Alliances in graphs

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Abstract

A defensive alliance in a graph $G = (V, E)$ is a set of vertices $S \subseteq V$ satisfying the condition that every vertex $v \in S$ has at most one more neighbor in $V - S$ than it has in S. Because of such an alliance, the vertices in S, agreeing to mutually support each other, have the strength of numbers to be able to defend themselves from the vertices in $V - S$. In this paper we introduce this new concept, together with a variety of other kinds of alliances, and initiate the study of their mathematical properties.

1 Introduction

In this paper we introduce the study of alliances in graphs. In its simplest form, an *alliance* is nothing other than a set of vertices having some collective property. But, as in the real world there are different types of alliances, so shall we define different types of alliances.

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Alliances are found in many varieties, including those formed:

(i) by people who unite by kinship or friendship;
(ii) by confederations between sovereign states;
(iii) by members of different political parties;
(iv) in botany, by groups of natural orders of plants;
(v) in ecology, by groupings of closely related associations;
(vi) in business, by companies with common economic interests;
(vii) in times of war, by nations for mutual support, usually defensive in nature, where allies are obligated to join forces if one or more of them are attacked, but also offensive, as a means of keeping the peace, e.g. NATO troops in a war-torn country.

With (vii) as our primary motivation, we define several types of graph theoretic alliances. But first we will need some definitions and notation. Let $G = (V, E)$ be a *graph* having *vertex set* V and *edge set* E. If $|V| = n$ and $|E| = m$, we say that G is of *order* n and *size* m. For any vertex $v \in V$, the *open neighborhood of* v is the set $N(v) = \{u : uv \in E\}$, while the *closed neighborhood of* v is the set $N[v] = N(v) \cup \{v\}$. The open and closed neighborhoods of sets of vertices $S \subseteq V$ are defined as follows: $N(S) = \bigcup_{v \in S} N(v)$, and $N[S] = N(S) \cup S$. Similarly, for a set S, the *boundary* of S is the set $\partial(S) = \bigcup_{v \in S} N(v) - S$.

A graph $G' = (V', E')$ is a *subgraph* of a graph $G = (V, E)$, written $G' \subseteq G$, if $V' \subseteq V$ and $E' \subseteq E$. If $S \subseteq V$ is a subset of the vertex set, the *subgraph induced by* S is the graph $G[S] = (S, E \cap S \times S)$.

We say that a set $S \subseteq V$ is a *dominating set* if $N[S] = V$, and is an *independent set* if no two vertices in S are adjacent.

2 Alliances in graphs

Definition 1. A non-empty set of vertices $S \subseteq V$ is called a *defensive alliance* if and only if for every $v \in S$, $|N[v] \cap S| \geq |N(v) \cap (V - S)|$. In this case, by strength of numbers, we say that every vertex in S is *defended* from possible attack by vertices in $V - S$. A defensive alliance is called *strong* if for every vertex $v \in S$, $|N[v] \cap S| > |N(v) \cap (V - S)|$. In this case we say that every vertex in S is *strongly defended.*

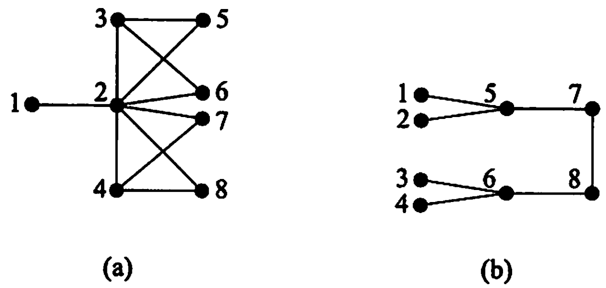

Figure 1: (a) The set $S = \{1,2,3,4\}$ is a 1-critical defensive alliance, but not a critical defensive alliance. (b) The set $S = \{1,2,3,4\}$ is a 1-critical offensive alliance, but not a critical offensive alliance.

Definition 2. A non-empty set of vertices $S \subseteq V$ is called an *offensive alliance* if and only if for every $v \in \partial(S)$, $|N(v) \cap S| \geq |N[v] \cap (V - S)|$. In this case we say that every vertex in $\partial(S)$ is *vulnerable* to possible attack by vertices in S. An offensive alliance is called *strong* if for every vertex $v \in \partial(S)$, $|N(v) \cap S| > |N[v] \cap (V - S)|$. In this case we say that every vertex in $\partial(S)$ is *very vulnerable.*

Any two vertices u, v in an alliance S are called *allies (with respect to S)*; we also say that u and v are *allied.*

An alliance S (defensive or offensive) is called *critical* if no proper subset of S is an alliance (of the same type). Note that the property of being an alliance (of any type) is not necessarily *hereditary.* For example, the defensive alliance $S = \{1,2,3,4\}$ in Figure 1(a) is 1-*critical* in the sense that for any single vertex $v \in S$, the set $S - \{v\}$ is not a defensive alliance. However, the set $S - \{2,3,4\}$ is a defensive alliance. In Figure 1(b) the set $T = \{1,2,3,4\}$ is 1-critical offensive alliance, but not a critical offensive alliance, since the proper subset $T' = \{1,2\} \subset T$ is also an offensive alliance.

