E9, 0A3E4C27F8619D5B, 0C846A571BE29F3D, EA5792D3B418F60C.
17. 02BFD9CE58A74163, 04CA319F86E27BD5, 06A5B1E493F2C8D7, 0826C3DA17EB4F59, 0AE9642D15CF738B, 0C79B6F18E352A4D, EC9B87D542F3160A.
18. 0259C8EBF714DA63, 049E6FC283D1B7A5, 06D5C3B2EA84F197, 08F274CE53A16BD9, 0A42DF581E37C69B, 0CA29F315B864E7D, EC69B725D3F4180A.
19. 02C8FD957A14BE63, 04E3F68A2B91D7C5, 06A429C3D5E81FB7, 08BDA352E7461CF9, 0AC4D8371E962F5B, 0CEA5827F4931B6D, E5DB81C947F3620A.
20. 0241C7E9DB58FA63, 04BE8D6C93F17A25, 06E5A4CFB3D18297, 0831EADF5C2B7469, 0A8CE372F4D5916B, 0C3A157F68B94E2D, EB65832D4FC1A709.
21. 02F9BEC7AD584163, 042BD9C68AF1E375, 064F5C39EA2D81B7, 083ACF71D4E6B529, 0A51C26947E8FD3B, 0C4A6FB82E53197D, EA7FB9D2C5148603.
22. 0295F8CE14B7AD63, 04F72B16C9D3AE85, 06FA53B9E481D2C7, 08D425BE6AC1F379, 0A47E3826915CFDB, 0C46B8A139F2E57D, E7AC9D81642F350B.
23. 02B71495FDACE863, 046C19F827AEBD35, 06A1D4C92FB85E37, 08D7CF4E2A5B6139, 0A481F3C52D6E79B, 0C26FA83B4751E9D, EB4A2D9FC6318507.
24. 02DBF97C58EA4163, 04F72A81D6E9C3B5, 06C8B924D53FA1E7, 082BECF647DA3159, 0AC1F2693D84E57B, 0C2E386B4917A5FD, EB57C2A3FD614908.
25. 0259CF7E8BDA1463, 049B2E37C6AF81D5, 06F93A2C85E1B4D7, 08D27ACE4F3B5169, 0A831C57426E9DFB, 0C35F179284AEB6D, E9FDCA8B61574302.
26. 0241EC597AFB8D63, 04C91FD37EA86B25, 06F49D2CA3E815B7, 083FC64DB172A5E9, 0A47D1C8F53926EB, 0C7F96A13B4E285D, E6FC87154BD93A02.
27. 02DB4179C5FA8E63, 042E968FC1B7AD35, 06A49FD51E3B28C7, 085EB6C4F372A1D9, 0AE7F184D62C395B, 0CE475A316F29B8D, E958624D31CFA70B.
28. 028514BEC9DF7A63, 0479E3AD268CF1B5, 06FB938D1A4C2E57, 0842F3C7DB61EA59, 0A8F94E6D371C52B, 0CA27E8B31964F5D, E895D3CB12F7640A.
29. 02B7F9CE85AD1463, 04961AE72CFD8B35, 062FA483C5E19BD7, 08C7A6D24F315BE9, 0AC1F86E3D92574B, 0C4E28A3971B6F5D, E8A7D9C3152F640B.
30. 02EB8514DF7AC963, 04C79BF83E1D62A5, 06B391CFA8E5D247, 086E4AD72B5C31F9, 0A3D9EC6F257184B, 0C8294F537EA61BD, EACF694BD5172803.

