# An Extremal Problem in Distance-$k$ Domination

Mark S. Anderson
Department of Mathematical Sciences,
Rollins College, Winter Park FL 32789
Robert C. Brigham
Department of Mathematics,
University of Central Florida, Orlando FL 32816
Julie R. Carrington
Department of Mathematical Sciences,
Rollins College, Winter Park FL 32789
Richard P. Vitray
Department of Mathematical Sciences,
Rollins College, Winter Park FL 32789
Donna J. Williams
Department of Computer Science, Wake Forest University
Winston-Salem NC 27109
Jay Yellen
Department of Mathematical Sciences,
Rollins College, Winter Park FL 32789

**Abstract**

The distance-$k$ domination number of graph $G$, $\gamma_{\leq k}(G)$, is the cardinality of a smallest set of vertices, $S$, such that every vertex not in $S$ is no more than distance $k$ from at least one vertex of $S$. Carrington, Harary, and Haynes showed $|V^0| \geq 2|V^+|$ where $V^0 = \{v \in V : \gamma_{\leq 1}(G - v) = \gamma_{\leq 1}(G)\}$ and $V^+ = \{v \in V : \gamma_{\leq 1}(G - v) > \gamma_{\leq 1}(G)\}$. This paper extends the result to distance-$k$ domination, with the obvious change in definition of $V^0$ and $V^+$, to show $|V^0| \geq \frac{2}{2k-1}|V^+|$. Extremal graphs are characterized when $k = 1$ and some progress is mentioned on the characterization problem when $k > 1$.

# 1 Introduction

Let $G = (V, E)$ be a graph. Its *domination number*, $\gamma(G)$, is the size of a smallest set, $S$, of vertices such that every vertex not in $S$ is adjacent to a vertex in $S$. The *distance-k domination* number of $G$ is defined similarly, except that vertices not in $S$ must be within distance $k$ of some vertex in $S$. We designate the distance-$k$ domination number as $\gamma_{\leq k}(G)$. An excellent introduction and references to distance-$k$ domination can be found in Henning [4]. Harary [3] introduced the names *changing* and *unchanging* to refer to how a graphical invariant is affected by the deletion or addition of an edge or by the deletion of a vertex. Brigham, Chinn, and Dutton [1] studied graphs for which the domination number decreases when a vertex is removed; and Carrington, Harary, and Haynes [2] investigated all the possible situations for changing and unchanging of domination.

The work in [2] includes a partition of the vertices $V$ into three sets $V^0 = \{v \in V : \gamma(G - v) = \gamma(G)\}$, $V^+ = \{v \in V : \gamma(G - v) > \gamma(G)\}$, and $V^- = \{v \in V : \gamma(G - v) < \gamma(G)\}$. The following results were shown in that paper.

**Proposition 1** *If $v \in V^+$ and $w \in V^-$, then $v$ and $w$ are not adjacent.*

**Proposition 2** *Every graph satisfies $|V^0| \geq 2|V^+|$.*

In this paper we generalize both results to distance-$k$ domination and investigate graphs for which equality in the generalization of Proposition 2 is achieved. Such graphs are called *extremal.* A complete characterization of extremal graphs is provided when $k = 1$, and one situation when $k > 1$ is mentioned. Section 2 generalizes the inequality, Section 3 establishes necessary conditions for equality, and Section 4 characterizes extremal graphs when $k = 1$.

## 2 An Inequality Between $V^0$ and $V^+$

As with ordinary domination, we partition the vertex set $V$ of graph $G$ into sets $V^+$, $V^0$, and $V^-$ according to whether $\gamma_{\leq k}(G-v) > \gamma_{\leq k}(G)$, $\gamma_{\leq k}(G-v) = \gamma_{\leq k}(G)$, or $\gamma_{\leq k}(G-v) < \gamma_{\leq k}(G)$, respectively. The partition of a graph arises from the partitions of the components and it is clear that $\gamma_{\leq k}(G-v) > \gamma_{\leq k}(G)$ if and only if it is true for the component of $G$ containing $v$. Thus we may assume that $G$ is connected. If $D$ is a minimum distance-$k$ dominating set of $G$ and $v$ is a vertex of $G$, let $W_D(v) = \{x \in V(G-v)$ : every path in $G$ from $D$ to $x$ of length $k$ or less must contain $v\}$. Observe that $W_D(v) \cap D = \emptyset$.

**Lemma 3** *Let $D$ be a minimum distance-$k$ dominating set of $G$. If $v \in V^+$, then there exists a vertex $x \notin V^+$ such that $x \in W_D(v)$.*

**Proof:** Since $v \in V^+$, $D - \{v\}$ is not a distance-$k$ dominating set of $G - v$, so $W_D(v) \neq \emptyset$. Let $x \in W_D(v)$ be of maximum distance in $G$ from $D$. If $x \in V^+$, then there exists $z \in W_D(x)$. Since every path in $G$ of length $k$ or less from $D$ to $x$ includes $v$ and every such path to $z$ includes $x$, it follows that $z \in W_D(v)$, which contradicts the choice of $x$ since $dist(z, D) > dist(x, D)$. Thus, $x \notin V^+$. □

Because we consider both $G$ and $G - v$, where $v \in V(G)$, we use the notation $dist_G(F, v)$ to represent the length of a shortest path in $G$ from $v$ to a vertex of $F \subseteq V(G)$.

**Lemma 4** *Let $w \in V^-$ and $F$ be a minimum distance-$k$ dominating set of $G - w$. If $v$ is adjacent to $w$, then $dist_G(F, v) = k$.*

**Proof:** Clearly $dist_G(F, v) \leq k$. If $dist_G(F, v) < k$, then $dist_G(F, w) \leq k$, implying $F$ distance-$k$ dominates $G$, a contradiction. □

The next result generalizes Proposition 1.