An offensive alliance is not necessarily a defensive alliance. Let the complete bipartite graph $K_{2,2}$ be defined by the independent sets V_1 and V_2. Since all vertices in V_1 and in V_2 have more neighbors not in their own set than they have in their own set, both V_1 and V_2 are (strong) offensive alliances, but not defensive alliances.

For a graph G we will consider the following classes of alliances:

$\mathcal{A}(G)$, the class of critical defensive alliances in G,
$\hat{\mathcal{A}}(G)$, the class of critical strong defensive alliances in G,
$\mathcal{A}_o(G)$, the class of critical offensive alliances in G, and
$\hat{\mathcal{A}}_o(G)$, the class of critical strong offensive alliances in G.

Associated with each of these classes of critical alliances, are two invariants, as follows:

$$\begin{aligned}
a(G) &= \min\{|S| : S \in \mathcal{A}(G)\}, \text{ the \textit{alliance number of } } G,\\
A(G) &= \max\{|S| : S \in \mathcal{A}(G)\}, \text{ the \textit{upper alliance number}},\\
\hat{a}(G) &= \min\{|S| : S \in \hat{\mathcal{A}}(G)\}, \text{ the \textit{strong alliance number}},\\
\hat{A}(G) &= \max\{|S| : S \in \hat{\mathcal{A}}(G)\}, \text{ the \textit{upper strong alliance number}},\\
a_o(G) &= \min\{|S| : S \in \mathcal{A}_o(G)\}, \text{ the \textit{offensive alliance number}},\\
A_o(G) &= \max\{|S| : S \in \mathcal{A}_o(G)\}, \text{ the \textit{upper offensive alliance number}},\\
\hat{a}_o(G) &= \min\{|S| : S \in \hat{\mathcal{A}}_o(G)\}, \text{ the \textit{strong offensive alliance number}},\\
\hat{A}_o(G) &= \max\{|S| : S \in \hat{\mathcal{A}}_o(G)\}, \text{ the \textit{upper strong offensive alliance number}}.
\end{aligned}$$

The following inequalities are consequences of these definitions:

$$\begin{aligned}
a(G) \le \; & \hat{a}(G) \le \hat{A}(G),\\
& a(G) \le A(G),\\
a_o(G) \le \; & \hat{a}_o(G) \le \hat{A}_o(G),\\
& a_o(G) \le A_o(G).
\end{aligned}$$

In the remainder of this paper we will only present some of the mathematical properties of the four (defensive) alliance numbers. Consequently, when we say *alliance*, we will always mean defensive alliance.

If S is a critical alliance of a graph G and $|S| = a(G)$, then we say that S is an *a-set* of G, and similarly, if $|S| = A(G)$, we say that S is an *A-set* of G. Likewise, if S is a critical strong alliance, and $|S| = \hat{a}(G)$, then S is an *$\hat{a}$-set*, or if $|S| = \hat{A}(G)$, then S is an *$\hat{A}$-set.*

One important kind of alliance that appears often is a global alliance.

Definition 3. A defensive alliance S is called *global* if it effects every vertex in $V - S$, that is, every vertex in $V - S$ is adjacent to at least one member of the alliance S. In this case, S is a dominating set, and one can define the global alliance number, denoted $\gamma_a(G)$, to equal the minimum cardinality of a global defensive alliance in G.

Definition 4. An offensive alliance S is called *global* if for every vertex $v \in V - S$, $|N(v) \cap S| \geq |N[v] \cap (V - S)|$. Thus, global offensive alliances are also dominating sets, since every vertex in $V - S$ is adjacent to at least one vertex in the alliance S.

Both global defensive alliances and global offensive alliances are new kinds of dominating sets. A fairly complete listing of different kinds of dominating sets can be found in the two books by Haynes, Hedetniemi and Slater [22, 21].

3 Concepts similar to alliances

Although the concept of an alliance in a graph is newly defined here, concepts similar to defensive and offensive alliances can be found in the literature. Among the earliest of these is that of an *unfriendly 2-partition* of a graph $G = (V, E)$, which is a partition $\Pi = \{V_0, V_1\}$, having the property that every vertex $v \in V_i$ is adjacent to at least as many vertices in V_{1-i} as it is to vertices in V_i. Stated in terms of alliances, unfriendly 2-partitions almost correspond to a partition of V into two offensive alliances. If the definition of an unfriendly 2-partition is changed as follows, then it corresponds to a partition of V into two global offensive alliances. A 2-partition $\Pi = \{V_0, V_1\}$ is called *very unfriendly* if every vertex $v \in V_i$ is adjacent to *more* vertices in V_{1-i} than it is to vertices in V_i. A simple example of a very unfriendly 2-partition can be seen by 2-coloring the vertices of a connected bipartite graph G having a minimum degree of at least two. The two color classes in such a coloring, say V_0 and V_1, define a very unfriendly 2-partition; each of V_0 and V_1 is a global strong offensive alliance.

