On Boundary Vertices in Graphs

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ABSTRACT

A vertex v of a connected graph G is an eccentric vertex of a vertex u if v is a vertex at greatest distance from u; while v is an eccentric vertex of G if v is an eccentric vertex of some vertex of G. The subgraph of G induced by its eccentric vertices is the eccentric subgraph of G.

A vertex v of G is a boundary vertex of a vertex u if $d(u, w) \leq d(u, v)$ for each neighbor w of v. A vertex v is a boundary vertex of G if v is a boundary vertex of some vertex of G. The subgraph of G induced by its boundary vertices is the boundary of G. A vertex v is an interior vertex of G if for every vertex u distinct from v, there exists a vertex w distinct from v such that $d(u, w) = d(u, v) + d(v, w)$. The interior of G is the subgraph of G induced by its interior vertices. A vertex v is a boundary vertex of a connected graph if and only if v is not an interior vertex. For every graph G, there exists a connected graph H such that G is both the center and interior of H.

Relationships between the boundary and the periphery, center, and eccentric subgraph of a graph are studied. The boundary degree of a vertex v in a connected graph G is the number of vertices u in G having v as a boundary vertex. We study, for each pair r, n of integers with $r \geq 0$ and $n \geq 3$, the existence of a connected graph G of order n such that every vertex of G has boundary degree r. We also study the boundary vertices of a connected graph from different points of view.

Keywords: central vertex, peripheral vertex, eccentric vertex, interior vertex, boundary vertex.

AMS Subject Classification: 05C12

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[1]Research supported in part by a Western Michigan University Faculty Research and Creative Activities Fund

1 Introduction

Let G be a nontrivial connected graph. The *distance* $d(u,v)$ between two vertices u and v of G is the length of a shortest $u-v$ path in G. A $u-v$ path of length $d(u,v)$ is called a $u-v$ *geodesic*. For a vertex v of G, the *eccentricity* $e(v)$ is the distance between v and a vertex farthest from v. The minimum eccentricity among the vertices of G is the *radius* $\text{rad}(G)$ of G and the maximum eccentricity is its *diameter* $\text{diam}(G)$. A vertex v in G is a *central vertex* if $e(v) = \text{rad}(G)$, and the subgraph induced by the central vertices of G is the *center* $Cen(G)$ of G. A vertex v in a connected graph G is a *peripheral vertex* if $e(v) = \text{diam}(G)$. The subgraph of G induced by its peripheral vertices is the *periphery* $Per(G)$.

A vertex v is *an eccentric vertex of a vertex* u if $d(u,v) = e(u)$, that is, every vertex at greatest distance from u is an eccentric vertex of u. A vertex v is an *eccentric vertex* of G if v is an eccentric vertex of some vertex of G (see [5, 7]). Consequently, if v is an eccentric vertex of u and w is a neighbor of v, then $d(u,w) \leq d(u,v)$. A vertex v may have this property, however, without being an eccentric vertex of u. In [3] a vertex v is defined to be a *boundary vertex of* u if $d(u,w) \leq d(u,v)$ for all $w \in N(v)$. A vertex v is *a boundary vertex of* G if v is a boundary vertex of some vertex of G. The subgraph of G induced by its eccentric vertices is called the *eccentric subgraph* $Ecc(G)$ of G, while the subgraph of G induced by its boundary vertices is called the *boundary* $\partial(G)$ of G. We write $H \leq G$ to indicate that H is a subgraph of G. Thus for every connected graph G,

$$Per(G) \leq Ecc(G) \leq \partial(G) \leq G.$$

Among all geodesics in G, let P_1 be one of greatest length. If P_1 is a $u-v$ geodesic, then u and v are peripheral vertices of G. For a fixed vertex u of G, let P_2 be a geodesic of greatest length having initial vertex u. If P_2 is a $u-v$ geodesic, then v is an eccentric vertex of u. For a fixed vertex u of G, let P_3 be a geodesic with initial vertex u that cannot be extended to a longer geodesic with initial vertex u. If P_3 is a $u-v$ geodesic, then v is a boundary vertex of u.

