AN IMPROVED LOWER BOUND FOR $g^{(4)}(18)$

M. GRÜTTMÜLLER, I.T. ROBERTS, AND R.G. STANTON

ABSTRACT. The cardinality of the minimal pairwise balanced designs on v elements with largest block size k is denoted by $g^{(k)}(v)$. It is known that $30 \leq g^{(4)}(18) \leq 33$. In this note, we show $31 \leq g^{(4)}(18)$.

1. INTRODUCTION

Let K be a set of positive integers. Then a *pairwise balanced design* PBD(v, K) of order v with block sizes from K is a pair $(V, \mathcal{B})$, where V is a finite set (the *point set*) of cardinality v and $\mathcal{B}$ is a family of subsets (called *blocks*) of V which satisfy the following properties:

(i) every pair of distinct elements of V occurs in exactly one block of $\mathcal{B}$;

(ii) if $B \in \mathcal{B}$, then $|B| \in K$.

The value $g^{(k)}(v)$ is the minimum number of blocks in a pairwise balanced design on v elements with largest block size k. The value $g^{(4)}(v)$ was investigated in [6, 2] and was determined for all v with the exception of 17 and 18. Stinson and Seah showed $g^{(4)}(17) \leq 31$ by exhibiting a PBD$(17, \{2,3,4\})$ with 31 blocks (reported in [7]). Also, from [5] we know that $g^{(4)}(17) \geq 30$. Lower and upper bounds for $v = 18$ are established by Stanton in [4, 3] as $30 \leq g^{(4)}(18) \leq 33$. The study of bounds on $g^{(k)}(v)$ for arbitrary k has been subject of numerous papers. The paper by Rees and Stinson [1] is a good survey of known results.

In this paper, we study $g^{(4)}(18)$ and prove that a PBD$(18, \{2,3,4\})$ has at least 31 blocks.

2. PRELIMINARIES

We begin by introducing some terminology and notation. Let g_i be the number of blocks of size i for $i = 2, 3, 4$. Then counting pairs of points in two ways gives

$$g_2 + 3g_3 + 6g_4 = \binom{18}{2}.$$

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If $g_2 + g_3 + g_4 = 30$, then it has been shown in [4, Case 4] that $g_2 = 0, g_3 = 9$ and $g_4 = 21$. We remark here, that if there is a PBD(18, $\{3,4\}$) with 30 blocks, then by deleting one point one obtains a PBD(17, $\{2,3,4\}$) with 30 blocks. Unfortunately, as noted above, the existence of such a PBD has neither been proved nor disproved so far.

Let $\mathcal{B}'$ be a subset of the block set $\mathcal{B}$. The *volume(frequency)* of a point x in $\mathcal{B}'$, denoted by $V(x;\mathcal{B}')$, is the number of blocks in $\mathcal{B}'$ which contain x. Similarly, if X' is a subset of the point set, then $V(X';\mathcal{B}') = \sum_{x\in X'} V(x;\mathcal{B}')$.

A point x has *point type* $3^\alpha 4^\beta$ if x is contained in exactly α blocks of size 3 (triples) and β blocks of size 4 (quadruples). We know from [4, 3] that in a PBD(18, $\{3,4\}$) with $g_3 = 9, g_4 = 21$ there are 3 points of type $3^4 4^3$, say $1,2,3$, and 15 points of type $3^1 4^5$, say $4,5,\dots,18$.

There is a unique way to arrange the 9 triples such that each of the 18 points occurs in either one or four of these triples

$$124 \quad 135 \quad 236 \quad 178 \quad 1yy \quad 2yy \quad 2yy \quad 3yy \quad 3yy$$

where every $y \in Y := \{9,10,\dots,18\}$ is used exactly once. Moreover, 9 quadruples $Q_a,\dots,Q_i$ contain exactly one of $1,2,3$. They are of the form

$$\begin{array}{lll} Q_a : 16xx & Q_b : 1xxx & Q_c : 1xxx \\ Q_d : 25xx & Q_e : 2xxx & Q_f : 2xxx \\ Q_g : 34xx & Q_h : 3xxx & Q_i : 3xxx \end{array}$$

where every x from $X := \{7,\dots,18\}$ occurs exactly twice. The remaining 12 quadruples $Q_A,\dots,Q_L$ contain no point $1,2,3$. Here, we distinguish between two main configurations. In Configuration 1 is a quadruple Q_A that contains all three points $4,5,6$. In Configuration 2 occur the pairs 45, 46 and 56 in three distinct quadruples.