**Lemma 5** *If $v \in V^+$ and $w \in V^-$, then $v$ and $w$ are not adjacent.*

**Proof:** Suppose $v$ is adjacent to $w$. Let $F$ be a minimum distance-$k$ dominating set of $G-w$ so $D = F \cup \{w\}$ is a minimum distance-$k$ dominating set of $G$. By Lemma 3, there exists $x \in W_D(v)$. Clearly, $x \neq w$ because $w \in D$ and $W_D(v) \cap D = \emptyset$. Since $F$ distance-$k$ dominates $G - w$, there is a path from $F$ to $x$ of length $k$ or less. Such paths in $G - w$ also exist in $G$ from $D$ to $x$ and so must contain $v$. Thus, $dist_G(F, v) \leq k - 1$ which contradicts Lemma 4. □

The generalization of Proposition 2 can be achieved by the following construction. Let $D$ be a minimum distance-$k$ dominating set of $G$ and assume that $V^0$ is not empty. Let $v_1 \in V^0$ and let $P_1$ be a minimum length path from $D$ to $v_1$. Observe that $P_1$ necessarily contains exactly one vertex $d_1 \in D$. Suppose we have constructed paths $P_1, P_2, \ldots, P_t$ where $P_i$ is a minimum length path from $D$ to $v_i \in V^0$ and contains exactly one vertex $d_i \in D$. If there exists $v_{t+1} \in V^0 - \cup_{i=1}^{t} V(P_i)$, construct a path $P_{t+1}$ of minimum length from $D$ to $v_{t+1}$ necessarily containing exactly one vertex $d_{t+1} \in D$. Otherwise, $V^0 \subseteq \cup_{i=1}^{t} V(P_i)$, and the construction is complete. Note that the $d_i$ are not necessarily distinct, but the $v_i$ are. Also, while the paths of the construction are distinct, they are not necessarily vertex disjoint. We assume this construction and the associated notation for the rest of this paper.

Before proceeding to the general result we need three lemmas. The first is a technical result used in the proofs of the other two.

**Lemma 6** *Let $D$ be a minimum distance-$k$ dominating set of $G$, and let $x \in W_D(v)$. Let $P$ be a minimum-length path from $D$ to $x$, and let $z \neq v$ be a vertex on the subpath of $P$ from $v$ to $x$. Then every minimum-length path from $D$ to $z$ must contain $v$.*

**Proof:** Let $Q$ be a minimum-length path from $D$ to $z$ and $P'$ be the subpath of $P$ from $z$ to $x$. Clearly, $Q \cup P'$ is a path from $D$ to $x$ not longer than $P$, and, since $x \in W_D(v)$, $Q \cup P'$ must contain $v$. By the minimality of $P$, $P'$ does not contain $v$, and so $Q$ must contain $v$. □

Our second lemma is critical for establishing the desired relation between $V^0$ and $V^+$

**Lemma 7** $V^+ \subseteq \cup_{i=1}^t V(P_i)$.

**Proof:** Suppose $V^+ \neq \emptyset$ and let $v \in V^+$. By Lemma 3, there exists $x \in W_D(v) - V^+$. Let $P$ be a minimum-length path from $D$ to $x$. Lemma 5 implies that there is a vertex $z \in V^0$ on the subpath of $P$ from $v$ to $x$. By the construction, $z$ is on path $P_i$ for some $i$, and it follows from the minimality of $P_i$ that the subpath of $P_i$ from $D$ to $z$ must be of minimum length. Hence, by Lemma 6, $P_i$ must contain $v$. □

For each vertex $d \in D$, we let $q_d = |\{i : d \in P_i\}|$, that is, the number of paths of the construction that contain $d$. The third lemma eliminates a troublesome case from the general proof.

**Lemma 8** *If* $d \in V^+ \cap D$, *then* $q_d \geq 2$

**Proof:** By Lemma 7, $d$ is on path $P_i$ for some $i$. Since $d \in V^+$ and $v_i \in V^0$, $P_i$ contains a vertex $y$ adjacent to $d$. Furthermore, since $d \in V^+$, $(D - \{d\}) \cup \{y\}$ does not distance-$k$ dominate $G - d$. Hence, there exists a vertex in $W_D(d)$ that is not distance-$k$ dominated in $G - d$ by $y$. Among such vertices, choose $x$ to be of maximum distance from $d$, and let $P$ be a minimum-length path from $d$ to $x$. Observe, as in the proof of Lemma 3, that $x \notin V^+$ and that $P$ is also a minimum-length path from $D$ to $x$. By Lemma 5, $P$ contains a vertex $z \in V^0$. By the construction, $z$ is on $P_j$ for some $j$, and, by Lemma 6, $d$ is also on $P_j$. If $i = j$, the minimality of $P_i$ and $P$ implies that the two subpaths from $d$ to $z$ are of equal length. Hence, $y$ distance-$k$ dominates $x$ in $G - d$, contrary to the choice of $x$. Therefore, $i \neq j$, and $q_d \geq 2$. □

We proceed to the main result of this section.