A vertex in a graph is called *complete* (or *extreme*, or *simplicial*) if the subgraph induced by its neighborhood is complete. In particular, every end-vertex is complete. The following two results appeared in [3].

Theorem A *Let G be a connected graph. A vertex v of G is a boundary vertex of every vertex of G distinct from v if and only if v is a complete vertex of G.*

Theorem B *No cut-vertex of a connected graph G is a boundary vertex of G.*

2 Interior Vertices of a Graph

There is another set of vertices in a connected graph G that is of interest and that is related to the set of boundary vertices of G. Let x and z be two distinct vertices in G. A vertex y distinct from x and z is said to lie *between* x and z if $d(x,z) = d(x,y) + d(y,z)$. A vertex v is an *interior vertex* of G if for every vertex u distinct from v, there exists a vertex w such that v lies between u and w. Let $\mathcal{I}(G)$ be the set of all interior vertices of G. The *interior* $Int(G)$ of G is the subgraph of G induced by $\mathcal{I}(G)$. We now see that the interior vertices are precisely those vertices that are not boundary vertices.

Theorem 2.1 *Let G be a connected graph. A vertex v is a boundary vertex of G if and only if v is not an interior vertex of G.*

Proof. Let v be a boundary vertex of G and assume, to the contrary, that v is also an interior vertex. Suppose that v is a boundary vertex of the vertex u. Since v is an interior vertex of G, there exists a vertex w distinct from u and v such that v lies between u and w. Let

$$P : u = v_1, v_2, \cdots, v = v_j, v_{j+1}, \ldots, v_k = w$$

be a $u - v$ path, where $1 < j < k$. However, $v_{j+1} \in N(v)$ and $d(u, v_{j+1}) = d(u,v) + 1$, a contradiction.

For the converse, let v be a vertex that is not an interior vertex of G. Hence there exists some vertex u such that for every vertex w distinct from u and v, the vertex v does not lie between u and w. Let $x \in N(v)$. Then

$$d(u,x) \leq d(u,v) + d(v,x) = d(u,v) + 1.$$

Since v does not lie between u and x, this inequality is strict and so $d(u,x) \leq d(u,v)$, that is, v is a boundary vertex of u. ∎

By Theorem 2.1, every vertex of a connected graph G is either an interior vertex or a boundary vertex of G. If $\mathcal{I}(G) = \emptyset$, then $G = \partial(G)$ and G is referred to as a *self-boundary graph.*

The following theorem gives a sufficient condition (in terms of boundary vertices) for every vertex of a connected graph to be a peripheral vertex.

Proposition 2.2 *If G is a connected graph with the property that a vertex v is a boundary vertex of a vertex u if and only if u is a boundary vertex of v, then every vertex of G is a peripheral vertex.*

Proof. Assume, to the contrary, that not all vertices of G are peripheral vertices. Then there exists a nonperipheral vertex v that is adjacent to a peripheral vertex u. Let u' be a vertex of G such that $d(u, u') = \text{diam}(G)$. Since v is not a peripheral vertex, $d(u', v) = \text{diam}(G) - 1$. Since $u \in N(v)$ and $d(u, u') > d(v, u')$, it follows that v is not a boundary vertex of u'. By hypothesis, u' is not a boundary vertex of v; so there is a vertex $v' \in N(u')$ such that

$$d(v, v') = d(v, u') + 1 = \text{diam}(G).$$

This implies that v is a peripheral vertex, contrary to our assumption. ■

The converse of Proposition 2.2 is not true. For example, let G be the graph obtained from an even cycle $C_{2k} : v_1, v_2, \cdots, v_{2k}, v_1$, where $k \geq 2$, by adding a new vertex x and joining x to v_1 and v_2. Then $\text{rad}(G) = \text{diam}(G) = k$ and so G is self-centered. Thus every vertex of G is a peripheral vertex. Since x is a complete vertex, x is a boundary vertex of v_1, but v_1 is not a boundary vertex of x.

