Cutting a graph into small components

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Abstract: For some fixed $n_0 \geq 0$ we study the minimum number of vertices or edges that have to be removed from a graph such that no component of the rest has more than n_0 vertices.

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1 Introduction

We consider finite and simple graphs $G = (V, E)$ and use standard terminology as in [3]. It is a well-known folklore result that the deletion of a *centroid* (or *median*) vertex [5, 7] in a tree of order n leads to a forest all components of which have at most $\frac{n}{2}$ vertices. Our main aim in this note is to discuss the following related question: *How many vertices (edges) do we have to delete from a tree (graph) such that all components of the arising graph have at most some given order?*

This question leads to a kind of connectivity number whose definition was proposed in [4]. In view of the above-mentioned property of the centroid it is obvious that repeatedly deleting centroid vertices of arising components reduces the order of the components by a factor of two in each step. Nevertheless, the resulting estimate on the number of vertices that has to be deleted is too large in general.

A *separator* of a graph G of order n is a set of vertices V' such that no component of $G - V'$ has more than $\frac{2}{3}n$ vertices. There are several deep results on the existence of small separators for special graph classes [1, 2, 6]. As above, repeated deletion of small separators will finally lead to components having at most some given order.

In the next section we give best-possible answers to the above question (mainly) for trees.

2 Results

Theorem 1 *Let $T = (V, E)$ be a tree of order n and $n_0 \geq 0$. There is a set $V' \subseteq V$ of at most $\lfloor \frac{n}{n_0+1} \rfloor$ vertices such that all components of $T - V'$ have at most n_0 vertices.*

Proof: We prove the statement by induction on $\sum_{d_T(u)\geq 3} d_T(u)$, where $d_T(u)$ denotes the degree of the vertex u in T. If $\sum_{d_T(u)\geq 3} d_T(u) = 0$, then T is a path and the result is obvious.

Now let $\sum_{d_T(u)\geq 3} d_T(u) > 0$, i.e. there is at least one vertex of degree at least three in T. There are vertices $u_1, u_2, ..., u_r, v_1, v_2, ..., v_s, w \in V$ for $r, s \geq 1$ such that $u_i u_{i+1} \in E$ for $1 \leq i \leq r-1$, $v_i v_{i+1} \in E$ for $1 \leq i \leq s-1$, $u_r w, v_s w \in E$, $d_T(u_1) = d_T(v_1) = 1$, $d_T(u_i) = 2$ for $2 \leq i \leq r$, $d_T(v_i) = 2$ for $2 \leq i \leq s$ and $d_T(w) \geq 3$. Let $\tilde{T} = (V, (E \setminus \{v_s w\}) \cup \{u_1 v_s\})$ (cf. Figure 1).

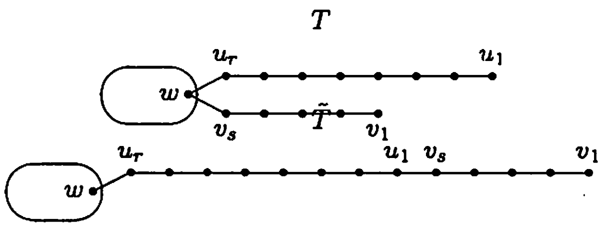

Figure 1.

Since $\sum_{d_{\tilde{T}}(u)\geq 3} d_{\tilde{T}}(u) < \sum_{d_T(u)\geq 3} d_T(u)$, there is a set $\tilde{V}'$ of at most $\lfloor \frac{n}{n_0+1} \rfloor$ vertices such that all components of $\tilde{T} - \tilde{V}'$ have at most n_0 vertices. If $w \in \tilde{V}'$, then all components of $T - \tilde{V}'$ have at most n_0 vertices and we are done. Hence suppose $w \notin \tilde{V}'$. Let W be the set of those vertices of the component of $\tilde{T} - \tilde{V}'$ that contains w which do not lie in $\{u_1, ..., u_r, v_1, ..., v_s\}$ and let $t = |W|$. Clearly, $t \leq n_0$. Let $r = r_1(n_0+1) + r_2$ and $s = s_1(n_0+1) + s_2$ such that $0 \leq r_2, s_2 \leq n_0$. We have

$$|\tilde{V}' \cap \{u_1, ..., u_r, v_1, ..., v_s\}| \geq \left\lfloor \frac{t+r+s}{n_0+1} \right\rfloor = \left\lfloor \frac{t+r_2+s_2}{n_0+1} \right\rfloor + r_1 + s_1.$$

Let

$$\begin{aligned} V'' &= (\tilde{V}' \setminus \{u_1, ..., u_r, v_1, ..., v_s\}) \cup \{u_{\nu(n_0+1)} \mid 1 \leq \nu \leq r_1\} \\ &\cup \{v_{\nu(n_0+1)} \mid 1 \leq \nu \leq s_1\}. \end{aligned}$$