31. 02BE4158C97ADF63, 04C1A83D269E7BF5, 06CFA53B92E18D47, 0846E3AC725BD1F9, 0AEC5D73F249168B, 0C2A6B4F8E57139D, EB8CA9DF25413607.
32. 02C9FD85A7BE4163, 048C1B697EAF3D25, 062B8AC5F49D1E37, 08E2F6A3517C4DB9, 0AD746CE5928F13B, 0CF75B42A1839E6D, E97CAD82614F350B.
33. 02FD9CA7EB415863, 0426D8EAF1C39B75, 06A4E5C8B29F31D7, 08A16B3D2EC74F59, 0A25381E97F6C4DB, 0CF8273E6491B5AD, E7CA8D91463F250B.
34. 02EB4158C9FD7A63, 04D1E862AF3C7B95, 06F42D8B193ECA57, 082C13BD5F7EA469, 0A8F1735C4E6D92B, 0C6BF25E97481A3D, E6789B1234D5FA0C.
35. 02A7C9DF8514EB63, 04AE1962B83D7FC5, 06E81C2DA5BF3947, 086C4F5DB1A372E9, 0AF61D842975EC3B, 0CA8259F13E7B46D, E7F5D3A1CB698402.
36. 02DF85C9BE7A4163, 046BD8E193ACF725, 06CE492AFB8135D7, 0824D3C75B1AE6F9, 0A869D173E5F2C4B, 0C8479E2B3F15A6D, EC68F92B35D71A04.
37. 029C7AFD4158EB63, 04E3B8F9627D1AC5, 06A5B94C2E18D3F7, 083AD24FC61B7E59, 0A8CE931746D52FB, 0C35791F6EA482BD, ECA7BF836D241905.
38. 02417AEB9C58DF63, 04AC827E9D6B31F5, 06E59381AD4CF2B7, 086A5D74FB1C3E29, 0A8E1D3F752C694B, 0CE4615B8F973A2D, E965832C41FD7B0A.
39. 02A7EB8514C9FD63, 04D8C719B3AF26E5, 0682CAD594E31FB7, 083D75CF6A42B1E9, 0AE84793F52D1C6B, 0CE2735B4F8A169D, E89C6A4F125D3B07.
40. 02C97AFD58EB4163, 047D8AC39F1B6E25, 069BD15CF283A4E7, 08FB5A17CE3D2649, 0A29D4C6F735E18B, 0C84F572B319EA6D, EC4F298B65D71A03.
41. 02BE9CA74158DF63, 04B92683AD7EC1F5, 06A1E8BD942F3C57, 08F4C731D2AE6B59, 0AF72E5D48C6193B, 0C253E4A817BF96D, EB8D6A492751F30C.
42. 02A79CFD8514EB63, 048A62FB3D917EC5, 06EADB594C281F37, 08C6F4B1A35D72E9, 0A4692D1C38E5F7B, 0CA5742B8F93E16D, E351D28C6BF9740A.
43. 0285147ADF9CEB63, 04DB17FA38E926C5, 06F8B53E2C1A49D7, 08D164B2F5EAC739, 0A257E1F48C3D69B, 0C4E68A5913FB72D, E84B2DA91763F50C.
44. 02FDBE7A8514C963, 04973B1F8D62ECA5, 06C25E48B9D13AF7, 0824B57C16EAD3F9, 0A68C3E9174F5D2B, 0C5938E1A27BF64D, E895A3BD1246F70C.
45. 029CFDA758BE4163, 04C7BF9E3D681A25, 06C8E5FA3B19D427, 082CAE137D5B6F49, 0A846E71D2F3C59B, 0CE2B4A697F1538D, EC697A24F35D1B08.
46. 0258A7149CFDBE63, 0482FB3ACE1697D5, 06A5F1D29B4C8E37, 08D4F62EA1B7C359, 0AD9F315C27E468B, 0C6B24AF75E9183D, ECA97BD4F3251608.
47. 028514C9EB7AFD63, 04F917DA3B8C26E5, 06A294D53FB18EC7, 08FCA16472E3DB59, 0A4E7F25C6831D9B, 0C3975AE1F6B248D, ECD7BF93A6841205.