Corollary 2.3 *If G is a connected graph with the property that a vertex v is a boundary vertex of a vertex u if and only if u is a boundary vertex of v, then G is a self-boundary graph.*

The converse of Corollary 2.3 is not true. For example, consider the graph G of Figure 1, where each vertex of G is labeled with its eccentricity. Then $\{x, x', v, v'\}$ is the set of peripheral vertices of G. Since u and u' are boundary vertices of each other and y and y' are boundary vertices of each other, it follows that G is a self-boundary graph. On the other hand, the vertex v is a boundary vertex of a vertex u, but u is not a boundary vertex of a vertex v.

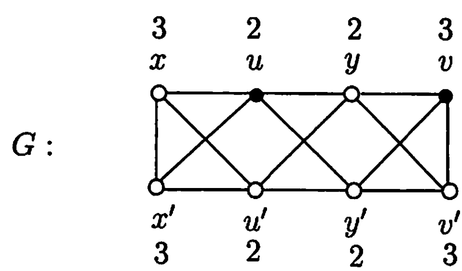

Figure 1: A self-boundary graph G

Consider the graph G of Figure 2. The boundary vertices of G are shaded, while the interior vertices are not. Also, each vertex of G is labeled with its eccentricity. Hence $\text{diam}(G) = 5$ and $\text{rad}(G) = 3$. Thus the central vertices of G are those having eccentricity 3. Consequently, some central vertices are interior vertices, while others are boundary vertices.

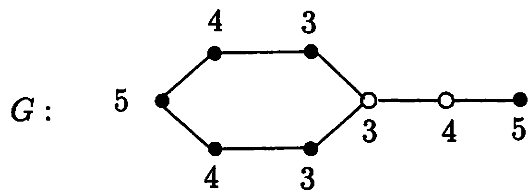

Figure 2: The graph G

There also exist graphs G with $\text{diam}(G) \neq \text{rad}(G)$ such that *every* central vertex is a boundary vertex (see Figure 3). In the graph G of Figure 3, the boundary vertices of G are precisely the peripheral vertices and central vertices of G. Hence, for the graph G of Figure 2, $Cen(G)$ and $Int(G)$ are disjoint. By modifying the graph G of Figure 2, we are able to obtain a more general result.

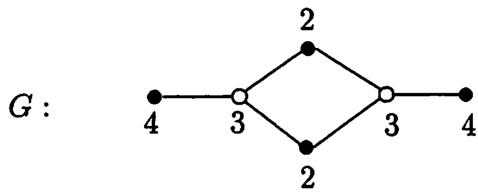

Figure 3: The graph G

Let G be a connected graph and let S and T be two subsets of $V(G)$. The *distance between* S *and* T is defined as

$$d(S,T) = \min\{d(s,t) : s \in S, t \in T\}.$$

Let $C(G)$ be the set of central vertices of G.

Proposition 2.4 *For each positive integer* $N \geq 1$*, there exists a graph that is not self-boundary such that*

$$d(C(G),\ \mathcal{I}(G)) = N.$$

Proof. Let $C_{4N} : v_1, v_2, \ldots, v_{4N}, v_1$ be a cycle of order $4N$ and let G be the graph obtained from C_{4N} by attaching paths of length N at v_1 and at v_{2N+1}, namely

$$P : v_1 = u_0, u_1, \cdots, u_N \text{ and } Q : v_{2N+1} = w_0, w_1, \cdots, w_N.$$

Since $C(G) = \{v_{N+1}, v_{3N+1}\}$ and

$$\mathcal{I}(G) = \{u_i, w_i : 0 \le i \le N-1\},$$

it follows that $d(\mathcal{C}(G), \mathcal{I}(G)) = N$. ∎

While it may be somewhat unexpected that the center and interior of a graph can be arbitrarily far apart, it should not be surprising to learn that these two subgraphs can be close to each other. Hedetniemi (see [2]) proved that for every graph G, there exists a connected graph H such that $Cen(H) = G$. The construction used in that proof is indicated in Figure 4. While G is the center of the graph H, it is not the interior of H since the cut-vertices v_1 and v_2 of H are necessarily interior vertices. Furthermore, some vertices of G may be boundary vertices as well. However, for each graph G, there does exist a connected graph, both of whose center *and* interior are G.

