Some families of 3-Equitable Graphs

Mukund V. Bapat, N.B.Limaye

Abstract: Let G be a simple graph with vertex set V and edge set E. A vertex labeling $f : V \longrightarrow \{0,1,2\}$ induces an edge labeling $f : E \longrightarrow \{0,1,2\}$ defined by $\bar{f}(uv) = \mid f(u) - f(v) \mid$. Let $v_f(i)$ denotes the number of vertices v with $f(v) = i, i = 0,1,2$. Similarly $e_f(i)$ denotes the number of edges uv with $\bar{f}(uv) = i, i = 0,1,2$. A graph is said to be 3-equitable if there exists a vertex labeling f such that $\mid v_f(i) - v_f(j) \mid \leq 1$ and $\mid e_f(i) - e_f(j) \leq 1$ for all $i \neq j, i,j = 0,1,2$. In which case f is called a **3-equitable** labeling.

In this paper we prove that following graphs are three equitable: (1) Helm graph H_n ($n \geq 4$), (2) A Flower graph FL_n, (3) One point union $H_n^{(k)}$ of k-copies of $H_n, k \geq 1$, (4) One point union $K_4^{(k)}$ of k copies of K_4, (5) A K_4-snake of n blocks, each equal to K_4, (6) A C_t- snake of n blocks $t = 4,6$ and $t = 5$ with n not congruent to 3 modulo 6.

Introduction

Through out this paper all graphs are finite, simple and undirected. Let G be a graph with vertex set V and edge set E. A vertex labeling $f : V \longrightarrow \{0,1,2\}$ induces an edge labeling $\bar{f} : E \longrightarrow \{0,1,2\}$ defined by $\bar{f}(uv) = \mid f(u) - f(v) \mid$. By $v_f(0)$, $v_f(1)$ and $v_f(2)$. We mean the number of vertices u with $f(v) = 0$ $f(v) = 1$ and $f(v) = 2$ respectively. Similarly by $e_f(0)$, $e_f(1)$ and $e_f(2)$ we mean the number of edges labeled $0,1$ and 2 respectively. A Graph G is said to be **3-equitable** if there exists a vertex labeling $f : V \longrightarrow \{0,1,2\}$ of G such that $| v_f(i) - v_f(j) | \leq 1$ and $| e_f(i) - e_f(j) | \leq 1$ for all $i \neq j, i, j = 0,1,2$. I. Cahit [1] introduced the concept of 3-equitable labelings of Graphs and he proved that the cycle C_n and the wheel W_n are 3-equitable iff n is not $\equiv 3(\bmod 6)$.

3-Equitable Labeling of Helms

Helm H_n is a graph defined as follows:

$$V(H_n) = \{v_0, v_1, v_2, \cdots, v_n, w_1, w_2, \cdots, w_n\}$$

$$E(H_n) = \{v_0 v_i / 1 \leq i \leq n\} \cup \{v_i v_{i+1} / 1 \leq i \leq n\} \cup \{v_i w_i / 1 \leq i \leq n\}$$

where $(i+1)$ is taken modulo n. Thus H_n has $(2n+1)$ vertices and $3n$ edges.

Theorem 1: Helm graph H_n is 3-equitable.

Proof: Let $n = 3t + r$ $r = 1,2,3$.

Case (i): Let $r = 1$. We define $f : V(H_{3t+1}) \longrightarrow \{0,1,2\}$ as follows:

$f(v_0) = 0$,

$f(v_i) = 0 \quad 1 \leq i \leq t, \quad f(v_{t+2k-1}) = 1 \quad 1 < k \leq t$

$f(v_{t+2k}) = 2 \quad 1 \leq k \leq t, \quad f(3t+1) = 2$

$f(w_1) = 1, \quad f(w_i) = 2, \quad 2 \leq i \leq t,$

$f(w_{t+2k-1}) = 1, \quad 1 \leq k \leq t, \quad f(w_{t+2k}) = 0, \quad 1 \leq k \leq t$

$f(w_{3t+1}) = 2.$

Case (ii): $r = 2$ We define $f : V(H_{3t+2}) \longrightarrow \{0,1,2\}$ as follows:

$f(v_0) = 0, \quad f(v_i) = 0 \quad 1 \leq i \leq t,$

$f(v_{t+2k-1}) = 1 \quad 1 \leq k \leq t, \quad f(v_{t+2k}) = 2 \quad 1 \leq k \leq t,$

$f(v_{3t+1}) = 2, \quad f(v_{3t+2}) = 2, \quad f(w_1) = 1,$

$f(w_i) = 2, \quad 2 \leq i \leq t, \quad f(w_{t+2k-1}) = 1, \quad 1 \leq k \leq t,$