**Theorem 9** *For any graph $G$, $|V^0| \geq \frac{2}{2k-1}|V^+|$.*

**Proof:** If $V^0 = \emptyset$, then Lemmas 3 and 5 imply $V^- = V$, and the inequality is trivially true. Otherwise, for each $d \in D$, let $P_{i_1}, P_{i_2}, ..., P_{i_{q_d}}$ be the $q_d$ paths of the construction which contain $d$. Let $V_d^+ = V^+ \cap \left(\cup_{j=1}^{q_d} V(P_{i_j})\right)$, that is, the vertices in $V^+$ that are in the union of the paths containing $d$, and let $V_d^0 = \{v_{i_1}, v_{i_2}, \ldots, v_{i_{q_d}}\}$. By the construction, $|V_d^0| = q_d$. Also, each path $P_i$ contains at most $k-1$ vertices other than $d$ and $v_i$. If $q_d = 1$, then, by Lemma 8, $d \notin V^+$ and $|V_d^+| \leq k-1$. In this case $\frac{|V_d^0|}{|V_d^+|} \geq \frac{1}{k-1} > \frac{2}{2k-1}$. For $q_d \geq 2$, first observe that $|V_d^+| \leq q_d(k-1)+1$, which implies $|V_d^0| \geq \frac{q_d}{q_d(k-1)+1}|V_d^+|$. Also observe that the function $f(x) = \frac{x}{x(k-1)+1}$ is strictly monotonically increasing. Therefore, $|V_d^0| \geq \frac{2}{2(k-1)+1}|V_d^+|$. Thus, if $q_d \geq 1$, then $|V_d^0| \geq \frac{2}{2k-1}|V_d^+|$. Summing this inequality over all $d \in D$ such that $q_d \geq 1$, the left hand sum equals the number of paths in the construction which is bounded above by $|V^0|$. Also, by Lemma 7, the sum of $|V_d^+|$ over all $d$ with $q_d \geq 1$ is bounded below by $|V^+|$. The result follows. □

Note that the inequality of the theorem is strict if there is a vertex $d \in D$ with $q_d > 0$ and $q_d \neq 2$, or if a path contains more than one vertex not in $V^+$, or if two paths intersect at a vertex not in $D$, or if the length of $P_i$ is less than $k$ for some $i$. We summarize this remark in the following corollary.

**Corollary 10** *Let $D$ be a minimum distance-$k$ dominating set of a graph $G$. If $V^0 \neq \emptyset$ and $|V^0| = \frac{2}{2k-1}|V^+|$, then*

1. *if $i \neq j$, then $V(P_i) \cap V(P_j) = \begin{cases} \{d_i\} & \text{if } d_i = d_j \\ \emptyset & \text{otherwise} \end{cases}$,*
2. *$q_d = 0$ or $2$ for all $d \in D$,*
3. *for each $i$, $P_i$ is a path of length $k$ containing $v_i \in V^0$ and $k$ vertices in $V^+$, and*

*4. if $v \in V^0$, then $v = v_i$ for some $i$ and $dist(v, D) = k$.*

# 3 Conditions Satisfied by Extremal Graphs

The connected extremal graphs for Theorem 9, that is, ones for which equality holds, can be divided into two classes depending on whether or not $V^0 = \emptyset$. As was noted in the proof to Theorem 9, if $V^0 = \emptyset$, then Lemmas 3 and 5 imply $V^- = V$ and equality holds. Investigating the structure of such graphs presents an interesting and difficult problem which we will not address here. An initial discussion appears in [1]. Instead, we turn our attention to the companion problem of investigating connected extremal graphs for which $V^0 \neq \emptyset$. This section establishes some important structural properties for such graphs.

A *$V$-structure* is a path of length $2k$. The middle vertex of the path is the *base* vertex of the $V$-structure, and the two subpaths emanating from the base vertex are its *legs*. Figure 1 illustrates two $V$-structures and also establishes the notation for their vertices. *Structure $i$* refers to the $V$-structure whose base vertex is $v_{i0}$.

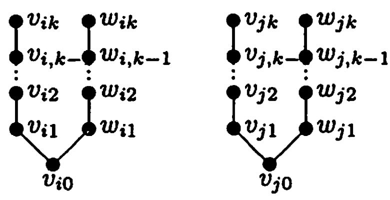



Figure 1: Structures $i$ and $j$

The next two lemmas describe elementary properties for extremal graphs.

**Lemma 11** *Let $D$ be a minimum distance-$k$ dominating set of a connected graph $G$, and let $H$ be the union of the paths $P_i$ from the construction in Section 2. If $G$ satisfies $|V^0| = \frac{2}{2k-1}|V^+| \neq 0$, then*

1. *$H$ is the union of vertex-disjoint $V$-structures whose base vertices are vertices in $D$,*

2. *the two end vertices of each of the* $V$*-structures are in* $V^0$ *and the rest of their vertices are in* $V^+$*,*

3. $H$ *spans* $G$*, and*

4. $D$ *is the unique minimum distance-*$k$ *dominating set of* $G$*.*

**Proof:** Results 1 and 2 follow immediately from Corollary 10. Suppose $V^- \neq \emptyset$, so there is a vertex $w \in V^-$. The shortest path from $D$ to $w$ must have length no more than $k$ and by Lemma 5 it must contain a vertex $v \in V^0$. In this event, $d(v, D) < k$ which contradicts Corollary 10. Thus, $V^- = \emptyset$ and result 3 follows from the construction and Lemma 7. Finally, suppose $D'$ is a minimum distance-$k$ dominating set which contains a vertex $v \notin D$. Since $H$ spans $G$, $v$ is in a $V$-structure of $H$ and since $v \notin D$ there is a vertex in $V^0$ whose distance from $v$ is less than $k$. Apply the construction of Section 2 based on $D'$. Since $G$ is extremal, $v$ will appear as the base vertex of a $V$-structure and by Corollary 10 no vertex of $V^0$ will be distance less than $k$ from $v$. This contradiction implies $D' \subseteq D$ and result 4 follows by the minimality of $D$. □

We define a graph $G$ to be *extremal-feasible* if $G$ is connected and contains a spanning subgraph $H$ which is the union of vertex disjoint $V$-structures whose set $B$ of base vertices forms a unique minimum distance-$k$ dominating set of $G$. According to Lemma 11, a connected extremal graph with $V^0 \neq \emptyset$ must be extremal-feasible. As we shall see, however, not all extremal-feasible graphs are extremal. Further characterization requires consideration of the edges in $G - H$.