48. 0258FDA7149CEB63, 0482E97FBD3A61C5, 06D159FC8AE42B37, 08E5AF31B726C4D9, 0A4F18D7C296E35B, 0C39B8647E1A2F5D, E9D8B7F531C4A206.
49. 027AEB4158DF9C63, 049DAF183C26E7B5, 06D5A3E91B2FC847, 08FBD4A2E537C169, 0A64CE1D397F528B, 0CA8E4F6B317592D, EA4F2C8D6B195703.
50. 02C9DFA74158BE63, 04B183AEC79F26D5, 06A849E53CF1DB27, 08FB371C64E2DA59, 0A4C82961E3D7F5B, 0C52A139B7E86F4D, E94F27B5D1C6A803.
51. 02EB8514DF7AC963, 048FC7B9E3D162A5, 068CE425DAFB1937, 08A4C2BD6E7531F9, 0AE59746C18D2F3B, 0C5B4F6A38E1729D, EA8C9FBD24163705.
52. 0258FDA714EB9C63, 047FAC3D9E2816B5, 06D1AE835CF94B27, 08BF46A31CE57D29, 0A24C7951E3F68DB, 0C26E7B391F5A84D, EBF7AD93C6418502.
53. 02BE4158FD9C7A63, 0462EAD3C18B7F95, 069E8C53FA4B1D27, 086BD74FC2A5E139, 0A38D6EC425791FB, 0CA8492F617E3B5D, EA87DFC32B196405.
54. 02EBDF58A7C94163, 046F79D1A3B8EC25, 06AE4CF835B192D7, 08472AD3FB6C1E59, 0A5C3718D6E9F42B, 0C8269B4AF13E75D, E5F9D18B63C7A402.
55. 02A7149CEBFD5863, 04FC1928E6BDA375, 069E3FA15C2B48D7, 083C4D1B5AE726F9, 0A6C81F5E2479D3B, 0CA8B7F25931E46D, E6D5A31CFB497208.
56. 029CEB857AFD4163, 04E3F9BD1A68C725, 06CA4F183D95E2B7, 08D6EA5BF2471C39, 0AD5FC28E731964B, 0C53B1E948A26F7D, E5DAC1B9F7364208.
57. 028514A7C9DFBE63, 0493EAFC268B71D5, 069F2EC8D4B1A357, 08F13C46BD7E5A29, 0AD6C524F791E83B, 0CA816F374E95B2D, EC7D8FA24B691305.
58. 029C85BEA7FD4163, 046C7BD9F83E1A25, 068B91FA35D2CE47, 08AC3D6E751B2F49, 0A5E97D826F31C4B, 0CF596B3718E24AD, ED798CA2436F510B.
59. 029C4158BEA7DF63, 0462C7B39DA1E8F5, 06C835E91B2D4AF7, 08475BD6A2E3C1F9, 0ACE731D59682F4B, 0CF3A524E6B9718D, E67D9C12A48F530B.
60. 02BEA7C9FD584163, 0473B8D91CFA6E25, 06FB95381DA2C4E7, 08CA1B5F7246D3E9, 0A5EC68F4D29317B, 0C579628E1F3A4BD, E5FC91BDA4863702.
61. 029C4158FDA7EB63, 04A1F79E3B8D26C5, 06D48A5B2F931EC7, 0837DB42E5FAC169, 0A6F3C2574E81D9B, 0C86EA271BF4953D, E98A6D4B2137F50C.
62. 0285FDA7BEC94163, 04C83AE971F6BD25, 062F9D5C3B8A1E47, 08E6AFB4D351C729, 0A4695E318D7FC2B, 0CA5B1937E24F86D, EDAF87C54B296103.
63. 02BE7AFD4158C963, 048D1CE3F97B62A5, 068A4FC5E291BD37, 08BF75D6C24E1A39, 0A6E94C7D28F135B, 0C3B952F64718EAD, E678BF123DC4A905.
64. 029CFD58EB7A4163, 04FB938C6EA17D25, 06D9E13F28AC5B47, 08B642AD1F5EC379, 0A35196F7C2E48DB, 0C49F81B26A57E3D, EC7BD98265F3140A.

65. 02BEA7FD4158C963, 048FC1B79D3E62A5, 06FB918AD5E42C37, 08B4AC6D135F27E9, 0AF164938D2EC75B, 0C529F471E86A3BD, EA8FC9BD25164703.

66. 02A7C9FD85EB4163, 04C391AF7E8B6D25, 0692E13BF84AC5D7, 08CF4273E6A51DB9, 0AEC18264D3F597B, 0C2B53A86F174E9D, EBF897C534D16A02.

67. 02FD9C7A4158EB63, 04268B93AE7D1FC5, 06CAD813E2BF4957, 084D27B53F6A1CE9, 0A5F73C8291E46DB, 0C4B1796E52AF83D, E6CB7D14AF832509.

68. 02A74158C9EBDF63, 04EC38B71926DAF5, 06B4F9D31A5E82C7, 08AE24D73FC5B169, 0AC6E3527948D1FB, 0C46A39B2F81E75D, ECDBA98F65431207.

69. 029CEB85147AFD63, 04839FBDA17C26E5, 069E28A4B1FC5D37, 08E4D1C3A52F6B79, 0A61E35F8C49D72B, 0CAE7F3B5918642D, E3A1F7C5DB296804.

70. 029CEB85147ADF63, 04C79E62A1BD83F5, 06A819D5C24FB3E7, 086B4D2731FCAE59, 0A3C1E48F257D69B, 0C8E2B5AF716493D, ED85A3CB2F476109.

71. 02C94158FDA7BE63, 04A3B1E9268CF7D5, 06BFA1D28E4C3957, 08B4D35CA617E2F9, 0A527CE38D64F19B, 0C6974813F5EA2BD, EC472BF3AD851906.

72. 028514EBFDA79C63, 049E3D17FC26A8B5, 064D95E2B18FA3C7, 08D61CEA4B3572F9, 0AC425F19386E7DB, 0C8E1374F6B92A5D, EADF8C9B46175203.

73. 029CA74158DFBE63, 04FC83E71A269BD5, 06D4913FA5CE28B7, 08A37C1D24E5B6F9, 0A6C481E97D35F2B, 0C27593B461F8EAD, EB5CA28F6D413907.