We will show in the following sections that both Configuration 1 and Configuration 2 are impossible and, therefore, there is no PBD on 18 points with 30 blocks of size 3 or 4.

3. Configuration 1

Suppose that there is a quadruple containing $4,5,6$ and w.l.o.g. let 7 be the fourth point in this quadruple. Then the quadruples $Q_A,\dots,Q_L$ have the following form

$$\begin{array}{llll} Q_A : 4567 & Q_B : 48yy & Q_C : 4yyy & Q_D : 4yyy \\ & Q_E : 58yy & Q_F : 5yyy & Q_G : 5yyy \\ & Q_H : 68yy & Q_I : 6yyy & Q_J : 6yyy \\ & & Q_K : 7yyy & Q_L : 7yyy \end{array}$$

where every $y \in Y$ occurs three times. Now complete the blocks Q_a, Q_d, Q_g as $16y_1y_2$, $25y_3y_4$ and $34y_5y_6$. Clearly all $y_i \neq 7$ and all y_i are pairwise distinct since otherwise for some y^* that occurs more than once in Q_a, Q_d, Q_g we have $V(y^*; Q_B, \ldots, Q_J) = 1$ and, therefore, $V(y^*; Q_K, Q_L) = 2$ a contradiction. Thus, we can complete Q_K, Q_L as $7y_1y_3y_5$ and $7y_2y_4y_6$. Moreover, $Q_e : 27yy$, $Q_f : 28y_1y$, $Q_h : 37yy$ and $Q_i : 38y_2y$ since y_1, y_2 cannot occur a second time together with 7 or as a pair. Also, $Q_F : 5y_1yy$ and $Q_G : 5y_2yy$.

If we count the volume of $y_3, \ldots, y_6$ in Q_b, Q_c we obtain 2 cases.

Case 1 $V(y_3, \ldots, y_6; Q_b, Q_c) = 4$.
Case 2 $V(y_3, \ldots, y_6; Q_b, Q_c) \leq 3$.

In Case 1 all 6 pairs y_iy_j $(i, j \in \{3, \ldots, 6\})$ are covered in $Q_b, Q_c, Q_d, Q_g, Q_K, Q_L$. But $V(y_3, \ldots, y_6; Q_H, Q_I, Q_J) = 4$ implies that there is another pair which occurs twice, a contradiction.

In Case 2 at least one of $y_3, \ldots, y_6$ occurs in Q_f or Q_i, say in Q_f. Then only y_6 is possible since $2, y_3$ and y_4 appear in Q_d and y_1 and y_5 appear in Q_K. If $y_6 \in Q_f$, then we cannot insert y_6 in Q_E, Q_F, Q_G as points $8, y_1, y_2$ appear there which contradicts $V(y_6; Q_E, Q_F, Q_G) = 1$.

Therefore, neither Case 1 nor Case 2 is possible and we conclude that there is no PBD(18, $\{3, 4\}$) with 30 blocks containing a quadruple with three points each of which is also in a block of size 3 with two of the points of type $3^4 4^3$.

4. Configuration 2

We next consider Configuration 2 in which there are 3 quadruples, say Q_A, Q_B, Q_E, each containing a pair of 4, 5, 6. Then every $x \in X$ occurs three times in blocks $Q_A, \ldots, Q_L$

$Q_A : 45xx$	$Q_B : 46xx$	$Q_C : 4xxx$	$Q_D : 4xxx$
$Q_E : 56xx$	$Q_F : 5xxx$	$Q_G : 5xxx$	
$Q_H : 6xxx$	$Q_I : 6xxx$		
$Q_J : xxxx$	$Q_K : xxxx$	$Q_L : xxxx$	.