Figure 4: Constructing a graph with a given center

The *corona* $cor(G)$ of a graph G is that graph obtained from G by adding a pendant edge to each vertex of G.

Theorem 2.5 *For every graph G, there exists a connected graph H such that $Cen(H) = Int(H) = G$.*

Proof. Let $V(G) = \{u_1, u_2, \cdots, u_n\}$ and let

$$V(cor(G)) = V(G) \cup \{v_1, v_2, \cdots, v_n\},$$

where v_i is adjacent to u_i for $1 \le i \le n$. Let P and Q be two copies of the path P_5, where $P : x_1, x_2, \cdots, x_5$ and $Q : y_1, y_2, \cdots, y_5$. Let H be the graph obtained from $cor(G)$, P, and Q by joining each end-vertex of P and Q to every vertex of G. The graph H is shown in Figure 5. Since $e(u_i) = 3$ and $e(v_i) = 4$ for $1 \le i \le n$, $e(x_1) = e(x_5) = e(y_1) = e(y_5) = 4$, $e(x_2) = e(x_4) = e(y_2) = e(y_4) = 5$, and $e(x_3) = e(y_3) = 6$, it follows that $Cen(H) = G$.

It remains to show that $Int(H) = G$. Since every vertex of G is a cut-vertex of H, it follows by Theorem B that $V(G) \subseteq \mathcal{I}(H)$. Thus it suffices to show that every vertex of $V(H) - V(G)$ is a boundary vertex of H. By Theorem A, each vertex v_i $(1 \le i \le n)$ is a boundary vertex of H. Certainly, the peripheral vertices x_3 and y_3 are boundary vertices

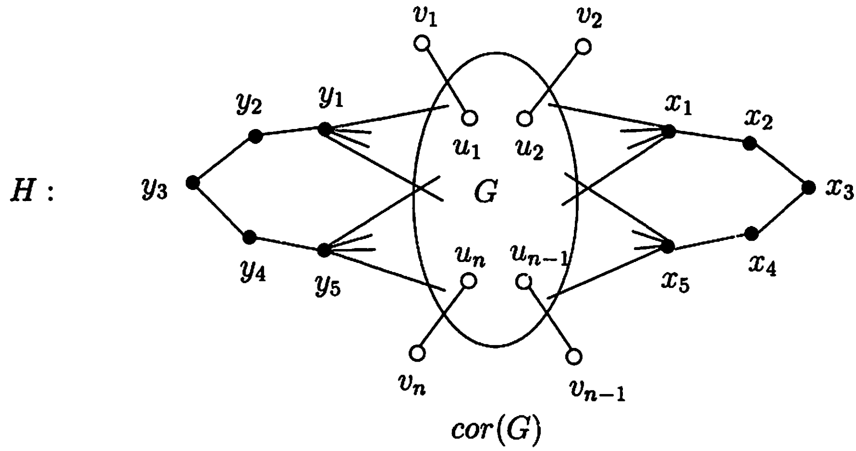

Figure 5: A graph H with $Cen(H) = Int(H) = G$

of H as well. Next we show that x_1 and x_4 are boundary vertices of each other. Observe that $d(x_1, x_4) = 3$. Since $N(x_4) = \{x_3, x_5\}$, and $d(x_1, x_3) = d(x_1, x_5) = 2$, it follows that x_4 is a boundary vertex of x_1. On the other hand, $N(x_1) = \{x_2\} \cup V(G)$ and $d(x_4, v) = 2$ for all $v \in \{x_2\} \cup V(G)$. Thus x_1 is a boundary vertex of x_4. By symmetry, it follows that x_2 and x_5, y_1 and y_4, and y_2 and y_5 are boundary vertices of each other as well. Therefore, $Int(H) = G$. ∎