74. 029C8514BE7AFD63, 04268DB7C3F91AE5, 064FCE2A318B9D57, 08AC1D2735BF6E49, 0A5F793D48E16C2B, 0C528F1B3E96A47D, E7F8C9B135D64A02.

75. 02DFBE85149C7A63, 047DB862A3E1CF95, 06FACE2B19384D57, 08D1F5A4E6C273B9, 0AD3C42F81697E5B, 0C8AE925317F4B6D, EA8F6C4B2D175903.

76. 0285EB9C7AFD4163, 04713ECAD69F8B25, 06C15A83BD9E24F7, 084CF3D1EA26B579, 0A37E468C5D291FB, 0C3594A1B72F6E8D, E8C9BFAD13642705.

77. 02A7EB85DFC94163, 047C3AD8FB91E625, 06D29ECA8351B4F7, 0846F31AE5C2BD79, 0AF573E824C1D96B, 0C86A5B39F172E4D, E8FDA9CB15476302.

78. 029CA7DFBE584163, 04CF1AD68E739B25, 069EC1D8B3A42F57, 08C713E2DB5A6F49, 0AE195D46C283F7B, 0C351B6E472A8F9D, E86DBC2917F4530A.

79. 02C9BE7A4158FD63, 0462B18D937FACE5, 06AE4835C19FBD27, 08A247CF3D1E6B59, 0AD5796C4F13E28B, 0C3B71A52F68E94D, E69ABF1DC2348705.

80. 027AEB4158C9FD63, 04817BFAC39D26E5, 06CF4A5DB928E137, 08D75B24CE3F1A69, 0AD4E7C1952F683B, 0C5F8B16479E2A3D, EDC593BFA4862107.

81. 029CFD4158A7BE63, 04BF3E97C81A62D5, 064F95E1C3ADB827, 0835CEAF6D71B249, 0AC2F574E8D9316B, 0C684A52E7F19B3D, EDA7CB93F6254108.

82. 02DFBE7A85149C63, 04A3CF8D1E6297B5, 06A139D52EC8B4F7, 08EADB1642F37C59, 0A5384E9F1C27D6B, 0C471869B2AF5E3D, E98BD7F521C3A406.

83. 029C4158FD7AEB63, 0462E81AD9B37FC5, 069E1D4835BFAC27, 082D6C1B47E3A5F9, 0A8D5E493C716F2B, 0CE68B7952A4F13D, E94F27A5C16D8B03.

84. 02BE4158C97ADF63, 048A1C7BD39E62F5, 06CEA4F18B92D537, 083FB5E164CA27D9, 0A6D194752CF8E3B, 0C59682E7FA31B4D, E74C2AB1DF563809.

85. 02C97AFD8514BE63, 049F1B62A83CE7D5, 06DB2F5C8E913A47, 08F6425EAD1C7B39, 0AC4F753E2D9618B, 0C68273FB95A1E4D, E79D8BA14265F30C.

86. 027A8514DFC9EB63, 04FB816297DAE3C5, 069DB4A13F52E8C7, 08F6A5EC42B71D39, 0AC1F7594E6D283B, 0C6847E1B9F2A35D, E9C76B43D1F5280A.

87. 02A7BE8514C9DF63, 047D268C19B3EAF5, 069F38D4B2CA1E57, 08B1F4A5D6C37E29, 0AD16EC72F84935B, 0C52813A64E97FBD, E74D29B1A586F30C.

88. 02DF8514EB7AC963, 047FB926E1D3C8A5, 064B83A1FCE25D97, 082F53BD61C7EA49, 0AD86F4C5E39172B, 0C6B18E9573FA24D, E3C18BAD4F652709.

Appendix 2

We list here invariants for the 88 perfect 1-factorizations of K_{16}.

	train invariants	tricolour invariants
1.	540,810,360,90	135,150,105,60,5
2.	602,736,322,140	140,154,91,56,14
3.	581,778,329,84,28	112,189,98,49,7
4.	616,743,315,91,21,14	91,210,112,42
5.	602,757,308,105,28	98,189,133,35
6.	620,720,335,90,35	120,165,120,45,5
7.	565,785,355,80,10,5	100,195,120,35,5
8.	595,790,275,100,40	100,195,115,45
9.	535,850,305,105,0,5	95,195,130,35
10.	590,780,285,130,15	105,195,105,45,5
11.	525,855,324,90,3,3	107,186,120,34,8
12.	606,741,333,93,21,6	105,201,100,38,10,1