$f(w_{t+2k}) = 0, \quad 1 \leq k \leq t, \quad f(w_{3t+1}) = 2, \quad f(w_{3t+2}) = 1.$

Case (iii): $r = 3$ We define $f : V(H_{3t+3}) \longrightarrow \{0, 1, 2\}$ as follows:

$f(v_0) = 0, \quad f(v_i) = 0 \quad 1 \leq i \leq t,$

$f(v_{t+2k-1}) = 1 \quad 1 \leq k \leq t, \quad f(v_{t+2k}) = 2 \quad 1 \leq k \leq t-1,$

$f(v_{3t}) = 0, \quad f(v_{3t+1}) = f(v_{3t+2}) = f(v_{3t+3}) = 2, \quad f(w_1) = 1,$

$f(w_i) = 2, \quad 2 \leq i \leq t, \quad f(w_{t+2k-1}) = 1, \quad 1 \leq k \leq t,$

$f(w_{t+2k}) = 0, \quad 1 \leq k \leq t-1, \quad f(w_{3t}) = 1,$

$f(w_{3t+1}) = 0, \quad f(w_{3t+2}) = 2, \quad f(w_{3t+3}) = 1.$

The chart below shows that the labeling f defined above is a 3-equitable labeling.

Value of r	$e_f(0)$	$e_f(1)$	$e_f(2)$	$v_f(0)$	$v_f(1)$	$v_f(2)$
1	$3t+1$	$3t+1$	$3t+1$	$2t+1$	$2t+1$	$2t+1$
2.	$3t+2$	$3t+2$	$3t+2$	$2t+1$	$2t+2$	$2t+2$
3.	$3t+3$	$3t+3$	$3t+3$	$2t+2$	$2t+3$	$2t+2$

□

A graph similar to Helm H_n is **flower graph** FL_n. Now we proceed to show that Fl_n, a flower graph is 3-equitable. A flower graph FL_n is obtained from Helm graph H_n by joining each pendant vertex to the central vertex by a separate edge. Thus for $FL_n, \mid V(FL_n) \mid = 2n + 1$ and $\mid E(FL_n) \mid = 4n$.

Theorem 2: A flower graph FL_n is 3-equitable for all $n \geq 3$.

Proof: Let $n = 3t + r; r = 1, 2, 3$.

Case (i): $n = 3$. Define $f : V(FL_3) \longrightarrow \{0, 1, 2\}$ as follows: $f(v_0) = 0$, $f(v_1) = 0, f(v_2) = 2 = f(v_3), f(w_1) = 0, f(w_2) = 1 = f(w_3)$. One can see that $v_f(0) = 3, v_f(1) = 2, v_f(2) = 2$ and $e_f(0) = e_f(1) = e_f(2) = 4$.

Case (ii) : $r = 1$ and $t > 0$. Theorem 1 we give 3-equitable labeling of H_{3t+1}. The same labeling is also a 3-equitable labeling of FL_{3t+1}.

Case (iii) $r = 2$. Suppose $t = 1$, i.e. $n = 5$. We define the function $f : V(FL_5) \longrightarrow \{0, 1, 2\}$ as follows: $f(v_0) = 0, f(v_1) = f(w_4) = f(w_5) = 0$, $f(v_2) = f(v_3) = f(w_2) = f(w_3) = 1, f(v_4) = f(v_5) = f(w_1) = 2$. This gives $v_f(0) = 4, v_f(1) = 4, v_f(2) = 3$ and $e_f(0) = 7, e_f(1) = 6, v_f(2) = 7$. Thus f is 3-equitable labeling.

For $t \geq 2$ we define $f : V(FL_{3t+2}) \longrightarrow \{0, 1, 2\}$ as follows.

$f(v_0) = 0, \quad f(v_i) = 0 \quad 1 \leq i \leq t, \quad f(v_{t+2k-1}) = 1, \quad 1 \leq k \leq t$

$f(v_{t+2}) = 1, \quad f(v_{t+2k}) = 2, \quad 2 \leq k \leq t, \quad f(v_{3t+1}) = f(v_{3t+2}) = 2,$

$f(w_i) = 2, \quad 1 \leq i \leq t, \quad f(w_{t+2k-1}) = 1, 1 \leq k \leq t, \quad f(w_{t+2}) = 2,$

$f(w_{t+2k}) = 0, 2 \leq k \leq t, \quad f(3t + 1) = 0, \quad f(3t + 2) = 1.$

Case (iv): $r = 3, t > 0$.

