Optimal fast solutions to the gossip problem

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Abstract

We present a new proof that the optimal fast solutions to the gossip problem, for an even number of participants $n > 2^{\lceil \log_2 n \rceil} - 2^{\lfloor \lceil \log_2 n \rceil / 2 \rfloor}$, require exactly $\frac{n}{2} \lceil \log_2 n \rceil$ calls.

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The *gossip problem* can be stated as follows: there are n persons, each knowing a piece of information not known by the others. They communicate by telephone (conference calls are not allowed), every call takes a time unit, each person can participate in only one call at a time, anybody can speak to anybody else and in each call the two parties exchange all the information they know at that time. The task is to minimize the number of calls (*short* solutions) or the time (*fast* solutions) needed to spread all of the information to everyone.

A *round* of calls is defined to be any set consisting of exactly all the simultaneously performed calls, hence the total time equals the number of rounds.

The *inverse* of a solution S, i.e. the sequence of calls obtained by reversing the order of the rounds of S, is itself a solution.

The number of calls required by the short solutions to the gossip problem (for a number of persons $n \geq 4$) is exactly $2n - 4$ [1].

The time required for the fast solutions is exactly $\lceil \log_2 n \rceil$ for even n and $\lceil \log_2 n \rceil + 1$ for odd n (Knödel's solution, which for even n requires exactly $\frac{n}{2} \lceil \log_2 n \rceil$ calls) [3].

The solutions which are both short and fast at the same time are called *ideal* solutions. They exist iff $n \leq 9$ or $n = 11, 13, 15$ [5].

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The fast solutions which minimize the number of calls are called *optimal* fast solutions. They have been determined for some values of n only: optimal solutions, in $2n-3$ calls, are known for $n=10$ [4], $n=12$ [8], $n=17,19,21$ and 23 [7]. Moreover, the number of calls for the optimal fast solutions is $\Theta(n\log_2 n)$ [2].

In [4, Theorem 3], Labahn states that, for even

$$n > 2^{\lceil \log_2 n \rceil} - 2^{\lceil \lceil \log_2 n \rceil /2 \rceil}, \tag{1}$$

the number of calls required by the optimal fast solutions to the gossip problem is exactly $\frac{n}{2}\lceil \log_2 n \rceil$, but this does not always occur. A counterexample is provided by $n=6$, which satisfies condition (1) and for which $\frac{n}{2}\lceil \log_2 n \rceil = 9$, whereas ideal solutions in $2n-4=8$ calls exist [4, 5].

In fact, the proof of [4, Theorem 3] contains an error. In it, Labahn proves that all the persons must take part in all rounds from the first to the $\lfloor \lceil \log_2 n \rceil /2 \rfloor$-th and claims that, by considering the inverse solution, the same also holds for the remaining rounds. However, the latter is incorrect; by applying the procedure to the inverse solution, indeed, we only deduce that all the persons must take part also in all rounds from the $\lceil \lceil \log_2 n \rceil /2 \rceil + 1$-st to the last and hence, if $\lceil \log_2 n \rceil$ is odd (as for $n=6$), this provides that all the persons must participate in all rounds from the first to the $\lfloor \lceil \log_2 n \rceil /2 \rfloor$-th and from the $\lceil \lceil \log_2 n \rceil /2 \rceil + 1$-st to the last, but not that all the persons must participate in the $\lfloor \lceil \log_2 n \rceil /2 \rfloor + 1 = \lceil \lceil \log_2 n \rceil /2 \rceil$-th round (in the case $n=6$, the second round).

On the other hand, the proof of [4, Theorem 3], with a slight modification [6], can be adapted to the following

Theorem 1 *For even $n > 2^{\lceil \log_2 n \rceil} - 2^{\lfloor \lceil \log_2 n \rceil /2 \rfloor}$, the number of calls $C(n)$ required by the optimal fast solutions to the gossip problem is exactly $\frac{n}{2}\lceil \log_2 n \rceil$.*

Here, we present an alternative 'complementary'

Proof: (i) $C(n) \le \frac{n}{2}\lceil \log_2 n \rceil$. For even n, Knödel's solution requires exactly $\frac{n}{2}\lceil \log_2 n \rceil$ calls, therefore these are sufficient.

(ii) $C(n) \ge \frac{n}{2}\lceil \log_2 n \rceil$. We prove that, for even $n > 2^{\lceil \log_2 n \rceil} - 2^{\lfloor \lceil \log_2 n \rceil /2 \rfloor}$, all the n persons must take part in all rounds from the $\lfloor \lceil \log_2 n \rceil /2 \rfloor + 1$-st to the $\lceil \log_2 n \rceil$-th, and, by applying this result to the inverse solution, we obtain that the same holds for the rounds from the first to the $\lceil \lceil \log_2 n \rceil /2 \rceil$-th, from which the statement will follow.

Before the r-th round ($r = 1, \ldots, \lceil \log_2 n \rceil$), each person knows at most, 2^{r-1} pieces of information. If a person i does not take part in the r-th round, that is, being n even, if two persons i, j do not take part in it, after this round they know 2^{r-1} pieces of information, at most, and the remaining persons 2^r, at most. Therefore, after $\lceil \log_2 n \rceil$ rounds, i and j know, at most,

$$2^{r-1} + 2^r + 2^{r+1} + \ldots + 2^{\lceil \log_2 n \rceil - 1} = \sum_{t=r-1}^{\lceil \log_2 n \rceil - 1} 2^t =$$

$$= \sum_{t=0}^{\lceil \log_2 n \rceil - 1} 2^t - \sum_{t=0}^{r-2} 2^t = (2^{\lceil \log_2 n \rceil} - 1) - (2^{r-1} - 1) = 2^{\lceil \log_2 n \rceil} - 2^{r-1}$$

pieces of information. Consequently, if i and j do not participate in the r-th round, with $r \geq \lfloor \lceil \log_2 n \rceil / 2 \rfloor + 1$, at the end they know, at most, $2^{\lceil \log_2 n \rceil} - 2^{r-1} \leq 2^{\lceil \log_2 n \rceil} - 2^{\lfloor \lceil \log_2 n \rceil / 2 \rfloor} < n$ pieces of information. Hence, no one can skip any round starting from the $\lfloor \lceil \log_2 n \rceil / 2 \rfloor + 1$-st, from which the statement follows. □

Thus, for even $n > 2^{\lceil \log_2 n \rceil} - 2^{\lfloor \lceil \log_2 n \rceil / 2 \rfloor}$, Knödel's solution is optimal.

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