The *height* of a vertex is defined to be its distance from $B$. An edge is *horizontal* if it joins two vertices of the same height. An *exterior* edge is one which joins two vertices in different $V$-structures. The following lemma restricts the types of edges in an extremal-feasible graph.

**Lemma 12** *If a connected graph* $G$ *satisfies* $|V^0| = \frac{2}{2k-1}|V^+| \neq 0$*, then all of the edges in* $E(G) - E(H)$ *are horizontal and exterior.*

**Proof:** The following are the three types of edges that are prohibited by the definition of horizontal and exterior. In each case, we reach a contradiction by showing a vertex known to be in $V^+$ would not be if the edge were present.

1. $v_{it}v_{is}$ with $s \geq t+2$: vertex $v_{i,s-1} \notin V^+$ since $B$ distance-$k$ dominates $G - v_{i,s-1}$.

2. $v_{it}w_{is}$ with $s \geq t \geq 1$: vertex $v_{i0} \notin V^+$ since $(B - \{v_{i,0}\}) \cup \{v_{it}\}$ distance-$k$ dominates $G - v_{i0}$.

3. $v_{it}v_{js}$ with $s > t$: vertex $v_{j,s-1} \notin V^+$ since $(B - \{v_{j,0}\}) \cup \{w_{j1}\}$ distance-$k$ dominates $G - v_{j,s-1}$.

□

Lemma 12 directs our attention to the horizontal, exterior edges of $G$. The following lemma provides a further reduction.

**Lemma 13** *Suppose $G$ is an extremal-feasible graph and $e \in E(G) - E(H)$ is a cut edge of $G$. If $G$ is extremal, then $G - e$ is extremal.*

**Proof:** Note that for any edge e, $\gamma_{\leq k}(G - e) \geq \gamma_{\leq k}(G)$. Also, if $G$ is extremal then $e$ being an exterior edge implies $B$ dominates $G - e$; hence, $\gamma_{\leq k}(G-e) = \gamma_{\leq k}(G)$. It follows that, if $v \in V^+(G)$, then $v \in V^+(G-e)$; so $|V^+(G - e)| \geq |V^+(G)|$. By Theorem 9, $|V^0(G - e)| \geq \frac{2}{2k-1}|V^+(G - e)| \geq \frac{2}{2k-1}|V^+(G)| = |V^0(G)|$ where the last equality is due to $G$ being extremal. The result follows since $|V(G - e)| = |V(G)|$. □

Lemma 13 leads us to consider horizontal exterior edges which are not cut edges, that is, edges contained in cycles of $G$. We define an *outer cycle* of an extremal-feasible graph $G$ to be a cycle of $G$ with no edges joining base vertices of $G$. An outer cycle $C$ is *proper* if $v_{i,0}$ in $C$ implies that $C \cap V_i$ is connected. Let $C$ be a proper outer cycle and let $V_0, V_1, \ldots, V_a$ be the $V$-structures with base vertices in C, where the indices are chosen to obey one of the cyclic orderings of $C$. Furthermore, by appropriately labeling the legs of the $V$-structures, we may assume that the shortest path in $C - \{v_{0,0}\}$ which has a vertex in $V_i$ and a vertex in $V_{i+1}$ has end vertices $v_{i,s}$ and $w_{i+1,t}$. We define $K(C)$ to be the graph obtained by taking the union of $C$ with $V_0, V_1, \ldots, V_a$. A vertex in $V(C) \cap B$ is said to be *anchored* if it is adjacent to a vertex in $B - V(C)$.

Let $V_i$ be a $V$-structure contained in $K(C)$ where $C$ is a proper outer cycle. For each $V_i$ there are non-negative integers $s$ and $t$ such that $C \cap V_i$ is the path $< v_{i,s}, v_{i,s-1}, \ldots, v_{i,1}, v_{i,0}, w_{i,1}, \ldots, w_{i,t-1}, w_{i,t} >$. We define the *base subpath* of $V_i$, denoted $bsp_i(C)$, to be the path $< v_{i,s-1}, \ldots, v_{i,1}, v_{i,0}, w_{i,1}, \ldots, w_{i,t-1} >$. The definition of outer cycle implies that neither $s$ nor $t$ equals 0, hence $bsp_i(C)$ is not empty (it always contains at least $v_{i,0}$). Also, if $G$ is extremal, then every vertex in $bsp_i(C)$ must be in $V^+$. A vertex $w \in bsp_i(C)$ is $V^+$-feasible if $K(C) - w$ contains an anchored vertex free path $P$ such that $|V(P)| > (2k+1)(|V(P) \cap (B - \{v_{i,0}\})| + 1)$. $C$ satisfies the *cycle condition* if every vertex in $\cup_{i=0}^{a} bsp_i(C)$ is $V^+$-feasible. Note that, if $V(C) \cap B$ is empty, then $C$ vacuously satisfies the cycle condition.

The following theorem provides a necessary but not sufficient condition for extremality.