If $t + r_2 + s_2 \leq n_0$, then let $V' = V''$ and if $t + r_2 + s_2 \geq n_0 + 1$, then let $V' = V'' \cup \{w\}$. Clearly, all components of $T - V'$ have at most n_0 vertices. Since

$$|V'| = |V''| + \left\lfloor \frac{t+r_2+s_2}{n_0+1} \right\rfloor$$

$$
\begin{aligned}
&= |\tilde{V}' \setminus \{u_1, ..., u_r, v_1, ..., v_s\}| + r_1 + s_1 + \left\lfloor \frac{t + r_2 + s_2}{n_0 + 1} \right\rfloor \\
&= |\tilde{V}'| - |\tilde{V}' \cap \{u_1, ..., u_r, v_1, ..., v_s\}| + r_1 + s_1 + \left\lfloor \frac{t + r_2 + s_2}{n_0 + 1} \right\rfloor \\
&\leq |\tilde{V}'| - \left\lfloor \frac{t + r_2 + s_2}{n_0 + 1} \right\rfloor - r_1 - s_1 + r_1 + s_1 + \left\lfloor \frac{t + r_2 + s_2}{n_0 + 1} \right\rfloor \\
&= |\tilde{V}'|,
\end{aligned}
$$

the proof is complete. □

The following lemma easily implies the existence of arbitrarily many trees for which the bound in Theorem 1 is tight.

Lemma 2 *Let $n_0 \geq 0$. Let T_1 be a tree of order n_1 such that a minimum set V_1' of vertices for which all components of $T_1 - V_1'$ have at most n_0 vertices has cardinality $\lfloor \frac{n_1}{n_0+1} \rfloor$. Let T_2 be any tree of order $n_0 + 1$. Let the tree $T = (V, E)$ consist of T_1 and T_2 together with one additional edge.*

Then a minimum set $V' \subseteq V$ of vertices for which all components of $T - V'$ have at most n_0 vertices has cardinality $\lfloor \frac{n_1}{n_0+1} \rfloor + 1$.

Proof: By Theorem 1, there is a set $V' \subseteq V$ as in the statement of the lemma with $|V'| \leq \lfloor \frac{n_1+n_0+1}{n_0+1} \rfloor = \lfloor \frac{n_1}{n_0+1} \rfloor + 1 = |V_1'| + 1$. Since T_2 has order n_0+1, the set V' contains at least one vertex u of T_2. One component of the tree $T - u$ contains the tree T_1 as a subgraph and hence $|V'| - 1 = |V' \setminus \{u\}| \geq |V_1'|$. Altogether, we obtain $|V'| = \lfloor \frac{n_1}{n_0+1} \rfloor + 1$. □

For general graphs we obtain the following corollary.

Corollary 3 *Let $G = (V, E)$ be a graph of order n and $n_0 \geq 0$. Let X be a set of vertices such that $G - X$ has no cycles.*

There is a set $V' \subseteq V$ of at most $\lfloor \frac{n-|X|}{n_0+1} \rfloor + |X|$ vertices such that all components of $G - V'$ have at most n_0 vertices.

Proof: Applying Theorem 1 to the components of the forest $G - X$ yields a set V'' of at most $\lfloor \frac{n-|X|}{n_0+1} \rfloor$ vertices such that all components of $G - X - V''$ have at most n_0 vertices. The conclusion follows for $V' = X \cup V''$. □

Corollary 3 is tight for arbitrary $n_0 \in \mathbb{N}_0 = \{0, 1, 2, ...\}$. For $n_0 = 0$ this is obvious and for $n_0 \geq 1$ consider a connected graph G that consists of l disjoint cycles C_{n_0+2} of length $n_0 + 2$ and $l - 1$ additional edges. Clearly, a set V' as in Corollary 3 which is of minimum cardinality contains at least two vertices in each of the l cycles, i.e. $|V'| \geq 2l$. Now Corollary 3 implies $|V'| \leq 2l$ and thus $|V'| = 2l$.

The graphs in the above example are cacti, i.e. all their cycles are edge-disjoint. The following figure shows graphs which are not cacti and for which the bound in Corollary 3 is also tight. (We leave the details and generalizations to the reader.)

Figure 2.

For cacti we can deduce a corollary of Theorem 1 also in the following way.

Corollary 4 *Let $G = (V, E)$ be a cactus of order n and $n_0 \geq 0$. There is a set $V' \subseteq V$ of at most $2\lfloor \frac{n}{n_0+1} \rfloor$ vertices such that all components of $G - V'$ have at most n_0 vertices.*

Proof: Let E' be a set of edges that contains exactly one edge of each cycle of G. Applying Theorem 1 to the tree $T = (V, E \setminus E')$, we deduce that there is a set $V'' \subseteq V$ of at most $\lfloor \frac{n}{n_0+1} \rfloor$ vertices such that all components of $T - V''$ have at most n_0 vertices. Let the set $V' \subseteq V$ arise in the following way. For each vertex $u \in V''$ that does not lie in a cycle of G let $u \in V'$. For each vertex $u \in V''$ that lies in a cycle of G let $u, u' \in V'$ where u' is a vertex incident with the edge of the cycle in E'. Clearly, $|V'| \leq 2|V''|$ and all components of $G - V'$ have at most n_0 vertices. □

Now we turn to the deletion of edges. Since we have to delete $n - n_0$ edges from star $K_{1,n-1}$ to obtain components of order at most $n_0 \geq 1$, it is reasonable to impose an upper bound on the maximum degree in this case.