For $t = 1$, we define the map f as follows: $f(v_0) = 0 = f(v_1)$,

$f(v_2) = f(v_3) = 1, f(v_4) = f(v_5) = f(v_6) = f(w_1) = 2,$

$f(w_2) = f(w_3) = 1, f(w_4) = f(w_6) = 0, f(w_5) = 1.$

Clearly $e_f(0) = e_f(1) = e_f(2) = 8$. Also $v_f(0) = 4, v_f(1) = 5, v_f(2) = 4$.

For $t \geq 2$, we define f as follows:

$f(v_0) = 0, f(v_i) = 0, \quad 1 \leq i \leq t, \quad f(v_{t+2k-1}) = 1, \quad 1 \leq k \leq t,$

$f(v_{t+2}) = 1, f(v_{t+2k}) = 2 \quad 2 \leq k \leq t-1, \quad f(3t) = 0,$

$f(v_{3t+1}) = f(3t+2) = f(v_{3t+3}) = 2, \quad f(w_i) = 2, \quad 1 \leq i \leq t,$

$f(w_{t+2k-1}) = 1, \quad 1 \leq k \leq t, \quad f(w_{t+2}) = 2,$

$f(w_{t+2k}) = 0, 2 \leq k \leq (t-1),$

$f(w_{3t}) = 1, \quad f(w_{3t+1}) = f(w_{3t+2}) = 0, \quad f(w_{3t+3}) = 1.$

The table below shows that f is a 3-equitable labeling.

r	$v_f(0)$	$v_f(1)$	$v_f(2)$	$e_f(0)$	$e_f(1)$	$e_f(2)$
1	$2t+1$	$2t+1$	$2t+1$	$4t+1$	$4t+2$	$4t+1$
2	$2t+1$	$2t+2$	$2t+2$	$4t+3$	$4t+2$	$4t+3$
3	$2t+2$	$2t+3$	$2t+2$	$4t+4$	$4t+4$	$4t+4$

□

One point union $H_n^{(k)}$ of k-copies of Helm H_n is defined as follows: The k-copies of H_n are labeled as $H_{n,1}H_{n,2}, \ldots H_{n,k}$. The common vertex to all copies is the central vertex v_0.

$$V\left(H_n^{(k)}\right) = \{v_o, v_{1,i}, \ldots, v_{n,i}, w_{1,i}, \ldots, w_{n,i} \mid 1 \leq i \leq k.\}$$

$$E\left(H_n^{(k)}\right) = \{v_0v_{j,i}, v_{j,i}v_{j+1,i}, v_{j,i}w_{j,i} \mid j = 1, \ldots, n, i = 1 \cdots k\},$$

where $j+1$ is taken modulo n. Note that $\mid V(H_n^{(k)}) \mid = 2kn + 1$ and $\mid E(H_n^{(k)}) \mid = 3kn$. We prove that $H_n^{(k)}$ is 3-equitable for $k \geq 1$. We treat the cases $n = 4, 5, 6$ separately and give a common proof for $n \geq 7$.

Theorem 3: $H_n^{(k)}$ is 3-equitable for all positive integers $n \geq 4$ and $k \geq 1$.

Proof: First we discuss the 3-equitable labeling of $H_4^{(k)}$. In Theorem 1 we proved that H_4 is 3-equitable. For $k \geq 2$, we define four different labelings of H_4.

Type A: Define $f : V(H_4) \longrightarrow \{0,1,2\}$ as follows: $f(v_0) = f(v_4) = f(w_4) = 0, f(v_1) = f(v_3) = f(w_1) = 2, f(v_2) = f(w_3) = f(w_2) = 1$.

Type B: Define $f : V(H_4) \longrightarrow \{0,1,2\}$ as follows: $f(v_0) = f(v_4) = f(w_2) = f(v_1) = 0, f(v_3) = f(w_1) = f(w_3) = 1, f(v_2) = f(w_4) = 2$.

Type C: Define $f : V(H_4) \longrightarrow \{0,1,2\}$ as follows: $f(v_0) = f(v_4) = f(w_2) = f(v_2) = 0, f(v_1) = f(w_4) = f(w_1) = f(w_3) = 2, f(v_3) = 1$.

Type D: Define $f : V(H_4) \longrightarrow \{0,1,2\}$ as follows: $f(v_0) = f(v_1) = f(w_2) = 0, f(v_4) = f(v_3) = f(w_3) = f(w_4) = 1, f(v_2) = f(w_1) = 2$.

Case (i): $k = 2$. We give labeling A to both $H_{4,1}$ and $H_{4,2}$.

Case (ii): $k = 3t + r$ where $r = 0$ or $r = 2$. We give labeling A to $H_{4,1}$ and $H_{4,2}$. For $1 \leq \alpha \leq t$, we give labeling B to $H_{4,3\alpha}$, labeling C to $H_{4,3\alpha+1}$ and labeling D to $H_{4,3\alpha+2}$.