**Theorem 14** *If a connected graph $G$ satisfies $|V^0| = \frac{2}{2k-1}|V^+| \neq 0$, then every proper outer cycle in $G$ satisfies the cycle condition.*

**Proof:** Suppose $C$ is a proper outer cycle in $G$ which does not satisfy the cycle condition. Then there exists $w \in \cup_{i=0}^{a} bsp_i(C)$ which is not $V^+$-feasible. We will inductively construct a distance-$k$ dominating set of $G - w$ of size $|B|$ which will imply that $w \notin V^+$ and thus $G$ is not extremal. By relabeling $V$-structures and considering another cyclic ordering of $C$, if necessary, we may assume $w = v_{0,r}$ for some $r \in \{0, 1, \ldots, k-1\}$. Let $B_0 = B - \{v_{0,0}, v_{1,0}, \ldots, v_{a,0}\}$.

Let $L_0$ be the path $< v_{0,r+1}, v_{0,r+2}, \ldots, v_{0,k} >$, and let $L_i$ be the path $< v_{i,0}, v_{i,1} \ldots, v_{i,k} >$ for $1 \leq i \leq a$. Also, let $R_i$ be the path $< w_{i,1}, w_{i,2}, \ldots, w_{i,k} >$ for $1 \leq i \leq a$ and let $R_{a+1}$ be the path $< v_{0,r-1}, v_{0,r-2},$

$\ldots, v_{0,0}, w_{0,1}, \ldots, w_{0,k} >$. Finally, for $0 \leq i \leq a$ let $P_i$ be the shortest path in $C - w$ with end vertices in $L_i$ and $R_{i+1}$.

By construction, $L_0 \cup P_0 \cup R_1$ is a tree $T$. Since $w$ is not $V^+$-feasible the diameter of $T$ is no more than $2k$; hence, the radius of $T$ is at most $k$ and there exists a vertex in $T$ which distance k-dominates $T$. Among such vertices choose $x_0$ to be the one closest to $w_{1,1}$ in $T$. If $x_0$ does not distance k-dominate $v_{1,0}$, there exists a path, $Q$, in $T$ of length $2k$ with one end at $w_{1,1}$ which implies $|V(Q)| = 2k+1 = (2k+1)(|V(Q) \cap (B - \{v_{0,0})\}| + 1)$. Also, $V(Q) \cap B = \emptyset$, so, $Q$ is trivially anchor free.

Let $i \leq a$ and suppose we have found vertices $\{x_0, x_1, \ldots, x_{i-1}\}$ such that $B_0 \cup \{x_0, x_1, \ldots, x_{i-1}\}$ distance k-dominates $L_0 \cup R_1 \cup \ldots \cup L_{i-1} \cup R_i$. We choose $x_i$ in $L_i \cup P_i \cup R_{i+1}$ as close as possible to $w_{i+1,1}$ in the tree $L_i \cup P_i \cup R_{i+1}$ subject to the condition that $B_0 \cup \{x_0, x_1, \ldots, x_{i-1}, x_i\}$ distance k-dominates $L_0 \cup R_1 \cup \ldots \cup L_{i-1} \cup R_i \cup L_i$. There are two cases to consider.

Case 1. $B_0 \cup \{x_0, x_1, \ldots, x_{i-1}\}$ distance k-dominates $v_{i,0}$. As in the base case, since $w$ is not $V^+$-feasible, the vertex $x_i$ distance k-dominates $(L_i - v_{i,0}) \cup P_i \cup R_{i+1}$. Thus, $x_i$ distance k-dominates $R_{i+1}$ and $B_0 \cup \{x_0, x_1, \ldots, x_{i-1}, x_i\}$ distance k-dominates $L_0 \cup R_1 \cup \ldots \cup L_{i-1} \cup R_i \cup L_i \cup R_{i+1}$. If $x_i$ does not distance k-dominate $v_{i+1,0}$, then there exists an anchor free path, $Q$, in $L_i \cup P_i \cup R_{i+1}$ with one end at $w_{i+1,1}$ such that $|V(Q)| = 2k+1 = (2k+1)(|V(Q) \cap (B - \{v_{0,0}\})| + 1)$.

Case 2. $B_0 \cup \{x_0, x_1, \ldots, x_{i-1}\}$ does not distance k-dominate $v_{i,0}$ (which incidentally implies that $v_{i,0}$ is not an anchor). Inductively, there exists an anchor free path $Q$ in $L_0 \cup R_1 \cup \ldots \cup L_{i-1} \cup R_i$ with one end at $w_{i,1}$ such that $|V(Q)| = (2k+1)(|V(Q) \cap (B - \{v_{0,0}\})| + 1)$. If $x_i$ does not distance k-dominate $R_{i+1}$, then there exists a path, $Q'$, in $L_i \cup P_i \cup R_{i+1}$ of length $2k+1$. Since $w$ is not $V^+$-feasible, the tree $(L_i - v_{i,0}) \cup P_i \cup R_{i+1}$ has diameter at most $2k$. Thus, $v_{i,0}$ must be an end vertex of $Q'$ and $Q' \cup Q$ is an anchor free path with $|V(Q' \cup Q)| = 2k + 2 + (2k+1)(|V(Q) \cap$

$(B - \{v_{0,0}\})| + 1) = (2k+1)(|V(Q' \cup Q)| \cap (B - \{v_{0,0}\})| + 1) + 1$ vertices, contrary to $w$ not being $V^+$-feasible. Thus, $x_i$ distance k-dominates $R_{i+1}$ and $B_0 \cup \{x_0, x_1, \ldots, x_{i-1}, x_i\}$ distance k-dominates $L_0 \cup R_1 \cup \ldots \cup L_{i-1} \cup R_i \cup L_i \cup R_{i+1}$. If $B_0 \cup \{x_0, x_1, \ldots, x_{i-1}, x_i\}$ does not distance k-dominate $v_{i+1,0}$, then there exists a path, $Q'$, in $L_i \cup P_i \cup R_{i+1}$ with one end at $w_{i+1,1}$ such that $|V(Q')| = 2k+1$. If $v_{i,0}$ is not in $Q'$, then $Q'$ satisfies the inductive requirements. If $v_{i,0}$ is in $Q'$, then $Q \cup Q'$ is an anchor free path in $L_0 \cup R_1 \cup \ldots \cup L_i \cup R_{i+1}$ with $(2k+1)(|V(Q \cup Q') \cap (B - \{v_{0,0}\})| + 1)$ vertices.