Let $B'_2 = \{Q_a, Q_d, Q_g\}$, $B''_2 = \{Q_A, Q_B, Q_E\}$, $B_2 = B'_2 \cup B''_2$, $B'_3 = \{Q_b, Q_c, Q_e, Q_f, Q_h, Q_i\}$, $B''_3 = \{Q_C, Q_D, Q_F, \ldots, Q_I\}$, $B_3 = B'_3 \cup B''_3$ and $B_4 = \{Q_J, Q_K, Q_L\}$. Define for $x \in X$ $\alpha = V(x, B_2)$, $\beta = V(x, B_3)$ and $\gamma = V(x, B_4)$. Then, we obtain

$$\alpha + \beta + \gamma = 5 \quad \text{and} \quad \alpha + 2\beta + 3\gamma = 10.$$

There are three possible types (α, β, γ) for a point $x \in X$: $T_1 : (2, 1, 2)$, $T_2 : (1, 3, 1)$ and $T_3 : (0, 5, 0)$. Denote the number of points of type T_i

by λ_i $(i = 1,2,3)$. Then,

$$\lambda_1 + \lambda_2 + \lambda_3 = 12 \quad \text{and} \quad 2\lambda_1 + \lambda_2 = 2|\mathcal{B}_2| = 12.$$

At most 3 points can occur in $Q_J, \dots, Q_L$ twice, so $\lambda_1 \leq 3$ and we record the four cases for $(\lambda_1, \lambda_2, \lambda_3)$: $(3,6,3)$, $(2,8,2)$, $(1,10,1)$ and $(0,12,0)$.

Before we consider these cases in more detail, we note that no point of type T_1 can occur in two of $\mathcal{B}_2'$. Otherwise, the existence of a point x^* of type T_1 with $Q_a = 16x^*x_1$, $Q_d = 25x^*x_2$ would imply that $x^* \in Q_C, Q_J, Q_K$. So x_1, x_2 do not occur in Q_J or Q_K and hence x_1 and x_2 cannot be of Type T_1. So x_1, x_2 must be of Type T_2, with x_1, x_2 in Q_L. Thus, x_1, x_2 occur at least twice each in $Q_A, \dots, Q_I$, and so they cannot occur in $\mathcal{B}_2''$. So they both occur at least twice in $\mathcal{B}_3''$, and hence in Q_D (as x^* in Q_C). This is a contradiction as x_1, x_2 in Q_L. So a T_1-point occurs in exactly one block from each of block sets $\mathcal{B}_2'$, $\mathcal{B}_2''$, $\mathcal{B}_3'$ and in exactly two of the blocks from $\mathcal{B}_4$. Also, it is worth noting that if x_1, x_2 are of Type T_1 then they appear together in a block of $\mathcal{B}_1$ (as each occurs twice in the 3 blocks in $\mathcal{B}_4$), so they are not in any other block together (including the triples). Call this (*).

4.1. **The Case** $(\lambda_1, \lambda_2, \lambda_3) = (3,6,3)$. Let x_i be the points of type T_1 and z_i the points of type T_3 $(i = 1,2,3)$. As $z_j \notin \mathcal{B}_2$ and $V(x_i; \mathcal{B}_3) = 1$, $x_i \notin \mathcal{B}_3''$ every pair x_1z_i $(i = 1,2,3)$ must be covered in a triple (at most one pair) or a block from $\mathcal{B}_3$ (at most two pairs). Thus, x_1 must occur together with a pair z_iz_j in a block from $\mathcal{B}_3'$. Similarly, x_2, x_3 occur together with a pair z_iz_j. Thus, every pair z_iz_j is covered in $\mathcal{B}_3'$. Furthermore, $V(z_1, z_2, z_3; \mathcal{B}_3'') = 9$ implies that a pair z_iz_j appears again in a block from $\mathcal{B}_3''$ which is a contradiction.