Lemma 5 *Let $T = (V, E)$ be a tree of order n and maximum degree $\Delta \geq 2$. Let $n_0 \in \mathbb{N} = \{1, 2, 3, ...\}$ with $n_0 < n$. There is an edge e in T such that there is one component of order n' of $T - e$ with $\frac{n_0}{\Delta-1} \leq n' \leq n_0$.*

Proof: We root the tree at an endvertex r. This implies that every vertex has at most $\Delta - 1$ children. For $u \in V$ let the set $V_{\leq u}$ consist of u and all its descendants.

We mark all vertices $u \in V$ with $|V_{\leq u}| \geq \frac{n_0}{\Delta-1}$. Note that r is marked, since $|V_{\leq r}| = n > n_0 \geq \frac{n_0}{\Delta-1}$. Let $v \in V$ be a marked vertex with maximum distance from r. This implies that no child of v is marked. (Note that this statement is trivially true, if v has no children.)

If v' is a child of v, then $|V_{\leq v'}| < \frac{n_0}{\Delta-1}$ which implies $|V_{\leq v'}| \leq \frac{n_0-1}{\Delta-1}$. Now,

$$\frac{n_0}{\Delta - 1} \leq |V_{\leq v}| \leq |\{v\}| + \left| \bigcup_{v' \text{ is child of } v} V_{\leq v'} \right| \leq 1 + (\Delta - 1)\frac{n_0 - 1}{\Delta - 1} = n_0.$$

This implies that v is not the root r. The desired result now follows for the edge e between v and the parent of v and the proof is complete. □

Proposition 6 *Let $T = (V, E)$ be a tree of order n and maximum degree $\Delta \geq 2$. Let $n_0 \in \mathbb{N}$.*

There is a set E' of at most $\lfloor \frac{n}{\lceil \frac{n_0}{\Delta-1} \rceil} \rfloor$ edges such that no component of $T - E'$ has more than n_0 vertices.

Proof: Since $n - \lfloor \frac{n}{\lceil \frac{n_0}{\Delta-1} \rceil} \rfloor \lceil \frac{n_0}{\Delta-1} \rceil < \left\lceil \frac{n_0}{\Delta-1} \right\rceil$, the desired result follows easily by repeatedly applying Lemma 5. □

Note that if $n_0 \leq \Delta - 1$, then $\lfloor \frac{n}{\lceil \frac{n_0}{\Delta-1} \rceil} \rfloor = n$, i.e. the bound given in Proposition 6 is trivial, since a tree has $n - 1$ edges. Whereas Lemma 5 is best-possible for, for example, complete $(\Delta - 1)$-ary trees, Proposition 6 can still be improved. We will illustrate this for $n_0 = 2$. Our motivation to include the result is to show arguments that easily generalize to other special cases but will lead to extremely tedious and lengthy case analyses.

Proposition 7 *Let $T = (V, E)$ be a tree of order n and maximum degree $\Delta \geq 2$. There is a set E' of at most $\lfloor \frac{\Delta-1}{\Delta} n \rfloor$ edges such that no component of $T - E'$ has more than 2 vertices.*

Proof: Let T be a counterexample of minimum order. It is easy to see that the diameter of T is at least 3. Let $P : v_0 v_1 ... v_l$ be a longest path in T with $l \geq 3$.

Let $\tilde{T} = T - ((\{v_1\} \cup N_T(v_1)) \setminus \{v_2\})$. Since T is a counterexample of minimum order, there is a set $\tilde{E}'$ of at most $\lfloor \frac{\Delta-1}{\Delta}(n - d_T(v_1)) \rfloor$ edges of $\tilde{T}$ such that no component of $\tilde{T} - \tilde{E}'$ has more than two vertices. Now the set $E' = \tilde{E}' \cup \{uv_1 \mid u \in N_T(v_1) \setminus \{v_0\}\}$ has at most

$$\left\lfloor \frac{\Delta - 1}{\Delta}(n - d_T(v_1)) \right\rfloor + d_T(v_1) - 1 = \left\lfloor \frac{\Delta - 1}{\Delta} n + \frac{d_T(v_1)}{\Delta} - 1 \right\rfloor \leq \left\lfloor \frac{\Delta - 1}{\Delta} n \right\rfloor .$$

edges and no component of $T - E'$ has more than two vertices. □

It is easy to see that Proposition 7 is (asymptotically) best-possible for complete $(\Delta - 1)$-ary trees.

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