After $a+1$ iterations we will have generated vertices $\{x_0, x_1, \ldots, x_a\}$ such that $B_0 \cup \{x_0, x_1, \ldots, x_a\}$ distance k-dominates $L_0 \cup R_1 \cup \ldots \cup L_a \cup R_{a+1}$. This implies $B_0 \cup \{x_0, x_1, \ldots, x_i\}$ distance k-dominates $G - w$. We observe that $|B_0 \cup \{x_0, x_1, \ldots, x_a\}| = |B_0| + a + 1 = |B|$; so $w \notin V^+$ and $G$ is not extremal. □

As an application of this theorem we have the following corollary which provides further insight into the structure of extremal graphs.

**Corollary 15** *If a connected graph $G$ satisfies $|V^0| = \frac{2}{2k-1}|V^+| \neq 0$, the following pairs of edges cannot occur:*

1. *$v_{i,t}w_{j,t}$ and $v_{j,s}w_{i,s}$ with $s \geq t \geq 1$, and*

2. *$v_{i,s}w_{j,s}$ and $w_{j,t}w_{i,t}$ with $s \geq t \geq 1$.*

**Proof:**

1. Let $C$ be the proper outer cycle $v_{i,t}, w_{j,t}, w_{j,t-1}, \ldots, w_{j,1}, v_{j,0}, v_{j,1}, \ldots, v_{j,s}, w_{i,s}, w_{i,s-1}, \ldots, w_{i,1}, v_{i,0}, v_{i,1}, \ldots, v_{i,t}$. Then $K(C) = V_i \cup V_j \cup v_{i,t}w_{j,t} \cup v_{j,s}w_{i,s}$. Let $P$ be a path in $K(C) - v_{i,0}$. If $|V(P) \cap B| \geq 1$, then $V(P) \cap B = \{v_{j,0}\}$ and $(2k+1)(|V(P) \cap (B - \{v_{i,0}\})| + 1) = 2(2k+1) = |V(K(C))| \geq |V(P)|$. Otherwise, $V(P) \cap B = \emptyset$ and either $V(P)$ is a subset of $V(L_i - v_{i,0}) \cup V(R_j)$ or $V(P)$ is a subset of $V(L_j - v_{j,0}) \cup V(R_i)$. In either situation, $|V(P)| \leq 2k < (2k+$

$1)(|V(P) \cap (B - \{v_{i,0}\})| + 1)$. Thus, in all cases $v_{i,0}$ is not $V^+$-feasible which implies $C$ does not satisfy the cycle condition violating Theorem 14.

2. Let $C$ be the proper outer cycle $v_{i,s}, w_{j,s}, w_{j,s-1}, \ldots, w_{j,t}, w_{i,t}, w_{i,t-1}, \ldots, w_{i,1}, v_{i,0}, v_{i,1}, \ldots, v_{i,s}$. Here, $K(C) = V_i \cup < v_{i,s}, w_{j,s}, w_{j,s-1}, \ldots, w_{j,t}, w_{i,t} >$. Let $P$ be a path in $K(C) - v_{i,s-1}$. If $|V(P) \cap B| = 1$ then $(2k+1)(|V(P) \cap (B - \{v_{i,0}\})| + 1) = 2(2k+1) > 2k+1+s-t+1 = |V(K(C))| \geq |V(P)|$. On the other hand, suppose $P$ is a path in $K(C) - v_{i,s-1} - v_{i,0}$. If $P$ is the path $v_{i,1}, v_{i,1}, \ldots, v_{i,s-2}$, then $|V(P)| = s - 2 < k < 2k+1$. Otherwise, $P$ contains at most $k - s + 1$ vertices in the left leg of $V_i$ and at most $k$ vertices in the right leg of $V_i$. Since $K(C)$ has only $s - t + 1$ vertices not in $V_i$, we have $|V(P)| \leq (k-s+1)+k+(s-t+1) = 2k-t+2 \leq 2k+1$ (where the last inequality follows from the assumption that $t \geq 1$). Thus, $v_{i,s-1}$ is not $V^+$-feasible and $C$ does not satisfy the cycle condition, once again violating Theorem 14.

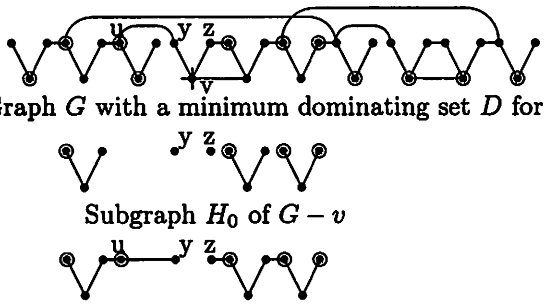
□

# 4 Characterization of Extremal Graphs When $k = 1$

In this section we show the necessary condition given in Theorem 14 is sufficient when $k = 1$, that is, for the case of normal domination (we will see in Section 5 that this is not true for all $k$). Since $k = 1$, we will use the terminology of normal domination. In particular, a *private neighbor* of a vertex $d$ in a dominating set is a vertex dominated only by $d$. Throughout this section we will assume $G$ is an extremal-feasible graph. We begin with a construction.