Unfriendly 2-partitions were perhaps first introduced by Borodin and Kostochka [3] in 1977, and have also been studied by Bernardi [2], Cowan and Emerson [6], Aharoni, Milner and Prikry [1], and Shelah and Milner [28]. Basic to these investigations is the simple observation that every finite graph has an unfriendly 2-partition. In fact, any 2-partition $\Pi = \{V_0, V_1\}$ of V which maximizes the number of edges between V_0 and V_1 is an unfriendly 2-partition. Shelah and Milner [28] showed that not all infinite graphs have an unfriendly 2-partition, but all graphs have an unfriendly 3-partition.

A similar concept has been studied by Gerber and Kobler [16], who define a vertex v in a set $A \subset V$ to be *satisfied* if it has at least as many neighbors in A as in $V - A$. A set A is called *cohesive* if every vertex in A is satisfied with respect to A. It is easy to see that every cohesive set is a strong defensive alliance. A graph is said to be *satisfiable* if there is a vertex partition into two or more non-empty sets so that every vertex is satisfied with respect to the set in which it occurs. Such a partition is called a *satisfactory partition*. Satisfactory partitions correspond, therefore, to partitions of V into disjoint, strong defensive alliances. It can be seen that not all graphs have satisfactory partitions, for example complete graphs. Shafique and Dutton [27] present some necessary and sufficient conditions for graphs to be satisfiable, and show that no forbidden subgraph characterization exists for this class of graphs.

Several authors have studied signed and minus dominating functions in graphs, which are defined as follows. A function $f : V \rightarrow \{-1, +1\}$ is called a *signed dominating function* if for every vertex $v \in V$, $f(N[v]) \geq 1$. Stated in other words, if $\Pi = \{V_{-1}, V_1\}$ is the 2-partition defined by f^{-1}, then the set V_1 is both a global strong defensive alliance and a global strong offensive alliance. Similarly, a function $f : V \rightarrow \{-1, 0, 1\}$ is called a *minus dominating function* if for every vertex $v \in V$, $f(N[v]) \geq 1$. Stated in other words, if $\Pi = \{V_{-1}, V_0, V_1\}$ is the 3-partition defined by f^{-1}, then the set V_1 is a global strong defensive alliance. Signed and minus domination have been studied in [8, 10, 9, 11, 13, 17, 18, 23, 29].

Another concept similar to that of alliances has been studied by Dunbar, Hoffman, Laskar and Markus [12], who define a set $S \subseteq V$ to be an *α-dominating set*, for some α, $0 < \alpha \leq 1$, if every vertex $v \in V - S$ satisfies the inequality: $|N(v) \cap S| \geq \alpha |N(v)|$. Thus, if $\alpha > 1/2$, then every vertex in $V - S$ has more neighbors in S than it has in $V - S$. This means that the set S is a global offensive alliance. However, if $\alpha \leq 1/2$, then every vertex in $V - S$ has at least as many neighbors in $V - S$ as it does in S, and therefore, $V - S$ is a strong defensive alliance. Recently, Langley, Merz, Stewart and Ward [25] have studied α-domination in tournaments.

Still another concept similar to that of alliances has been studied by Dunbar, Harris, Hedetniemi, Hedetniemi, Laskar and McRae [7], who define a set $S \subseteq V$ to be *nearly perfect* if for all $v \in V - S$, $|N(v) \cap S| \leq 1$. It is easy to see that if S is a nearly perfect set, then the complement $V - S$ is a defensive alliance.

A *perfect dominating set* is a dominating set having the property that for all $v \in V - S$, $|N(v) \cap S| = 1$. A set $S \subseteq V$ is a *2-packing* if for all $v \in V$, $|N[v] \cap S| \leq 1$.

It follows from these definitions that the complement of every perfect dominating set and the complement of every 2-packing is a defensive alliance (cf. [5, 22, 21] for discussion of perfect dominating sets and 2-packings).

One final type of set leads to alliances. A set $S \subseteq V$ is a *vertex cover* if for every edge $uv \in E$, $S \cap \{u, v\} \neq \emptyset$. It is well known that the complement $V - S$ of any vertex cover S is an independent set. It is easy to see, therefore, that every vertex cover of a connected graph is a global offensive alliance.

Since the introduction of the concept of alliances in graphs by the authors in August, 2001 [24], several papers have been written, and are in preparation, on this topic (cf. [4, 14, 15, 19, 20, 26]).

4 Properties of the alliance and strong alliance numbers, $a(G)$ and $\hat{a}(G)$

We first observe that any critical (strong) alliance S in a graph G must induce a connected subgraph of G. This is obvious, since any component of $G[S]$ is a strictly smaller alliance (of the same type).