Case (iii): $k = 3t + 1$. We give labeling A to $H_{4,1}$ For $1 \leq \alpha \leq t$, we give labeling to C to $H_{4,3\alpha-1}$, labeling D to $H_{4,3\alpha}$ and labeling B to $H_{4,3\alpha+1}$.

Since all the labelings A, B, C, D label the edges equitably, the resulting labeling is equitable on the edges of $H_4^{(k)}$. Following table shows that it labels the vertices also equitably.

k	$v_f(0)$	$v_f(1)$	$v_f(2)$
1	3	3	3
2	5	6	6
k=3t	3k-t	3k-t+1	3k-t
k=3t+1	3k-t	3k-t	3k-t
k=3t+2	3k-t-1	3k-t	3k-t

For $H_5^{(k)}$. We define below 3 types of labelings called B, C and D which suitably can be used to construct a 3-equitable labeling of $H_5^{(k)}$.

Type B: Define $f : V(H_5) \longrightarrow \{0,1,2\}$ as follows: $f(v_0) = 0, f(v_1) = f(w_3) = f(w_4) = 0, f(v_2) = f(v_3) = f(w_2) = 1, f(v_4) = f(v_5) = f(w_1) = f(w_5) = 2$.

Type C: Define $f : V(H_5) \longrightarrow \{0,1,2\}$ as follows: $f(0) = 0, f(v_1) = f(v_5) = f(w_4) = 0, f(v_3) = f(w_1) = f(w_3) = f(w_5) = 1, f(v_2) = f(v_4) = f(w_2) = 2$.

Type D: Define a function $f : V(H_5) \longrightarrow \{0,1,2\}$ as follows: $f(v_0) = 0, f(v_1) = 0, f(v_3) = 0, f(v_4) = f(v_5) = 1, f(v_2) = f(w_1) = 2, f(w_2) = f(w_3) = 0, f(w_4) = 1, f(w_5) = 2$.

One can easily see that B and C are 3-equitable labelings but D is not. However, each of them label the edges equitable.

That H_5 is 3-equitable, is proved in theorem 1.

Case (i): $k = 2$. Define f by assigning labeling B to one copy of H_5 and labeling C to the second copy of H_5.

Case (ii) $k \geq 3$. Let $k = 3t + r, r = 0,1,2$. This means the number of vertices is $10(3t + r) + 1$ and the number of edges is $15(3t + r)$.

We assign labeling B to $H_{5,1}$, labeling C to $H_{5,2}$. After this stage sequence of B, C and D will be repeated periodically, i.e. for $1 \leq \alpha \leq t$, we assign labeling B to $H_{5,3\alpha}$, labeling C to $H_{5,3\alpha+1}$ and labeling D to $H_{5,3\alpha+2}$. Since, all the labeling B C, and D label the edges equitably, f also labels the edges equitably.

The following table gives the label distribution of vertices which clearly shows that f is a 3-equitable labeling..

r	$v_f(0)$	$v_f(1)$	$v_f(2)$
0	$10t$	$10t$	$10t+1.$
1	$10t+3$	$10t+4$	$10t+4$
2	$10t+7$	$10t+7$	$10t+7$

For $(H_6)^{(k)}$ we define $f : V(H_6^{(k)}) \longrightarrow \{0,1,2\}$ as follows: $f(v_0) = 0$,

$f(v_{1,t}) = f(w_{1,t}) = f(w_{5,t}) = f(w_{6,t}) = 0$,

$f(v_{2,t}) = f(v_{4,t}) = f(w_{2,t}) = f(w_{4,t}) = 1$,

$f(v_{3,t}) = f(v_{5,t}) = f(v_{6,t}) = f(w_{3,t}) = 2$.

Clearly $v_f(0) = 4k+1, v_f(1) = 4k, v_f(2) = 4k$ and $e_f(0) = e_f(1) = e_f(2) = 6k$, i.e. f is a 3- equitable labeling of $H_6^{(k)}$.

Now let $n \geq 7$. Let $n = 3t + r, r = 1,2,3, t \geq 2$. Let the k copies of H_n be numbered as $H_{n,1}, \cdots, H_{n,k}$.