3 Relationships Between the Boundary and Other Subgraphs of a Graph

Since every peripheral vertex in a connected graph G is a boundary vertex of G, the periphery and boundary of G always have a nonempty intersection. We have also seen that the center of a connected graph and its boundary may have a nonempty intersection as well. We now present a sufficient condition for the set of boundary vertices of a connected graph to be the union of the sets of central vertices and peripheral vertices.

For a connected graph G, let $\mathcal{B}(G) = V(\partial(G))$ denote the set of boundary vertices, $\mathcal{E}(G) = V(Ecc(G))$ the set of eccentric vertices, and $\mathcal{P}(G) = V(Per(G))$ the set of peripheral vertices of G.

Proposition 3.1 *Let G and H be two graphs, where G does not have a unique vertex of eccentricity* 1 *and H is disconnected. Then there exists a* *connected graph F such that*

$$Cen(F) = G, \ Per(F) = H, \text{ and } \partial(F) = G \cup H.$$

Proof. Let $H = H_1 \cup H_2$, where H_1 is a component of H. Let F be the graph obtained from G and H by (1) adding two new vertices x and y and (2) joining x to each vertex in $G \cup H_1$ and joining y to each vertex in $G \cup H_2$. The graph F is shown in Figure 6.

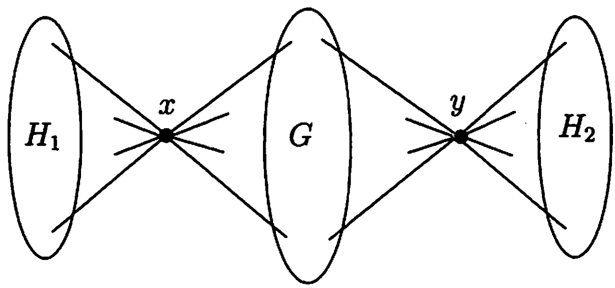

Figure 6: The graph F

We now show that F has the desired properties. Observe that $e(x) = e(y) = 3$, $e(u) = 2$ if $u \in V(G)$, and $e(u) = 4$ if $u \in V(H)$. Thus $C(F) = G$ and $Per(F) = H$. It remains to show that $\partial(F) = G \cup H$. Since $Per(F) \subseteq \partial(F)$, it suffices to show that $V(G) \subseteq \mathcal{B}(F)$. Let $u \in V(G)$. We consider two cases.

Case 1. $e_G(u) = 1$. Since G does not have a unique vertex of eccentricity 1, there exists a vertex v in G distinct from u such that $e_G(v) = 1$. Therefore, $d_F(u, v) = 1$ and $N_F(u) = V(G) \cup \{x, y\}$. Since $d_F(v, w) = 1$ for all $w \in N_F(u)$, it follows that u is a boundary vertex of v.

Case 2. $e_G(u) \neq 1$. Then there is a vertex v in G that is not adjacent to u. Since $d_F(u, v) = 2$ and $d_F(v, w) \leq 2$ for all $w \in N_F(u)$, it follows that u is a boundary vertex of v. Thus $\partial(F) = G \cup H$. ∎