Proposition 1. *For any critical alliance S in a graph G, the subgraph $G[S]$ induced by S is connected.*

It is obvious that there are only two kinds of alliances of cardinality one: an isolated vertex (a vertex of degree zero) and an end vertex (a vertex of degree one).

Proposition 2. *For any graph G,*

(i) $a(G) = 1$ *if and only if there exists a vertex* $v \in V$ *such that* $\deg(v) \leq 1$.

(ii) $\hat{a}(G) = 1$ *if and only if G has an isolated vertex.*

Corollary 1. *For any tree T and path P_n, $a(T) = a(P_n) = 1$.*

The following characterizations of graphs for which $a(G) = 2$ and graphs for which $\hat{a}(G) = 2$ are immediate consequences of Proposition 1, and depend on the minimum degree of a vertex in G, denoted $\delta(G)$.

Proposition 3. *For any graph G,*

(i) $a(G) = 2$ *if and only if* $\delta(G) \geq 2$ *and G has two adjacent vertices of degree at most three.*

(ii) $\hat{a}(G) = 2$ *if and only if* $\delta(G) \geq 1$ *and* G *has two adjacent vertices of degree at most two.*

The *wheel* W_n is the graph which consists of a cycle C_n of length n, and a central vertex which is adjacent to every vertex of the cycle.

Corollary 2. *For any cycle* C_n*, wheel* W_n*, and path* P_n*,*

(i) $a(C_n) = \hat{a}(P_n) = \hat{a}(C_n) = a(W_n) = 2$.

(ii) $\hat{a}(W_n) = \lceil \frac{n}{2} \rceil + 1$.

Proof.

(i) Any two adjacent vertices on C_n or on the cycle of the wheel W_n constitute an alliance of C_n and W_n, respectively.

(ii) If S is an $\hat{a}$-set of the wheel W_n, then no vertex of $G[S]$ can have degree one, since every vertex in W_n has degree at least three. Thus, the minimum degree of a vertex in $G[S]$ is at least two. The only induced subgraphs of W_n having minimum degree at least two are the cycle C_n or subgraphs containing the central vertex (of degree n). It follows, therefore, that the only $\hat{a}$-sets of W_n contain the central vertex and at least half of its neighbors on C_n. □

The following characterizations, of graphs for which $a(G) = 3$ and graphs for which $\hat{a}(G) = 3$, are again consequences of Proposition 1.

Proposition 4. *For any graph* G*,*

(i) $a(G) = 3$ *if and only if* $a(G) \neq 1$, $a(G) \neq 2$, *and* G *has an induced subgraph isomorphic to either (a)* P_3*, with vertices, in order,* u, v *and* w*, where* $\deg(u)$ *and* $\deg(w)$ *are at most three, and* $\deg(v)$ *is at most five, or (b) isomorphic to* K_3*, each vertex of which has degree at most five.*

(ii) $\hat{a}(G) = 3$ *if and only if* $\hat{a}(G) \neq 1$, $\hat{a}(G) \neq 2$, *and* G *has an induced subgraph isomorphic to either (a)* P_3*, with vertices, in order,* u, v *and* w*, where* $\deg(u)$ *and* $\deg(w)$ *are at most two, and* $\deg(v)$ *is at most four, or (b) isomorphic to* K_3*, each vertex of which has degree at most four.*

Similar characterizations of graphs for which $a(G) = k$ and $\hat{a}(G) = k$ are possible, in principle, since for any value of k, there are only a finite number of connected graphs of order k, and the vertices in each of these will have certain degree restrictions. Unfortunately, the number of connected graphs of order k grows exponentially with k.

The $m \times n$ *grid graph* is the Cartesian product $G_{m,n} = P_m \Box P_n$.

Theorem 1. *For the $m \times n$ grid graph $G_{m,n}$,*

(i) $a(G_{m,n}) = 1$ *if and only if* $\min\{m, n\} = 1$.

(ii) $a(G_{m,n}) = 2$ *if and only if* $\min\{m, n\} \geq 2$.

(iii) $\hat{a}(G_{m,n}) = 2$ *if and only if* $\min\{m, n\} < 3$.

(iv) $\hat{a}(G_{m,n}) = 3$ *if and only if* $\min\{m, n\} = 3$.

(v) $\hat{a}(G_{m,n}) = 4$ *if and only if* $\min\{m, n\} \geq 4$.

Proof.

(i) This is obvious from Proposition 1.

(ii) This follows from Proposition 2*(i)*.

(iii) This follows from Proposition 3*(ii)*, since any such grid graphs have two adjacent vertices of degree two.

(iv) This follows from Proposition 4*(ii)(a)*, since the first column of $G_{3,n}$ is an $\hat{a}$-set of cardinality three.