4.2. **The Case** $(\lambda_1, \lambda_2, \lambda_3) = (2,8,2)$. Let x_i $(i = 1,2)$ be the points of type T_1, y_j $(j = 1, \dots, 8)$ the points of type T_2, and z_k $(k = 1,2)$ the points of type T_3. W.l.o.g. we have quadruples

$$\begin{array}{lll} Q_a : 16x_1y_1 & Q_d : 25x_2y_4 & Q_g : 34 - - \\ Q_A : 45x_1y_2 & Q_B : 46x_2y_5 & Q_E : 56 - - \\ Q_J : x_1x_2y_7y_8 & Q_K : x_1y_1y_5y_6 & Q_L : x_2y_1y_2y_3 \end{array}$$

Since y_7 and y_8 are undifferentiated, we may write $Q_g : 34y_3y_7$ and $Q_E : 56y_6y_8$. Now, 4 requires elements $y_1, y_4, y_6, y_8, z_1, z_2$, 5 requires elements $y_1, y_3, y_5, y_7, z_1, z_2$ and 6 requires elements $y_2, y_3, y_4, y_7, z_1, z_2$. This forces blocks

$$\begin{array}{ll} Q_C : 4y_4y_8 - & Q_D : 4y_6 - - \\ Q_F : 5y_1y_7 - & Q_G : 5y_3 - - \\ Q_H : 6y_2y_7 - & Q_I : 6y_3 - - \end{array}$$

If we have $Q_H : 6y_2y_7y_4$ and $Q_I : 6y_3z_1z_2$, we cannot fill in the Q_G block. So we write $Q_H : 6y_2y_7z_1$ and $Q_I : 6y_3y_4z_2$. This forces $Q_F : 5y_1y_7z_2$, $Q_G : 5y_3y_5z_1$, and then $Q_C : 4y_4y_8z_1$, $Q_D : 4y_6y_1z_2$ or $Q_C : 4y_4y_8y_1$, $Q_D : 4y_6z_1z_2$. But the first possibility gives a repeated y_1z_2. So we have

$$\begin{array}{ll} Q_C : 4y_4y_8y_1 & Q_D : 4y_6z_1z_2 \\ Q_F : 5y_1y_7z_2 & Q_G : 5y_3y_5z_1 \\ Q_H : 6y_2y_7z_1 & Q_I : 6y_3y_4z_2 \end{array}$$

But x_1 must appear once more in triples, say $2x_1-$, and once more in quadruples $3x_1--$, and it is missing elements y_3, z_1, z_2. But all three pairs y_3z_1, y_3z_2, z_1z_2 have been used up. So we have a contradiction.

4.3. The Case $(\lambda_1, \lambda_2, \lambda_3) = (1, 10, 1)$. Let x be the point of type T_1, z the point of type T_3 and assume $x \in Q_a, Q_A, Q_J, Q_K$. If xz appears in a triple, then xy_1y_2 is contained in a block from $\mathcal{B}_3'$ where y_1, y_2 are of Type T_2. But, $V(x, y_1, y_2; \mathcal{B}_4) = 4$ which is a contradiction.

As $x \notin \mathcal{B}_3''$, xzy_1 (with y_1 of type T_2) occur in a block from $\mathcal{B}_3'$, and hence $V(z; \mathcal{B}_3'') = 3$. Then we can let $Q_A : 45xy_2$ and $Q_L : y_1y_2y_3y_4$ with the y_i of Type T_2. We know $V(y_1, \ldots, y_4; \mathcal{B}_2' \cup \mathcal{B}_3') = 8$ and, hence, $5 \leq V(y_1, \ldots, y_4; \mathcal{B}_3') \leq 6$. If $V(y_1, \ldots, y_4; \mathcal{B}_3') = 5$, then we obtain $V(y_1, \ldots, y_4; \mathcal{B}_3'') = 7$ and a pair y_iy_j which is already covered in Q_L. Assume $V(y_1, \ldots, y_4; \mathcal{B}_3') = 6$ and $V(y_1, \ldots, y_4; \mathcal{B}_3'') = 6$. As z is already paired with y_1, each pair zy_i $(i = 2, 3, 4)$ occurs in a block from $\mathcal{B}_3''$. So we can assume $Q_H : 6zy_2-$. Let $Q_d : 25u_1u_2$ and $Q_g : 34u_3u_4$. Note that $u_1, \ldots, u_4$ must be of Type T_2 (as $45x \subset Q_A$ and $z \notin B_2$). Also $u_1, \ldots, u_4$ cannot be in $B"_2$. As $y_2 \in Q_A$, $y_2 \notin Q_d$ or Q_g, so y_2 is distinct from $u_1, \ldots, u_4$. To be paired with 6, $V(u_1, \ldots, u_4; Q_H, Q_I) = 4$, which implies that three of $u_1, \ldots, u_4$ occur in Q_I, a contradiction.