13. 606,735,333,108,15,3	113,175,126,31,10
14. 654,684,327,93,27,15	117,178,111,36,13
15. 600,768,291,117,21,3	118,190,81,56,10
16. 564,792,336,96,12	133,152,109,54,7
17. 576,786,318,105,12,3	138,143,121,37,16
18. 588,762,339,87,21,3	128,156,126,29,15,1
19. 576,765,348,105,6	104,201,100,41,9
20. 579,777,312,129,3	119,168,115,50,3
21. 582,759,345,105,9	137,138,124,50,6
22. 570,798,315,96,21	132,150,109,64
23. 597,744,336,111,9,3	130,164,99,50,12
24. 585,777,318,96,21,3	112,181,115,40,6,1
25. 612,738,309,123,15,3	134,144,115,62
26. 588,756,351,81,21,3	108,185,114,45,3
27. 585,771,324,102,15,3	109,185,111,47,3
28. 594,744,345,102,15	103,192,118,36,6
29. 651,687,321,96,42,3	109,174,130,42
30. 604,752,312,108,20,4	108,178,129,36,4
31. 556,830,298,98,10,8	116,164,131,42,2
32. 616,728,334,94,20,6,2	116,182,97,56,4
33. 592,766,318,102,18,4	100,198,113,40,4
34. 570,794,324,92,18,2	90,204,131,26,4
35. 572,768,362,86,10,2	124,156,125,46,4
36. 602,762,296,120,14,6	110,182,115,44,4
37. 572,796,314,98,18,2	118,164,125,46,2
38. 566,780,354,88,12	104,190,119,36,6
39. 566,796,330,90,16,2	118,170,119,42,4,2
40. 630,702,338,100,28,2	104,178,137,36
41. 590,782,294,110,22,0,2	82,218,123,32

42. 596,768,302,108,26	108,186,117,36,8
43. 578,784,326,88,20,4	120,162,125,44,4
44. 556,802,344,86,8,4	98,196,119,42
45. 600,758,304,118,20	98,212,91,50,4
46. 612,742,310,108,26,2	98,196,123,34,4
47. 606,732,344,94,22,2	104,186,123,40,2
48. 604,722,368,86,16,4	116,176,109,50,4
49. 506,884,324,78,6,2	132,154,107,56,6
50. 536,834,336,82,12	116,170,119,48,2
51. 562,806,334,70,24,4	122,156,131,42,4
52. 600,760,318,88,30,4	114,190,89,56,6
53. 592,776,302,104,22,4	106,188,115,42,4
54. 604,756,300,118,20,2	110,180,121,38,6
55. 592,752,344,94,12,6	122,158,131,36,8
56. 598,748,334,102,12,6	108,178,129,36,4
57. 590,752,354,82,18,2,2	122,162,123,40,8
58. 558,806,330,90,16	126,158,119,46,4,2
59. 586,766,336,90,18,4	100,188,135,26,6
60. 592,756,324,116,12	104,198,105,40,8
61. 598,730,362,94,16	110,172,137,32,2,2
62. 568,792,334,88,14,4	106,180,131,34,4
63. 550,814,326,106,4	98,206,107,36,8
64. 540,838,310,106,6	130,162,105,44,14
65. 552,812,340,78,16,2	124,166,113,40,12
66. 602,748,322,106,20,2	110,180,125,30,10
67. 582,770,334,96,16,2	118,164,129,38,6
68. 554,792,362,84,8	108,174,135,36,2
69. 562,828,286,98,24,2	144,136,113,50,12
70. 554,794,362,78,12	98,198,115,44

71.	544,824,336,84,8,4	106,186,121,36,6
72.	576,788,310,112,14	114,174,121,40,6
73.	596,734,358,100,10,2	96,198,125,32,4
74.	580,768,348,82,20,2	108,194,103,40,10
75.	556,816,316,96,16	108,172,143,26,6
76.	586,766,322,114,12	94,190,147,20,4
77.	542,834,326,78,20	112,180,115,42,6
78.	554,834,292,102,14,4	114,180,109,46,6
79.	586,772,318,106,16,2	118,164,131,34,8
80.	578,776,326,108,12	120,160,133,34,8
81.	598,748,332,102,18,2	100,212,89,46,8
82.	590,766,320,108,10,6	118,160,133,42,2
83.	600,746,336,96,18,2,2	104,184,133,26,8
84.	582,778,322,96,20,2	108,192,103,46,6
85.	602,742,334,102,16,4	104,198,105,42,4,2
86.	602,730,356,90,22	98,192,129,34,2
87.	586,774,308,118,14	108,188,109,46,4
88.	602,740,332,110,14,2	110,176,127,38,4