(v) This follows from the fact that a grid graph $G_{m,n}$, where $\min\{m, n\} \geq 4$, does not have an induced subgraph as defined in Proposition 4*(ii)(a)* or *(ii)(b)*. □

For several values of k it is possible to completely determine the value of both $a(G)$ and $\hat{a}(G)$ for the class of k-regular graphs, that is, graphs in which every vertex has degree k. In order to present these results we need the following definition. The length of a smallest cycle in a graph G is called the *girth* of G, and is denoted girth(G). Note that, by definition, a cycle C of length girth(G) does not contain any chords, that is, an edge between two non-consecutive vertices on C.

Theorem 2. *For any graph $G = (V, E)$,*

(i) if G is 1-regular, then $a(G) = 1$ and $\hat{a}(G) = 2$.

(ii) if G is 2-regular, then $a(G) = 2$ and $\hat{a}(G) = 2$.

(iii) if G is 3-regular, then $a(G) = 2$ and $\hat{a}(G) = \text{girth}(G)$.

(iv) if G is 4-regular, then $a(G) = \text{girth}(G)$ and $\hat{a}(G) = \text{girth}(G)$.

(v) if G is 5-regular, then $a(G) = \text{girth}(G)$.

Proof.

(i) This follows from Propositions 2*(i)* and 3*(ii)*.

(ii) This follows from Proposition 3.

(iii) In a 3-regular graph, any two adjacent vertices form an alliance. Therefore, $a(G) = 2$. In order to show that $\hat{a}(G) = \text{girth}(G)$, let S be an $\hat{a}$-set of a 3-regular graph G. Consider the induced subgraph $G[S]$. This graph can have no vertex of degree one, since then it would not be strongly defended. Thus, $G[S]$ has minimum degree at least two, and must contain at least one cycle. Let C be a shortest cycle in $G[S]$. It is easy to see that C is a strong alliance in G. Therefore, $G[S] = C$. It follows that any shortest cycle is a strong alliance in G, and hence, $\hat{a}(G) = \text{girth}(G)$.

(iv) The same argument used in the proof of *(iii)* above can be used to show that $a(G) = \text{girth}(G)$ and $\hat{a}(G) = \text{girth}(G)$, if G is a 4-regular graph, since any smallest cycle in G is both a critical alliance and a critical strong alliance.

(v) Again, the same argument used in the proof of *(iii)* above can be used to show that $a(G) = \text{girth}(G)$ if G is a 5-regular graph, since any smallest cycle in G is a critical alliance (but not a strong alliance) in G. □

Corollary 2*(ii)* shows that, for the class of wheels W_n, the value of $\hat{a}(G)$ can be arbitrarily large. The next two observations provide two more classes of graphs for which $a(G)$ and $\hat{a}(G)$ can be arbitrarily large.

Proposition 5. *For the complete graph K_n,*

(i) $a(K_n) = \lceil \frac{n}{2} \rceil$.

(ii) $\hat{a}(K_n) = \lfloor \frac{n}{2} \rfloor + 1$.

Proposition 6. *For the complete bipartite graph $K_{m,n}$, $2 \leq m \leq n$,*

(i) $a(K_{m,n}) = \lfloor \frac{m}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$.

(ii) $\hat{a}(K_{m,n}) = \lceil \frac{m}{2} \rceil + \lceil \frac{n}{2} \rceil$.

In fact, while $a(T) = 1$ for all trees T, the value of $\hat{a}(T)$ can be arbitrarily large. Consider only the star $K_{1,n}$, where $\hat{a}(K_{1,n}) = \lceil \frac{n}{2} \rceil + 1$. Note in this case that an end vertex does not define a strong alliance. Thus, any connected subgraph which defines an $\hat{a}$-set of $K_{1,n}$ must contain the vertex of degree n. But since this vertex must be strongly defended, at least half of its neighbors must be in any strong alliance.

5 Bounds for $a(G)$ and $\hat{a}(G)$

We have seen that both $a(G)$ and $\hat{a}(G)$ can equal one for some graphs G. We also note that the entire vertex set V is always an alliance and a strong alliance. Thus,

Proposition 7. *For any graph G of order n,*

$$1 \leq a(G) \leq \hat{a}(G) \leq n.$$

Although the lower bound is sharp for graphs with isolated vertices or with end vertices, the upper bound can be improved slightly.

Proposition 8. *For any connected graph G of order n having minimum degree $\delta(G)$,*

(i) $a(G) \leq n - \lceil \delta(G)/2 \rceil$,

(ii) $\hat{a}(G) \leq n - \lfloor \delta(G)/2 \rfloor$,

and these bounds are sharp.

Proof.

(i) Let $S \subseteq V$ be any set of cardinality $n - \lceil \delta(G)/2 \rceil$. It is easy to see that every vertex in S is defended, and thus that S is an alliance.

(ii) Let $S \subseteq V$ be any set of cardinality $n - \lfloor \delta(G)/2 \rfloor$. It is easy to see that every vertex in S is strongly defended, and thus that S is a strong alliance.