4.4. The Case $(\lambda_1, \lambda_2, \lambda_3) = (0, 12, 0)$. Let $y_1, \ldots, y_6 \in X$ be the points in $\mathcal{B}_2'$ and $z_1, \ldots, z_6 \in X$ be the points in $\mathcal{B}_2''$. We start with counting pairs y_iy_j. Clearly, 3 pairs y_iy_j appear in $\mathcal{B}_2'$. Each block from $\mathcal{B}_3''$ contains exactly one point z_k and two points y_i, y_j. So 6 pairs y_iy_j are covered in $\mathcal{B}_3''$. Furthermore, each block from $\mathcal{B}_4$ contains exactly two points z_k, z_l and two points y_iy_j since otherwise the existence of a block $Q_L : z_kz_lz_my_i$ or $Q_L : z_kz_lz_mz_n$ would imply $V(Q_L; \mathcal{B}_3') > 6$, a contradiction. So 3 pairs y_iy_j appear in $\mathcal{B}_4$. For the remaining 3 pairs y_iy_j, let p be the number of pairs in the triples and hence there are $(3-p)$ pairs in $\mathcal{B}_3'$. There are three types of blocks in $\mathcal{B}_3'$: $uyyz$, $uyzz$ and $uzzz$ $(u \in \{1, 2, 3\})$. We refer to blocks of these types as A-blocks, B-blocks and C-blocks, respectively. A point y which is contained in a triple uyy must occur in a B-block and a point y contained in a triple uyz must occur in an A-block. Thus, there are exactly $(3-p)$

A-blocks, $2p$ B-blocks and $(3-p)$ C-blocks. Moreover, if a point z is contained in a triple uyz, then z occurs in a B-block and a C-block. We obtain at least $(3-p)$ B-blocks from the $(6-2p)$ uyz triples and, thus, $2p \geq 3-p$. So $1 \leq p \leq 3$ and we need to consider 3 cases for p.

In Case $p=3$ let the triples be

$$1y_1y_2, \quad 2y_3y_4, \quad 3y_5y_6, \quad 1z_1z_2, \quad 2z_3z_4 \quad \text{and} \quad 3z_5z_6.$$

Note, that we cannot have $1y_1y_2$, $1y_3y_4$ as this would imply that the pair y_5y_6 appears in a triple and in Q_a. Now, $Q_a : 16y_4y_5$, $Q_d : 25y_1y_6$ and $Q_g : 34y_2y_3$. Thus, $y_3, y_6 \in Q_b, Q_c$ and there is a unique way to fill in $y_1, \ldots, y_6$ into the blocks from B_3'' and B_4. In particular, $Q_L : y_3y_6 - -$. Moreover, z_1, z_2 cannot occur together with y_3, y_6 in B_3' and only with one of y_3, y_6 in B_3''. Hence, both z_1, z_2 must occur in Q_L, a contradiction.