Case i: $r = 1$. We define three types of labelings of H_n as given below:

Type A: Define $f : V(H_{3t+1}) \longrightarrow \{0,1,2\}$ as $f(v_0) = 0$,

$f(v_i) = 0, f(v_{t+2i-1}) = 1, f(v_{t+2i}) = 2 = f(v_{3t+1}), 1 \leq i \leq t$,

$f(w_1) = 1, f(w_i) = 2 = f(w_{3t+1}), 2 \leq i \leq t,$

$f(w_{t+2i-1}) = 1, f(w_{t+2i}) = 0, 1 \leq i \leq t.$

Type B: Define $f : V(H_{3t+1}) \longrightarrow \{0,1,2\}$ as $f(v_0) = 0,$

$f(v_i) = 0, 1 \leq i \leq t, \quad f(v_{t+1}) = 0, \quad f(v_{t+2}) = 2,$

$f(v_{t+2i-1}) = 1, f(v_{3t+1}) = f(v_{t+2i}) = 2, 1 < i \leq t,$

$f(w_1) = 1, \quad f(w_i) = 2, 1 < i \leq t,$

$f(w_{t+2i-1}) = 1, f(w_{t+2i}) = 0, 1 \leq i < t,$

$f(w_{3t-1}) = 0, \quad f(w_{3t}) = 1, \quad f(w_{3t+1}) = 2.$

Type C: Define $f : V(H_{3t+1}) \longrightarrow \{0,1,2\}$ as $f(v_0) = 0,$

$f(v_i) = 0, 1 \leq i \leq t,$

$f(v_{t+1}) = 1, \quad f(v_{t+2}) = 0, \quad f(v_{3t+1}) = 2,$

$f(v_{t+2i-1}) = 1, f(v_{t+2i}) = 2, 1 < i \leq t,$

$f(w_1) = 1, \quad f(w_{t+2i-1}) = 1, 1 \leq i \leq t,$

$f(w_i) = 2, f(w_{t+2i}) = 0, 1 < i \leq t,$

$f(w_{t+2}) = 2, \quad f(w_{3t+1}) = 0.$

Following table gives the label distributions of the labelings A, B, C.

Type	$v_f(0)$	$v_f(1)$	$v_f(2)$	$e_f(0)$	$e_f(1)$	$e_f(2)$
A	$2t+1$	$2t+1$	$2t+1$	$3t+1$	$3t+1$	$3t+1$
B	$2t+2$	$2t$	$2t+1$	$3t+1$	$3t+1$	$3t+1$
C	$2t+2$	$2t+1$	$2t$	$3t+1$	$3t+1$	$3t+1$

Label vertices of $H_{n,i}$ as in type A if $i \equiv 1 (\bmod 3)$, as in type B if $i \equiv 2 (\bmod 3)$ and as in type C if $i \equiv 0 (\bmod 3)$. Note that the common vertex receives label zero in all the three labelings A, B, C. Since all these labelings are equitable on the edges, the resulting labeling is equitable on the edges. Following table shows that the label distribution on the vertices is 3-equitable:

Value of k	$v_f(0)$	$v_f(1)$	$v_f(2)$
$3m+1$	$m(6t+2)+2t+1$	$m(6t+2)+2t+1$	$m(6t+2)+2t+1$
$3m+2$	$m(6t+2)+4t+2$	$m(6t+2)+4t+1$	$m(6t+2)+4t+2$
$3m$	$m(6t+2)+1$	$m(6t+2)$	$m(6t+2)$

Case ii: $r = 2$. Again we define three types of labelings for H_n as follows:

Type A: Define $f : V(H_{3t+2}) \longrightarrow \{0,1,2\}$ as $f(v_0) = 0$,

$f(v_i) = 0, f(v_{t+2i}) = 2, 1 \le i \le t, f(v_{t+1}) = 0,$

$f(v_{t+2i-1}) = 1, 1 < i \le t, f(v_{3t+1}) = 0, f(v_{3t+2}) = 2,$

$f(w_1) = 1, f(w_{t+2}) = f(w_i) = 2 =, 1 \le i \le t,$

$f(w_{t+1}) = 1 = f(w_{3t+2}) = f(w_{t+4}), \quad f(w_{t+3}) = 0 = f(w_{3t+1}),$

$f(w_{t+2i-1}) = 1, f(w_{t+2i}) = 0, 3 \le i \le t.$

Type B: Define $f : V(H_{3t+2}) \longrightarrow \{0,1,2\}$ as $f(v_0) = 0$,

$f(v_i) = 0, 1 \le i \le t, \quad f(v_{t+1}) = 0, \quad f(v_{t+2}) = 2,$

$f(v_{t+2i-1}) = 1, f(v_{3t+1}) = f(v_{3t+2}) = f(v_{t+2i}) = 2, 1 < i \le t,$

$f(w_1) = 1, \quad f(w_i) = 2, 1 < i \le t,$

$f(w_{t+1}) = f(w_{t+2}) = 1, \quad f(w_{t+3}) = f(w_{t+4}) = 0,$

$f(w_{t+2i-1}) = 1, f(w_{t+2i}) = 0, 2 \le i \le t,$

$f(w_{3t+2}) = 1, \quad f(w_{3t+1}) = 2.$

Type C: Define $f : V(H_{3t+2}) \longrightarrow \{0,1,2\}$ as $f(v_0) = 0$,

$$f(v_i) = 0, \quad f(v_{t+2i-1}) = 1, \quad f(v_{t+2i}) = 2, \quad 1 \leq i \leq t,$$

