Minimum Number of Bridges on Connected Almost Cubic Graphs with Given Deficiency

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ABSTRACT. Let G be a simple graph having a maximum matching M. The deficiency $def(G)$ of G is the number of vertices unsaturated by M. A bridge in a connected graph G is an edge of G such that $G-e$ disconnected. A graph is said to be almost cubic (or almost 3-regular) if one of its vertices has degree $3 + e$, $e \geq 0$, and the others have degree 3. In this paper we find the minimum number of bridges of connected almost cubic graphs with given deficiency.

1. Introduction

For our purposes, all graphs are finite, loopless and have no multiple edges. For most of our notation and terminology we follow that of Bondy and Murty [2]. Thus G is a graph with vertex set $V(G)$ and edge set $E(G)$. The number of vertices is $|V(G)|$.

A *matching* M in G is a subset of $E(G)$ in which no two edges have a vertex in common. M is a *maximum matching* if $|M| \geq |M'|$ for any other matching M' of G. A vertex v is *unsaturated* by M if there is no edge of M is incident with v. A matching M is called a *1-factor* (or a *perfect matching*) if there is no vertex of the graph is unsaturated by M.

The *deficiency* $def(G)$ of G is the number of vertices unsaturated by a maximum matching of G. Observe that $def(G) = |V(G)| - 2|M|$ for any maximum matching M in G. Consequently, $def(G)$ has the same parity as $|V(G)|$, and $def(G) = 0$ if and only if G has a 1-factor.

An *edge cut set* of a connected graph G is a subset of $E(G)$ whose deletion from G results in a disconnected graph. A graph G is *k-edge-connected* if there is no edge cut set of G of cardinality of less than k.

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A *bridge* is the element of an edge cut set of cardinality one, that is an edge of G such that $G-e$ disconnected.

Many problems concerning matchings have been studied in the literature, see for example [6]. The relations between edge cut-sets and perfect matching of regular graphs have been studied in the Chartrand and Nebesky [4] and Katerinis [5]. The following two theorems can be found in Chartrand and Nebesky [4].

Theorem 1.1 (Petersen). *Every cubic bridgeless graph has a 1-factor.* []

Theorem 1.2 (Chartrand and Nebesky). *If G is an r-regular,* $(r-2)$*-connected graph,* $r \geq 3$*, having even number of vertices and G contains at most* $r-1$ *edge cut set of cardinality* $r-2$*, then G has a 1-factor.* []

From Theorem 2, when $r=3$, we can say that every cubic graphs with at most two bridges has a perfect matching. Every cubic graph has even number of vertices. So, a cubic graph G has no perfect matching if and only if $def(G) \geq 2$. Theorem 2 has the following corollary.

Corollary 1.3. *Every cubic graph G with* $def(G) \geq 2$ *has at least three bridges.* []

A graph is said to be *almost cubic* (or *almost 3-regular*) if one of its vertices has degree $3+e$, $e \geq 0$, and the others have degree 3. In this paper we study the lower bound of the number of bridges of connected almost cubic graphs with given deficiency $def(G) = d \geq 0$. We show that that for every non-negative integer m not less than the bound there exists a connected almost cubic graph G with $def(G) = d$ and number of bridges m.

2. The Bounds

Let G be a graph. If S is a subset of $V(G)$, $G-S$ denotes the graph formed from G by deleting all the vertices in S together with their incident edges. A component of G is called *odd* or *even* according as its number of vertices is odd or even. The number of odd components of a graph G is denoted by $o(G)$. We need Berge's formula ([1], p159) to establish our results.

Berge's Formula:

$$def(G) = \max_{S \subseteq V(G)} \{o(G-S) - |S|\}.$$ []

Our first result is on the lower bound of the number of bridges of a connected almost cubic graph G when $def(G)$ is given.

Let G be a connected graph on n vertices, $n-1$ of which have degree 3 and one has degree $3+e$, $e \geq 0$, and let $def(G) = d$. If e is even, then every vertex of G has odd degree, hence n is even and so d is even; if e is odd, then only one vertex of G has even degree, hence n is odd and so d is odd. Thus d and e has the same parity.

When $d = 0$ or 1, there exists a graph with one vertex of degree $3 + e$ and the others have degree 3 having $def(G) = d$. The graph is formed from a vertex v and a cycle on $3 + e$ vertices and join v to all vertices of the cycle. So suppose $d \geq 2$.

Theorem 2.1. *Let G be a connected graph on n vertices, $n-1$ of which have degree 3 and one have degree $3+e$, $e \geq 0$, and $def(G) = d \geq 2$. Then G has at least $\frac{3d-e}{2}$ bridges.*

Proof. By Berge's formula, there exists a vertex set $S \subseteq V(G)$ such that

$$o(G-S) = |S| + d.$$

Since $d \geq 2$, then $|S| \geq 1$.

Let m be the number of odd components of $G-S$ each of which is joined to S by a bridge. Let v be the vertex of degree $3+e$ in G. If $v \in S$ or $e = 0$, then each odd component of $G-S$ is joined to S by odd number of edges. Hence

$$\begin{aligned} 3|S| + e &\geq m + 3(o(G-S) - m) \\ &= m + 3(|S| + d - m), \\ m &\geq \frac{3d-e}{2}. \end{aligned}$$

If $v \notin S$ and $e \geq 1$, then at most one odd component of $G-S$ is joined to S by even number of edges. Hence

$$\begin{aligned} 3|S| &\geq m + 2 + 3(o(G-S) - m - 1) \\ &= m + 2 + 3(|S| + d - m - 1), \\ m &\geq \frac{3d-1}{2} \end{aligned}$$

$$\geq \frac{3d-e}{2},$$

this completes the proof since the number of bridges of G is at least m.