In Case $p=2$ we have triples of the form

$$u_1y_1y_2, \quad u_2y_3y_4, \quad u_3y_5z_1 \quad \text{and} \quad u_4y_6z_2$$

where $u_i \in \{1,2,3\}$. It follows that y_5, y_6 must occur together in an A-block. If $u_3 = u_1$, then y_5, y_6 must also occur together in a block from B_2', a contradiction. Whence, $u_3 \neq u_1$. If $u_4 \neq u_1, u_2$ (say $u_4 = 3$), then beside the A-block containing y_5, y_6 we find from $V(y_1, \ldots, y_5; Q_g, \ldots, Q_i) = 5$ that there is a second A-block. This implies $p = 1$, again a contradiction. Finally, assume $u_1 = u_3, u_2 = u_4$. Then $V(y_1, \ldots, y_6; Q_g, \ldots, Q_i) = 6$ and we obtain again two A-blocks. Thus, $p = 2$ is not possible.

In the last Case $p = 1$ we can assume that we have triples of the form

$$1y_1y_2, \quad 1z_1z_2, \quad 2y_3z_3 \quad \text{and} \quad 2y_4z_4,$$

or of the form

$$1y_1y_2, \quad 1y_3z_3, \quad 2z_1z_2 \quad \text{and} \quad 2y_4z_4.$$

In both cases $V(y_1, y_2; Q_d, Q_g) = 2$, $V(y_1, \ldots, y_4; Q_g, \ldots, Q_i) = 4$ and $Q_g : 34y_1y_3$ as y_1, y_2 cannot occur together. Thus, Q_h is a B-block containing y_1 and Q_i is a B-block containing y_4 (y_5, y_6 occur together with 3 in a triple and cannot be in Q_i). But y_4 needs to belong to an A-block, a contradiction.

We conclude that there is no PBD$(18, \{3, 4\})$ with 30 blocks where the three points each of which is also in a block of size 3 with two of the points of type 3^14^3 do not occur as pairs in distinct quadruples.

5. Conclusion

In a PBD$(18, \{2, 3, 4\})$ with 30 blocks there must be three points of type 3^14^3 and a further three points each of which is also in a block of size 3 with two of the $3^4 4^3$-points. We have shown that the latter three points do not occur together in a common quadruple (Configuration 1), nor is the opposite true (Configuration 2), and have therefore established:

Theorem 5.1. *There does not exist a PBD on 18 points with 30 blocks of size at most 4. Thus,* $g^{(4)}(18) \geq 31$.

We remark that with this result we have also established:

Corollary 5.2. *There does not exist a PBD on 17 points with 30 blocks of size at most 4 in which the block set contains either a subset of four blocks of size 2 and three blocks of size 3, or a subset of one block of size 2 and five blocks of size 3 which are mutually non-intersecting.*

References

[1] R.S. Rees and D.R. Stinson. On the number of blocks in a perfect covering of v points. *Discr. Math.*, **83**:81–93, 1990.

[2] R.G. Stanton. The exact covering of pairs on nineteen points with block sizes two, three and four. *JCMCC*, 4:69–78, 1988.

[3] R.G. Stanton. An improved upper bound on $g^{(4)}(18)$. *Cong. Numer.*, **142**:29–32, 2000.

[4] R.G. Stanton. A lower bound for $g^{(4)}(18)$. *Cong. Numer.*, **146**:153–156, 2000.

[5] R.G. Stanton. An improved lower bound for $g^{(4)}(17)$. *unpublished*, :–, 2003.

[6] R.G. Stanton and D.R. Stinson. Perfect pair-coverings with blocks sizes two, three and four. *J. Combinatorics, Information and System Sciences*, **8**:21–25, 1983.

[7] R.G. Stanton and A.P. Street. Some achievable defect graphs for pair-packings on seventeen points. *JCMCC*, **1**:207–215, 1987.

M. Grüttmüller, Department of Mathematics, University of Rostock, 18051 Rostock, Germany

E-mail address: m.gruettmueller@mathematik.uni-rostock.de

I.T. Roberts, School of Engineering, Northern Territory University, Darwin, NT, 0909, Australia

E-mail address: ian.roberts@ntu.edu.au

R.G. Stanton, Department of Computer Science, University of Manitoba, Winnipeg, Canada R3T 2N2

E-mail address: stanton@cc.umanitoba.ca