Let $v$ be a base vertex of an extremal-feasible graph $G$ with neighbors $y$ and $z$ in the same $V$-structure, and let $D$ be a minimum dominating set of $G - v$. Define $H_0$, a subgraph of $G$, to be the union of the $V$-

structures of $G$ whose base vertices are not in $D$ along with $y$ and $z$. Let $U = V(H_0) - (B \cup D) = \{u_1, u_2, \ldots, u_k\}$ and let $W = \{w_1, w_2, \ldots, w_k\}$ be a set of not necessarily distinct vertices of $D$ such that $(w_i, u_i)$ is an edge of $G$ for $1 \leq i \leq k$ (such edges must exist since $D$ dominates $G - v$). Finally define $H_1$ to be the subgraph of $G$ with vertex set $V(H_0) \cup W$ and edge set $E(H_0) \cup \{w_i u_i : 1 \leq i \leq k\}$. Observe that this construction implies $V(H_1) \cap B \cap D = \emptyset$. Figure 2 illustrates these concepts.




Graph $G$ with a minimum dominating set $D$ for $G - v$

Subgraph $H_0$ of $G - v$

Subgraph $H_1$ of $G - v$ with vertices in $D$ circled

Figure 2: Development of subgraphs $H_0$ and $H_1$

The following lemma is immediate from the construction.

**Lemma 16** *No edge in $H_1$ joins two vertices of $D$ or two vertices of $B$. Furthermore, if $< x, y, z >$ is a path in $H_1$ and $\{x, y, z\} \cap B = \emptyset$, then $y \in W$.*

**Lemma 17** *If $G$ satisfies the cycle condition, then $y$ and $z$ are in different components of $H_1$.*

**Proof:** Suppose $y$ and $z$ are in the same component of $H_1$. Then $H_1$ contains a path $P$ joining $y$ and $z$. Adding edges $zv$ and $vy$ to $P$ creates an outer cycle $C$ in $G$. (Note that when $k = 1$ every outer cycle is proper). Since $G$ satisfies the cycle condition, $v$ is $V^+$-feasible so there exists an anchor-free path $Q =< q_1, q_2, \ldots, q_t >$ in $K(C) - v$ such that $t > 3(|V(Q) \cap (B - \{v\})| + 1) = 3(|V(Q) \cap B| + 1)$. We choose $Q$ to be of minimum length. If $\{q_1, q_2, q_3\} \cap B \neq \emptyset$ then $< q_4, \ldots, q_t >$ is an anchor free path with $t - 3$

vertices. This contradicts the minimality of $Q$ since $t-3 > 3(|V(Q) \cap B|+1)-3 = 3(|V(Q-\{q_1,q_2,q_3\}) \cap (B)|+1)$. Thus, $\{q_1,q_2,q_3\} \cap B = \emptyset$; and, by Lemma 16, $q_2 \in W \subset D$. Similarly, $q_{t-1} \in W$. Note that $t = |V(Q)| \geq 4$, so, $q_2$ and $q_{t-1}$ are distinct. Further, $< q_2, \ldots, q_{t-1} >$ is an anchor-free path between two vertices of $D$ neither of which dominates a base vertex of the path. Let $Q'$ be a shortest subpath of $Q$ with distinct end vertices $s'$ and $t'$ both in $D$ and neither dominating a base vertex of $Q'$. Suppose $V(Q') \cap D$ contains a vertex $r$ distinct from $s'$ and $t'$. Since $r \in D$, it is not a base vertex and hence dominates at most one base vertex. This implies that either $r$ does not dominate a base vertex in the subpath of $Q'$ from $r$ to $s'$, or $r$ does not dominate a base vertex in the subpath of $Q'$ from $r$ to $t'$, contradicting the minimality of $Q'$. Thus $V(Q') \cap D = \{s', t'\}$. Lemma 16 indicates $s'$ and $t'$ are not adjacent in $H_1$, and thus $Q'$ contains at least one base vertex $b$. The vertex in $D$ which dominates $b$ cannot be a base vertex of $H_1$ since $V(H_1) \cap B \cap D = \emptyset$. Also, the vertex of $D$ which dominates $b$ cannot be a base vertex outside of $H_1$ because $b$ is not an anchor. Therefore, the vertex of $D$ which dominates $b$ must be in $Q'$. As $b$ is not dominated by $s'$ or $t'$, we have contradicted the fact that $V(Q') \cap D = \{s', t'\}$ and conclude that $y$ and $z$ cannot be in the same component of $H_1$. □

**Lemma 18** *If $G$ satisfies the cycle condition and $E$ is a component of $H_1$, then* $|V(E) \cap D| \geq |B \cap V(E)| + 1$.

**Proof:** Suppose $w$ is a vertex of $E$. If $w \in (D \cup U)$, $w$ is dominated by $V(E) \cap D$. Otherwise $w$ is a base vertex and is dominated by a vertex in either $V(E) \cap D$ or $B - V(E)$. On the other hand, if $w$ is not in $E$, the entire $V$-structure containing it also is not in $E$ and hence $w$ is dominated by $B - V(E)$. It follows that $G$ is dominated by $(V(E) \cap D) \cup (B - V(E))$.