It is easy to see that these bounds are sharp for complete graphs. □

The example in Proposition 5*(i)* shows that $a(G)$ can be at least as large as $\lceil \frac{n}{2} \rceil$, for a graph of order n. This led the authors in [24] to conjecture that this is the maximum value that $a(G)$ can have for any graph G. This was subsequently proved in [14].

Theorem 3. *For any graph G of order n,*

$$a(G) \leq \left\lceil \frac{n}{2} \right\rceil.$$

The graph $G = \overline{K}_{2n} + K_n$, illustrated for $n = 4$ in Figure 2 is an example where $\hat{a}(G) = \lceil \frac{n}{2} \rceil + 1$. This example led the authors to conjecture in [24] that this is the maximum value that $\hat{a}(G)$ can achieve. This was subsequently proved and even improved slightly in [14].

Theorem 4. *For any graph G of order n,*

$$\hat{a}(G) = \left\lfloor \frac{n}{2} \right\rfloor + 1.$$

For the following class of graphs, the upper bound in Theorem 3 can be improved slightly. Let $\lambda(G)$ denote the *edge connectivity* of a connected graph G, which equals the minimum number of edges in a set, whose removal from G results in a disconnected graph. Note that for any connected graph G, $\lambda(G) \leq \delta(G)$, that is, a connected graph G can always be disconnected by removing all of the edges incident to a vertex of minimum degree $\delta(G)$.

Let $\Pi = \{V_1, V_2\}$ be a bipartition of the vertices of a connected graph G, such that there are $\lambda(G)$ edges between vertices in V_1 and vertices in V_2. If either $|V_1| = 1$ or $|V_2| = 1$, then we say that Π is a *singular λ-bipartition*, while if both V_1 and V_2 have at least two vertices, then we say that Π is a *non-singular λ-bipartition.*

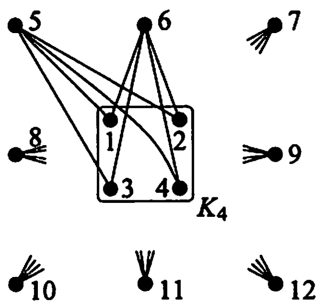

Figure 2: A graph G for which $\hat{a}(G) = \lceil \frac{n}{2} \rceil + 1$.

Proposition 9. *For any graph G of order n, for which $\lambda(G) < \delta(G)$,*

$$a(G) \leq \hat{a}(G) \leq \lfloor \tfrac{n}{2} \rfloor.$$

Proof. Let $\Pi = \{V_1, V_2\}$ be a λ-bipartition of G , and assume that $|V_1| \leq |V_2|$. Since $\lambda(G) < \delta(G)$, we know that Π is a *non-singular λ-bipartition.* Clearly, $|V_1| \leq \lfloor \frac{n}{2} \rfloor$. We claim that the set V_1 is a strong alliance, and therefore, $\hat{a}(G) \leq \lfloor \frac{n}{2} \rfloor$.

This follows from the observation that if V_1 contains a vertex u that is not strongly defended, then the modified partition $\Pi' = \{V_1 - \{u\}, V_2 \cup \{u\}\}$ is a bipartition having fewer edges between the two sets than Π, contradicting the assumption that Π is a λ-bipartition. □

Corollary 3. *For any graph G of order n having a non-singular λ-bipartition,*

$$a(G) \leq \hat{a}(G) \leq \lfloor \tfrac{n}{2} \rfloor.$$

One other upper bound can be given for $a(G)$ involving Cartesian products.

Proposition 10. *For any Cartesian product graph $G_1 \square G_2$,*

(i) $a(G_1 \square G_2) \leq \min\{a(G_1)\hat{a}(G_2), \hat{a}(G_1)a(G_2)\}$.

(ii) $\hat{a}(G_1 \square G_2) \leq \hat{a}(G_1)\hat{a}(G_2)$.

Proof.

(i) Let S_a and $S_{\hat{a}}$ be an a-set and $\hat{a}$-set of G_1,respectively, and let T_a and $T_{\hat{a}}$ be an a-set and $\hat{a}$-set of G_2, respectively. It is straightforward to see that the Cartesian product sets $S_a \times T_{\hat{a}}$, and $S_{\hat{a}} \times T_a$ are both alliances in $G = G_1 \Box G_2$.

(ii) As in the proof of *(i)*, it is easy to see that $S_{\hat{a}} \times T_{\hat{a}}$ is a strong alliance in $G_1 \Box G_2$. □

6 Alliances in infinite graphs

In this section we briefly consider the values of $a(G)$ and $\hat{a}(G)$ in several classes of infinite graphs.

Proposition 11.

(i) For the one-way infinite path $P_{1,\infty}$, $a(P_{1,\infty}) = 1$, while $\hat{a}(P_{1,\infty}) = 2$.