We have seen graphs G for which $\mathcal{I}(G) \neq \emptyset$. In each case, if $\partial(G) \neq Per(G)$, then $\partial(G)$ is disconnected. However, this need not happen in general. For an integer $n \geq 4$, consider the Cartesian product $P_3 \times P_n$ of P_3 and P_n, where $P_3 : u_1, u_2, u_3$ and $P_n : v_1, v_2, \ldots, v_n$. For each pair i, j of integers with $1 \leq i \leq 3$ and $1 \leq j \leq n$, let $w_{i,j} = (u_i, v_j)$. Let G_n be the graph obtained from $P_3 \times P_n$ by adding the edges $w_{2,1}w_{1,2}$, $w_{2,1}w_{3,2}$, $w_{2,n}w_{1,n-1}$, $w_{2,n}w_{3,n-1}$ and those edges in the set

$$\{w_{2,j}w_{1,j-1}, w_{2,j}w_{1,j+1}, w_{2,j}w_{3,j-1}, w_{2,j}w_{3,j+1} : 2 \leq j \leq n-1\}.$$

Note that

$$\begin{aligned}\mathcal{I}(G_n) &= \{w_{2,j} : 2 \leq j \leq n-1\}\\ \mathcal{P}(G_n) &= \{w_{1,1}, w_{2,1}, w_{3,1}, w_{1,n}, w_{2,n}, w_{3,n}\}.\end{aligned}$$

Hence $\partial(G_n) \neq Per(G_n)$. On the other hand, $\partial(G_n)$ contains C_{2n+2} as a spanning subgraph and so is connected.

4 The Boundary Degree of a Vertex

Since an interior vertex v of a nontrivial connected graph G is not a boundary vertex of G, it follows that v is not a boundary vertex of any vertex of G. On the other hand, every boundary vertex is the boundary vertex of at least one vertex of G. We define the ***boundary degree*** $b(v)$ of a vertex v in a connected graph G of order $n \geq 2$ as the number of vertices u in G having v as a boundary vertex. Thus $0 \leq b(v) \leq n-1$, where $b(v) = 0$ if and only if v is an interior vertex of G. At the other extreme, we have the following consequence of Theorem A.

Corollary 4.1 *Let G be a graph of order $n \geq 2$ and let v be a vertex of G. Then $b(v) = n-1$ if and only if v is complete.*

In the graph $G = C_5 + K_1$ of Figure 7, the vertex z is an interior vertex and so $b(z) = 0$. On the other hand, u is a boundary vertex of w, x, and z, but not of v and y. Hence $b(u) = 3$ and, by symmetry, $b(v) = b(w) = b(x) = b(y) = 3$ as well. Thus G contains an odd number of vertices of odd boundary degree.

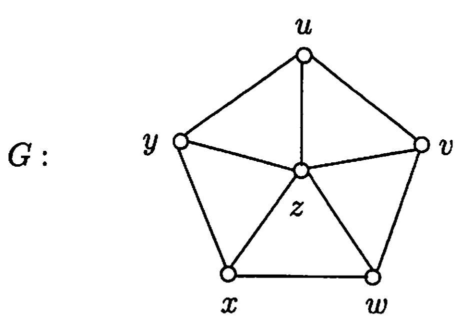

Figure 7: The graph $G = C_5 + K_1$

To show that no vertex in a connected graph G of order $n \geq 3$ has boundary degree $n-2$, we present the following lemma.

Lemma 4.2 *Let G be a graph of order $n \geq 3$ and let v be a vertex of G. If $x, y \in N(v)$ such that $xy \notin E(G)$, then v is not a boundary vertex of x or y.*

Proof. Since $d(v, x) = 1$ and $d(y, x) \geq 2$, it follows that $d(v, x) < d(y, x)$. Moreover, since $y \in N(v)$, it follows that v is not a boundary vertex of x. Similarly, v is not a boundary vertex of y. ■

Proposition 4.3 *Let G be a graph of order $n \geq 3$. If v is not a complete vertex of G, then $b(v) \leq n - 3$.*

Proof. If v is not a complete vertex of G, then there exist $x, y \in N(v)$ such that $xy \notin E(G)$. By Lemma 4.2, v is not a boundary vertex of x and y. Since v is also not a boundary vertex of itself, $b(v) \leq n - 3$. ■

Corollary 4.4 *Let G be a graph of order $n \geq 4$ and v be a vertex of G. Then*

$$b(v) \in \{0, 1, 2, \cdots, n-3, n-1\}.$$

Next we show that the restriction on the boundary degrees stated in Corollary 4.4 is the only restriction.