By the construction, $V(E) \cap D \neq \emptyset$. Since $V(H_1) \cap B \cap D = \emptyset$, $(V(E) \cap D) \cup (B - V(E)) \neq B$. By the uniqueness of $B$, $|(V(E) \cap D) \cup (B - V(E))| > |B|$. Since $V(E) \cap B \cap D = \emptyset$, $|(V(E) \cap D) \cup (B - V(E))| = |V(E) \cap D| + |B - V(E)|$. Furthermore, $|B| = |B - V(E)| + |B \cap V(E)|$. The result follows. □

**Lemma 19** *If $G$ satisfies the cycle condition and $D$ is a minimum dominating set of $G - v$ for some base vertex $v$, then $D$ is the disjoint union of $D \cap B$ with $D \cap V(H_1)$.*

**Proof:** Let $w$ be a vertex of $D - B$, $r$ be a private neighbor of $w$, and $b$ be the base vertex in the $V$-structure containing $w$. If $b \notin D$, then $w \in H_1$. If $b \in D$, then $r$ is not in the $V$-structure containing $w$. Since $r$ is a private neighbor of $w$ and $w \in D - B$, $r$ cannot be dominated by a base vertex in $D$ so $r \in H_0$ implying $w \in H_1$. In either case $w \in D \cap V(H_1)$ so $(D - B) \subseteq (D \cap V(H_1))$. But $V(H_1) \cap B \cap D = \emptyset$ implies $(D - B) = (D \cap V(H_1))$. Thus $D = (D \cap B) \cup (D - B) = (D \cap B) \cup (D \cap V(H_1))$ and the union is disjoint. □

**Lemma 20** *If $G$ satisfies the cycle condition and $v$ is a base vertex of $G$, then $v \in V^+$.*

**Proof:** Let $D$ be a minimum dominating set of $G - v$. By Lemma 19, $|D| = |D \cap B| + |D \cap V(H_1)|$. Lemma 18 implies $|D| \geq |D \cap B| + |B \cap V(H_1)| + n$ where $n$ is the number of components in $H_1$. By Lemma 17, $n \geq 2$, so $|D| \geq |D \cap B| + |B \cap V(H_1)| + 2$. Also, $B = \{v\} \cup (D \cap B) \cup (B \cap V(H_1))$ implying $|B| - 1 = |D \cap B| + |B \cap V(H_1)|$. Therefore, $|D| \geq |B| - 1 + 2 = |B| + 1 = \gamma(G) + 1$ and $v \in V^+$. □

We are now in a position to state the characterization theorem which follows immediately from Lemmas 11 and 20 and Theorem 14.

**Theorem 21** *For $k = 1$, $G$ is extremal if and only if $G$ is extremal-feasible and every cycle of $G$ satisfies the cycle condition.*

# 5 A Special Case Characterization for Arbitrary $k$

When $k$ is greater than one, the characterization problem appears to be significantly more difficult, apparently because of the intricate way in which cycles joining the $V$-structures can be constructed. As an indication of this, we state below without justification (the proof will be presented elsewhere) the relatively complex result for a simple case involving a single outer cycle $C$ with one additional edge $e$ joining two base vertices of $C$. This addition of $e$ creates two shorter cycles $C_1$ and $C_2$, using $e$ and some of the edges of $C$ in each. Let $s_i$ be the number of vertices in $C_i$ which are not in $V$-structures whose base vertices are in $C_i$, where $i = 1, 2$. The following theorem presents the characterization for this class of graphs.

**Theorem 22** *Let $k \geq 2$ and $G$ be a graph obtained from an extremal-feasible graph by adding a single proper outer cycle $C$ whose exterior edges join end vertices of the $V$-structures, edge $e$ between two base vertices of $C$, and arbitrary edges between base vertices not in $C$. Then $G$ is extremal if and only if*

1. *$G - e$ is extremal and*
2. *either*
   (a) *both $s_1 > 0$ and $s_2 > 0$ or*
   (b) *at least one of $s_1$ and $s_2$ is at least 2k-1.*

Consider the graph shown in Figure 3. It is of the type under consideration here. Notice that $k = 3$ and, without loss of generality, $s_1 = 0$, and $s_2 = 4$. Thus it fails to meet the requirements given in Theorem 22 for extremal graphs. On the other hand, it does obey the cycle condition. To see this, observe that $u$ and $v$ are the only base vertices in the outer cycle $C$. If any vertex of the $V$-structure containing $v$ that is at distance at most two from $v$ is removed, the vertices numbered 1 through 8 form a path with $8 = 2k+1+1$ vertices. Since this path contains no base vertices, the cycle condition is satisfied. A similar argument shows any vertex that is distance at most two from $u$ and is in the $V$-structure containing $u$ also leaves a path with $2k + 2$ vertices. It follows from this example that the necessary condition for extremality is not sufficient for $k \geq 2$.

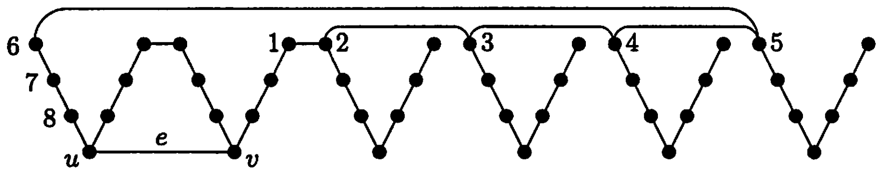


Figure 3: Example of a non-extremal graph

# 6 Concluding Remarks

There is much work remaining to be done on the question discussed in this paper. Although it appears that a complete characterization of extremal graphs for arbitrary $k$ may be difficult, it is reasonable to expect that subproblems will be amenable to solution. In particular, graphs containing a single cycle where all exterior edges in the cycle are at the same level might be classifiable. It does not appear that the general situation for $k = 2$ is appreciably easier than for arbitrary $k > 1$.

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