(ii) For the two-way infinite path $P_{\infty,\infty}$, $a(P_{\infty,\infty}) = \hat{a}(P_{1,\infty}) = 2$.

(iii) For the infinite grid $G_{\infty,\infty}$, $a(G_{\infty,\infty}) = \hat{a}(G_{\infty,\infty}) = 4$.

(iv) For the infinite d-dimensional grid $G^d_\infty = P_{\infty,\infty}\Box\ldots\Box P_{\infty,\infty}$ (d times), $a(G^d_\infty) = \hat{a}(G^d_\infty) = 2^d$.

(v) For the rooted, infinite binary tree $T_{2,\infty}$, $a(T_{2,\infty}) = 2$, while $\hat{a}(T_{2,\infty}) = \infty$.

Proof.

(i) Clearly the one end vertex in the one-way infinite path $P_{1,\infty}$ forms an alliance, while any two adjacent vertices in $P_{1,\infty}$ form a strong alliance.

(ii) Any two adjacent vertices in the two-way infinite path $P_{\infty,\infty}$ form both an alliance and a strong alliance.

(iii) Any four vertices forming a 4-cycle in the infinite grid $G_{\infty,\infty}$ form both an alliance and a strong alliance.

(iv) Any 2^d vertices forming a P_2^d in the infinite d-dimensional grid G^d_∞ form both an alliance and a strong alliance.

(*v*) In the infinite binary tree $T_{2,\infty}$, any two adjacent vertices form an alliance, but not a strong alliance. In $T_{2,\infty}$, every vertex has one parent and two children, that is, every vertex has degree three, except the root, which has degree two. It follows, therefore, that no finite, connected set S of vertices in $T_{2,\infty}$ can be an alliance, since in such a set S, the induced subgraph $G[S]$ would necessarily have to have an end vertex, since there are no cycles in $T_{2,\infty}$, and this vertex would not be strongly defended. Thus, $\hat{a}(T_{2,\infty})$ is not finite. However, any two-way infinite path in $T_{2,\infty}$ is a strong alliance. □

Theorem 5. *For the rooted, infinite ternary tree* $T_{3,\infty}$,

$$a(T_{3,\infty}) = \hat{a}(T_{3,\infty}) = \infty.$$

Proof. In the infinite ternary tree $T_{3,\infty}$, every vertex has one parent and three children, that is, every vertex has degree four, except the root, which has degree three. It follows, therefore, that no finite, connected set S of vertices in $T_{3,\infty}$ can be an alliance, since in such a set S, the induced subgraph $G[S]$ would necessarily have to have an end vertex, since there are no cycles in $T_{3,\infty}$, and this vertex would not be defended. Thus, $a(T_{3,\infty})$ is not finite. However, any two-way infinite path in $T_{3,\infty}$ is an alliance, and in fact, is a strong alliance. □

7 Properties of the upper alliance numbers, $A(G)$ and $\hat{A}(G)$

In this section we consider several results about the two upper alliance numbers $A(G)$ and $\hat{A}(G)$.

Table 1 provides an overview of results previously established for $a(G)$ and $\hat{a}(G)$, and indicates corresponding results for both $A(G)$ and $\hat{A}(G)$.

The facts that both $A(P_n) = 2$ and $\hat{A}(P_n) = 2$, for $n \geq 4$, follow from the observation that in a path, any two adjacent vertices, neither of which is an end vertex, form a critical alliance, and a critical strong alliance. Since any critical (strong) alliance S must induce a connected subgraph, the only possible critical (strong) alliances in a path are themselves paths. But since they would contain two adjacent vertices, S could not be critical if it contains more than two vertices.

The same reasoning applies to the observation that $A(C_n) = 2$ and $\hat{A}(C_n) = 2$, for $n \geq 3$, where again a largest critical (strong) alliance consists of two adjacent vertices.

Class	$a(G)$	$A(G)$	$\hat{a}(G)$	$\hat{A}(G)$
1-regular	1	1	2	2
Paths	1	$2, n \geq 4$	$2, n \geq 3$	$2, n \geq 3$
Trees	1	any k	any k	any k
Cycles	2	2	$2, n \geq 3$	$2, n \geq 3$
3-regular	2	2	girth(G)	lcc(G)
4-regular	girth(G)	lcc(G)	girth(G)	lcc(G)
5-regular	girth(G)	lcc(G)	?	?

Table 1: Overview of alliance numbers for several classes of graphs.

The fact that $A(G)$ can be arbitrarily large, even for trees, is perhaps best illustrated with the ***subdivided star***, $S(K_{1,n})$. This is the tree which is obtained from the star $K_{1,n}$ by subdividing every edge, that is, removing each edge uv and replacing it with two new edges ux and xv. It is easy to see that the set S obtained from a subdivided star by including the one vertex of degree n and $\lfloor \frac{n}{2} \rfloor$ of its neighbors is a critical alliance of maximum order.