$$f(v_{3t+1}) = 0, \quad f(v_{3t+2}) = 2,$$

$$f(w_1) = 1, \quad f(w_{3t+1}) = 0, f(w_{3t+2}) = 1 = f(w_{t+1}),$$

$$f(w_{t+2}) = 2,$$

$$f(w_i) = 2, f(w_{t+2i-1}) = 1, f(w_{t+2i}) = 0, 1 < i \leq t,$$

All the three labelings are equitable on the edges. Each label is received by $3t+2$ edges. The table below gives the label distributions on the vertices.

Type	$v_f(0)$	$v_f(1)$	$v_f(2)$
A	$2t+3$	$2t+1$	$2t+1$
B	$2t+2$	$2t+1$	$2t+2$
C	$2t+2$	$2t+2$	$2t+1$

Assign labeling B to $H_{n,1}$ and labeling C to $H_{n,2}$. For $i > 2$ to $H_{n,i}$ assign labeling B if $i \equiv 0(\bmod 3)$, labeling C if $i \equiv 1(\bmod 3)$ and labeling A if $i \equiv 2(\bmod 3)$. If $k = 1$ the labeling B is 3-equitable. Clearly, for $k > 1$ the resulting labeling is equitable on the edges and each label is assigned to $k(3t+2)$ edges. The table below shows that this labeling is equitable on the vertices also.

k	$v_f(0)$	$v_f(1)$	$v_f(2)$
$k = 3m$	$m(6t+4)$	$m(6t+4)$	$m(6t+4)+1$
$k = 3m+2$	$m(6t+4)+4t+3$	$m(6t+4)+4t+3$	$m(6t+4)+4t+3$
$k = 3m+4$	$m(6t+4)+8t+5$	$m(6t+4)+8t+6$	$m(6t+4)+8t+6$

Case iii: $r = 3$. Again we define a labelings for H_n as follows:

$f(v_0) = 0,$

$f(v_{3t}) = 0 = f(v_{3t-1}),\quad f(v_{3t+1}) = f(v_{3t+2}) = f(v_{3t+3}) = 2,$

$f(v_i) = 0, f(v_{t+2i-1}) = 1, f(v_{t+2i}) = 2, 1 \le i < t,$

$f(w_1) = 1,\quad f(w_{t+1}) = 0,\quad f(w_{t+2}) = 1,$

$f(w_{3t-1}) = f(w_{3t}) = f(w_{3t+3}) = 1,\quad f(w_{3t+1}) = 0,$

$f(w_{3t+2}) = 2,$

$f(w_i) = 2, 2 \le i \le t,$

$f(w_{t+2i-1}) = 1, f(w_{t+2i}) = 0, 1 < i < t.$

Clearly $v_f(0) = 2t+3, v_f(1) = v_f(2) = 2t+2$ and $e_f(0) = e_f(1) = e_f(2) = 3t+3$. In $H^{(k)}_{3t+3}$ we assign this labeling to all the blocks. One can easily see that the resulting labeling is 3-equitable. □

K_4-Snake And K_4-Star

A K_4 snake of length n with vertex set V and edge set E is a graph defined as follows: $V = \{v, v_2, \ldots, v_n\} \dot{\cup} \{a_i, b_i / i = 1, 2, \ldots, n\}$,
$E = \{v_i v_{i+1} / 1 \le i \le n\} \dot{\cup} \{a_i v_i, a_i v_{i+1}, a_i b_i, b_i v_i, b_i v_{i+1} / 1 \le i \le n\}$. Here $i+1$ is taken modulo n. Clearly, $|V| = 3n+1, |E| = 6n$.

Theorem 4: a K_4-Snake on n blocks is 3-equitable, $n \ge 2$.

Proof: Case (i): n is even. Define $f : V \longrightarrow \{0, 1, 2\}$ as follows:

$f(v_i) = 0, 1 \le i \le n, f(a_i) = f(b_i) = 1$ if i is odd and

$f(a_i) = f(b_i) = 2$ if i is even . Clearly, $v_f(0) = n+1, v_f(1) = v_f(2) = n$ and $e_f(0) = e_f(1) = e_f(2) = 2n$.

Case (ii): $n = 2m+1, m \in \mathbb{N}$. First we consider the case of $n = 3$. We define f as follows: $f(v_1) = 0, f(v_2) = 1, f(v_3) = 0 = f(v_4)$, $f(a_1) = 0, f(b_1) = 2, f(a_2) = 1 = f(b_2), f(a_3) = 2 = f(b_3)$. note that $v_f(0) = 4, v_f(1) = 3 = v_f(2), e_f(0) = e_f(1) = e_f(2) = 6$. Thus a K_4-Snake on 3-blocks of K_4 is 3-equitable.