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In Theorem 2.1, if $e = 0$, then we get the following corollary. This corollary is also a corollary of Lemma 2.2 of Caccetta and Purwanto in [3].

Corollary 2.2. *Every connected cubic graph G with def(G) =* $d \geq 2$ *has at least* $\frac{3d}{2}$ *bridges.* []

In Theorem 2.1, if $e = 0$ and def(G) = d is replaced by def(G) $\geq$ 2 (G has no 1-factor) then we get Corollary 1.3 (Petersen).

3. Constructions

In this section we will show that for every non negative integer m not less than the bound in Theorem 2.1, there exists a connected almost cubic graph G with *def*(G) = d and number of bridges m. This will imply that the bound in Theorem 2.1 is sharp. We will use the following graphs in our constructions.

Let p be a positive even integer, and $q = \frac{3p}{2}$. Construct a graph $H_{p,q}$ as follows. Take two empty graphs $\overline{K}_p$ and $\overline{K}_q$ with vertices $u_1, u_2, ..., u_p$ and $v_1, v_2, ..., v_q$ respectively. For every i, $1 \leq i \leq p$, join u_i to every v_j, $j \equiv i + k$ (mod q), $k = 0, \frac{p}{2}, p$. The resulting graphs $H_{p,q}$ has $p + q$ vertices, p of which of degree 3 and q others of degree 2, and has no bridge.

Construct a graph H_0 as follows. Take a cycle on five vertices $v_1, v_2, v_3, v_4, v_5, v_1$ and join v_2 to v_4 and v_3 to v_5. The resulting graph H_0 has one vertex of degree 2 and four others of degree 3. Then construct a graph H_i, $i \geq 1$, as follows. Take a cycle on four vertices v_1, v_2, v_3, v_4, v_1 and join v_2 to v_4. The resulting graph, say L, has two vertices of degree 2 and two others of degree 3. Then take a copy of H_0 with a vertex of degree 2 named u_0, and take i copies of L, say

$L_1, L_2, \ldots, L_i$ with vertices of degree 2 in L_j are u_j and w_j, $1 \le j \le i$. For every *j*, join w_j to u_{j-1}. The resulting graph H_i has an odd number of vertices, one of which of degree 2 and all others of degree 3, and has *i* bridges.

Construct a graph H_i^* as follows. Take a graph H_{i-1} and a cycle on four vertices. Then join the vertex of degree 2 in H_{i-1} to one vertex of the cycle. The resulting graph H_i^* has an odd number of vertices, three of which of degree 2 and all others of degree 3, and has *i* bridges.

Now we are ready to state and prove our result.

Theorem 3.1. *Let d, e and m be non-negative integers,* $e \ge 0$, $d \ge 2$, *d and e have the same parity, and* $m \ge \frac{3d-e}{2}$. *Then there exists a connected graph having one vertex of degree of degree* $3+e$ *and the others of degree* 3, *has* $def(G) = d$, *and has m bridges.*

Proof. We construct our graph according to the value of *d*.

Case $2 \le d \le \frac{e}{3}$:

Since $d \le \frac{e}{3}$, then $\frac{3d-e}{2} \le 0$. Take one cycle on $3+e-3d$ vertices. If $m \ge 1$, then take one copy of H_m^*, and $d-1$ cycles each of which on three vertices; if $m = 0$, then take *d* cycles each of which on three vertices. Join all the vertices of degree 2 in these graphs to one new vertex. The resulting graph G_1 is a connected graph having one vertex of degree $3+e$ and all other vertices of degree 3, has $def(G_1) = d$, and has *m* bridges.

Case $\frac{e}{3} \le d \le e+2$:

Take $\frac{e-d+2}{2}$ cycles each of which on three vertices. If $m = \frac{3d-e}{2}$, then take *m* copies of H_0; if $m > \frac{3d-e}{2}$, then take $\frac{3d-e}{2} - 1$ copies of H_0 and one copy of $H_{m-\frac{3d-e}{2}+1}$. Join all the vertices of degree 2 in these graphs to one new vertex. The resulting graph G_2 is a connected graph having one vertex of degree $3+e$ and all other vertices of degree 3, has $def(G_2) = d$, and has *m* bridges.

Case $d \geq e+4$:

If $m = \frac{3d-e}{2}$, then take m copies of H_0; if $m > \frac{3d-e}{2}$, then take $\frac{3d-e}{2} - 1$ copies of H_0 and one copy of $H_{m-\frac{3d-e}{2}+1}$. Say the vertices of degree 2 in these graphs are $u_1, u_2, \ldots, u_{\frac{3d-e}{2}}$ respectively. Take a graph $H_{d-e,3(\frac{d-e}{2})}$ with vertices of degree 2 are $v_1, v_2, \ldots, v_{3(\frac{d-e}{2})}$. For every i, $1 \leq i \leq 3(\frac{d-e}{2}) - 1$, join u_i to v_i, and for every $3(\frac{d-e}{2}) \leq i \leq \frac{3d-e}{2}$, join u_i to $v_{3(\frac{d-e}{2})}$. The resulting graph G_3 is a connected graph having one vertex of degree $3+e$ and all others of degree 3, has $def(G_3) = d$, and has m bridges. []

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