Since $\hat{a}(K_{1,n}) = \lfloor \frac{n}{2} \rfloor + 1$, and since $\hat{a}(G) \leq \hat{A}(G)$, it follows that $\hat{A}(G)$ can be arbitrarily large, even for trees.

In a 3-regular graph G, any two adjacent vertices form an alliance. Therefore, no larger critical alliances are possible, and $A(G) = 2$.

By lcc(G) we mean the maximum length of a chordless cycle in a graph G. We have previously observed that any chordless cycle in a 3-regular graph forms a critical strong alliance. Therefore, it follows that in a 3-regular graph G, $\hat{A}(G) \geq$ lcc(G). But $\hat{A}(G) \leq$ lcc(G), since any critical strong alliance S in a 3-regular graph G, that has cardinality greater than lcc(G), must necessarily contain a cycle in $G[S]$. Since a smallest cycle in $G[S]$ is a critical strong alliance, it follows that $G[S]$ itself must induce a chordless cycle.

Similar arguments can be used to show that for 4-regular graphs G, $A(G) =$ lcc(G) and $\hat{A}(G) =$ lcc(G), and for 5-regular graphs, $A(G) =$ lcc(G).

8 Other types of alliances

We have only considered defensive alliances in this introductory paper on *Alliances in Graphs*. Clearly, many other types of alliances can be defined. Of particular interest for further study are the following:

1. Offensive alliances and strong offensive alliances ([14]).

2. Global alliances ([19])

3. Dual alliances ([4])

   Definition 5. An alliance is called *dual* if it is both defensive and offensive.

4. Open alliances

   Definition 6. An alliance is called *open* if it is defined completely in terms of open neighborhoods, that is, a set $S \subseteq V$ is an *open defensive alliance* if for every vertex $v \in S$, $|N(v) \cap S| \geq |N(v) \cap (V - S)|$.

   Notice that with this definition, every strong alliance is also an open alliance. Notice also that an open alliance is a cohesive set, as defined earlier [16, 27].

5. Weighted alliances

   Definition 7. An alliance S is called *weighted* if associated with every vertex $v \in V$ is a weight (or a strength) $wt(v)$, and we require that for every vertex $v \in S$, $\sum_{u \in N[v] \cap S} wt(u) \geq \sum_{w \in N(v) \cap (V-S)} wt(w)$.

9 Algorithmic complexity questions

This paper has not considered any algorithmic complexity issues, but there are many to consider, including, for example, the following NP-completeness questions:

(CRITICAL) ALLIANCE
INSTANCE: A graph G and a positive integer k.
QUESTION: Does G have a (critical) alliance of size at most k?

(CRITICAL) STRONG ALLIANCE
INSTANCE: A graph G and a positive integer k.
QUESTION: Does G have a (critical) strong alliance of size at most k?

McRae, Goddard, Hedetniemi, Hedetniemi and Kristiansen have shown that ALLIANCE and STRONG ALLIANCE are both NP-complete, even when restricted to bipartite or chordal graphs [26]. But many complexity questions remain unanswered, including the following sample:

OFFENSIVE ALLIANCE
INSTANCE: A graph G and a positive integer k.
QUESTION: Does G have an offensive alliance of size at most k?

GLOBAL ALLIANCE
INSTANCE: A graph G and a positive integer k.
QUESTION: Does G have a global alliance of size at most k?

DUAL ALLIANCE
INSTANCE: A graph G and a positive integer k.
QUESTION: Does G have a dual alliance of size at most k?

CRITICAL (STRONG) ALLIANCE
INSTANCE: A graph G and a set $S \subseteq V$.
QUESTION: Is S a critical (strong) alliance of G?

In addition, the following questions are of some interest:

1. Given a graph G and a vertex $v \in V$, define the *alliance number of* v, $a(v)$, to equal the smallest defensive alliance containing vertex v. How difficult is it to determine $a(v)$?
2. Given a graph G and two vertices $u, v \in V$, what is $a(u, v)$, that is, the smallest cardinality of an alliance containing both u and v?
3. Given a graph G, define the *alliance packing number* $P_a(G)$ to equal the maximum number of pairwise-disjoint, defensive alliances contained in G. Similarly, define the *alliance partition number*, $\psi_a(G)$, to equal the maximum order of a partition $\Pi = \{V_1, V_2, \ldots, V_k\}$ of V, such that each block of the partition V_i is a defensive alliance? What is the alliance packing number or the alliance partition number of a grid graph $G_{m,n}$?

4. What does it mean to say that an alliance is *weak*? For example, one could say that an alliance S is *weak* (or 1-critical) if for every vertex $v \in S$, $S - \{v\}$ is no longer an alliance.

5. Define the *kth alliance number* $a_k(G)$ to equal the smallest cardinality of a defensive alliance having the property that it can defend itself from any k simultaneous attacks. What can you say about $a_k(G)$?

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