For K_4 snake with $2m+1$ blocks $m > 1$, we first take 3-equitable labeling of K_4-snake with $2m-2$ blocks, as defined in case (i). This labeling is extended to a 3-equitable labeling by using labeling of K_4-Snake of length 3 given above. Note that $v_f(0) = 2m+2, v_f(1) = 2m+1, v_f(2) = 2m+1, e_f(0) = e_f(1) = e_f(2) = 4m+2$. That completes the proof. □

A K_4-Star $K_4^{(n)}$ is one point union of n-copies of K_4. The vertex set is $\{v_0\}\dot{\cup}\{v_{i,k} \mid 1 \leq i \leq 3, 1 \leq k \leq n.\}$ and the edge set is $\{v_0\ v_{i,k} \mid 1 \leq i \leq 3, 1 \leq k \leq n\}\dot{\cup}\{v_{i,k}\ v_{i+1,k} \mid 1 \leq i \leq 3, 1 \leq k \leq n\}$. Here $i+1$ is taken modulo 3.

Theorem 5: $K_4^{(n)}$ is 3-equitable for all $n \geq 2$.

Proof: v_0 is the common vertex of all copies of K_4.

Case (i): n is even. Define $f : V\left(K_4^{(n)}\right) \longrightarrow \{0,1,2\}$ as follows: $f(v_{1,t}) = 1$, if t is even and $f(v_{1,t}) = 0$ if t is odd, $f(v_{2,t}) = 1$ if t is even and $f(v_{2,t}) = 2$ if t is odd, $f(v_{3,t}) = 0$ if t is even and $f(v_{3,t}) = 2$ if t is odd.

One can easily check that $v_f(0) = n+1, v_f(1) = n = v_f(2)$, $e_f(0) = e_f(1) = e_f(2) = 2n$.

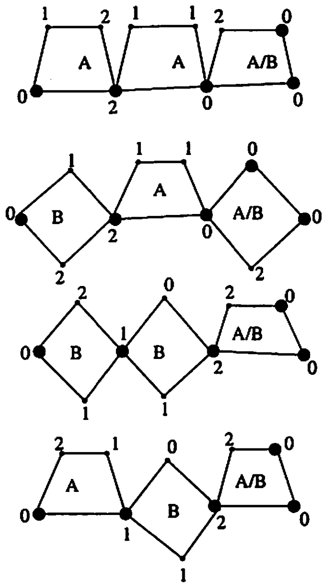
Case (ii) : $n = 2m+1$. For $K_4^{(3)}$, define $f : V\left(K_4^{(3)}\right) \longrightarrow \{0,1,2\}$ as follows. $f(v_{1,1}) = f(v_{2,1}) = f(v_{3,1}) = 1, f(v_{1,2}) = f(v_{2,2}) = 2, f(v_{3,2}) = 0$, $f(v_{1,3}) = 1, f(v_{2,3}) = 2, f(v_{3,3}) = 0$. Note that $e_f(0) = e_f(1) = e_f(2) = 6, v_f(0) = 3, v_f(1) = 4, v_f(2) = 3$.

To obtain an labeling of $K_4^{(2m+1)}$ first obtain the labeling of $K_4^{(2m-2)}$ using case (i) above. Extend this labeling by using labeling of $K_4^{(3)}$. Note that $v_f(0) = 2m + 1, v_f(1) = 2m + 2, v_f(2) = 2m + 1$, and $e_f(0) = e_f(1) = e_f(2) = 4m + 2$. This completes the proof. □

C_t-Snakes, $t = 4, 5, 6$

A C_4-snake $S(c_4, n)$ of length n is defined inductively as follows: For $n = 1$ we take C_4. If a C_4-snake of length $n - 1$ is constructed, then by identifying a non-cut vertex of an end block with one vertex of one more copy of C_4 we obtain a C_4-snake of length n. We note that every non-end block has two cut vertices and they may or may not be adjacent. A block is said to be of type A if the two cut vertices in a C_4 block are adjacent and of type B if they are at a distance two.

The figure below gives 3-equitable labelings of all possible C_4-snakes of length 3 or sections of length 3 of a bigger C_4-snake. The cut points and the potential cut points in the end blocks are indicated by bold circles around them. In each figure the right most block has two vertices labeled zero.



In all these labelings we have $v_f(0) = 4, v_f(1) = v_f(2) = 3$, $e_f(0) = e_f(1) = e_f(2) = 4$.

Theorem 6: C_4-snake $S(C_4, n)$ of length n is 3-equitable for all positive integers n.

Proof: When $n = 1, 2$ one can easily check that $S(C_4, n)$ is 3-equitable.

Let $n = 3t + r, r = 0, 1, 2$. A C_4-snake $S(C_4, n)$ of length n has $3n + 1$ vertices and $4n$ edges.

Case i: $r = 0$. If $t = 1$, we have already given 3-equitable labeling of $S(C_4, 3)$ above. Moreover, in these labelings both the end blocks have vertices with label 0 as potential cut vertices in a bigger snake.

Now let $n = 3t, t > 1$. We take a C_4-snake of length $3t$ and break it up into t copies of $S(C_4, 3)$. Each copy has one of the 8 possibilities $\{AAA, AAB, BAA, BBA, ABB, ABA, BAB, BBB\}$. We can now label them as shown in the figure above and then splice them again to form the original snake.

One can easily see that in the resulting labeling $v_f(0) = 3t + 1$, $v_f(1) = v_f(2) = 3t, e_f(0) = e_f(1) = e_f(2) = 4t$. Hence $S(C_4, 3t)$ is 3-equitable.

Case ii: $r = 1$. Consider a C_4 snake $S(C_4, 3t + 1)$. Label $3t$ consecutive blocks as in case 1. The cut vertex in the last block will have label zero. Label the last block as $\{0211\}$ in a cyclic manner. Here the label zero was already present. Clearly in this labeling $v_f(0) = 3t + 1 = v_f(2)$, and $v_f(1) = 3t + 2, e_f(0) = e_f(2) = 4t + 1, e_f(1) = 4t + 2$.

Case iii: $r = 2$. Consider C_4-snake of length $3t + 2$. The non-end blocks form a C_4-snake of length $3t$. Take a 3-equitable labeling of this truncated snake. At one end, label the last block as $\{0, 0, 2, 2\}$ in a cyclic manner. One of the vertex with label zero is from the labeling of the truncated snake. At the other end, label the last block as $\{0, 2, 1, 1\}$ in a cyclic manner. Again the vertex with label zero is from the labeling of the truncated snake. Clearly in the resulting labeling $v_f(0) = 3t + 2 = v_(1)$, $v_f(2) = 3t + 3, e_f(0) = e_f(2) = 4t + 3, e_f(1) = 4t + 2$.

Hence by $S(C_4, n)$ is 3-equitable for all n.

Remark: If we consider C_5-snake there are again two types of blocks; one in which two cut vertices are adjacent and another in which two cut vertices are at a distance 2. We have constructed basic labelings of all combinations of three such blocks. None of these labelings are 3-equitable. However by joining two such segments suitably we could construct 3-equitable labelings of every $S(C_5, 6)$. Using same techniques as in Theorem 6, we could construct 3-equitable labeling of $S(C_5, n), n \equiv 0, 1, 2, 4, 5 \pmod 6$. When $n = 6t + 3$ the number of edges is $30t + 15$. If we want a 3-equitable labeling then we must have $e_f(1) = e_f(0) = e_f(2) = 10t + 5$. However, in any block which is C_5 any labeling of the vertices with labels $0, 1, 2$ creates even number of edges with label 1. Hence $e_f(1)$ can never be $10t + 5$.

For a C_6-snake $S(C_6, n)$ of length n, in every block the distance between two cut points is 1 or 2 or 3, that is there are three types of blocks. We could give 3-equitable labeling for $S(C_6, n)$ using same techniques. Thus we have following:

Theorem 7: (a) A C_5-snake $S(C_5, n)$ is 3-equitable iff $n \equiv 0, 1, 2, 4, 5 \pmod 6$.
(b) A C_6-snake is 3-equitable for every natural number n.

The problem of C_t-snake becomes complicated since in any block two cut vertices can be at a distance $s, 1 \le s \le \left[\frac{t}{2}\right]$. However, if $t = 2s + 1$ and $n = 6m + 3$ the snake $S(C_t, n)$ has $3(2m + 1)(2s + 1)$ edges. That means for a 3-equitable labeling we must have $e_f(1) = (2m + 1)(2s + 1)$ which is not possible. Hence $S(C_t, n)$ is not 3-equitable when t is odd and

$n \equiv 3(\mod 6)$. These results motivates us to make following conjecture:

Conjecture: If $t \geq 3$ and n are two natural numbers every C_t-snake is 3-equitable if either t is odd and n is not congruent to 3 modulo 6 or t is even.

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N. B. Limaye
Department of Mathematics
Vidyanagari, University of Mumbai
Mumbai - 400098, INDIA

M. V. Bapat
Department of Mathematics
S. S. H. Kelkar College, Devgad
Maharashtra, INDIA