Theorem 4.5 *For each pair k, n of integers with $k \in \{0, 1, 2, \cdots, n - 3, n-1\}$ and $n \geq 3$, there exists a connected graph G of order n containing a vertex of boundary degree k.*

Proof. Since the result is true for $k = 0$ and $k = n - 1$, we may assume that $1 \leq k \leq n-3$. Let $G = K_{k+1} + \overline{K}_{n-k-1}$. Let $v \in V(K_{k+1})$. We show that $b(v) = k$. Since $V(\overline{K}_{n-k-1}) \subseteq N(v)$ and $V(\overline{K}_{n-k-1})$ is an independent set, it then follows by Lemma 4.2 that the vertex v is not a boundary vertex of any vertex in $V(\overline{K}_{n-k-1})$. Next we show that v is a boundary vertex of every vertex in $V(K_{k+1}) - \{v\}$. Let $u \in V(K_{k+1}) - \{v\}$. Then $N(u) = V(G) - \{u\}$. Since $d(u, v) = 1$ and $d(w, v) = 1$ for all $w \in N(u)$, it follows that $d(w, v) \leq d(u, v)$ for all $w \in N(u)$, which implies that v is a boundary vertex of u. Therefore, $b(v) = k$. ■

For each integer $n \in \{5, 6, 7\}$, Figure 8 shows a graph of order n for which the set of boundary degrees of its vertices is precisely $\{0, 1, 2, \ldots, n-3, n-1\}$. Whether such a graph exists for each integer $n \geq 8$ is not known.

A connected graph G is *r-boundary degree regular* (or *r-boundary regular*) if every vertex of G has boundary degree r.

Proposition 4.6 *For each integer $r \geq 1$, there exists an r-boundary regular graph.*

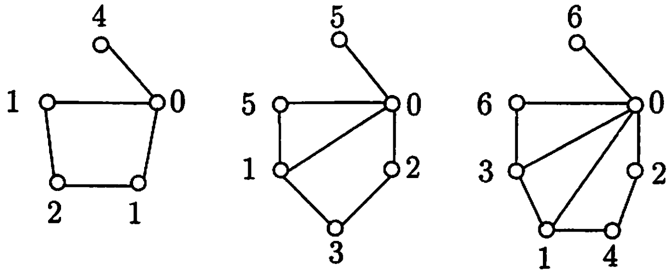

Figure 8: Graphs with boundary degrees $0, 1, 2, \ldots, n-3, n-1$

Proof. Let $G = K_{r+1} \times C_4$. For each integer i with $1 \le i \le 4$, let G_i be a copy of K_{r+1} in G with $V(G_i) = \{v_{i,1}, v_{i,2}, \cdots, v_{i,r+1}\}$. Then for each i $(1 \le i \le 4)$ and j $(1 \le j \le r+1)$, the vertex $v_{i,j}$ is adjacent to $v_{i,k}$ for all k with $1 \le k \le r+1$, $k \ne j$, and $v_{i,j}$ is adjacent to $v_{i+1,j}$ and $v_{i-1,j}$, where addition and subtraction are computed modulo 4. We show that G is an r-boundary regular graph. Let $v \in V(G)$. Assume, without loss of generality, that $v = v_{1,1}$. Since $d(v, w) = 3 = \text{diam}(G)$ for each $w \in V(G_3) - \{v_{3,1}\}$, it follows that v is a boundary vertex of w, and so $b(v) \ge |V(G_3) - \{v_{3,1}\}| = r$. Next, we show that v is not a boundary vertex of any vertex in $V(G) - (V(G_3) - \{v_{3,1}\})$. Let $u \in V(G) - (V(G_3) - \{v_{3,1}\})$. We consider three cases.

Case 1. $u = v_{3,1}$. Then $d(v, u) = 2$. Since $v_{1,2} \in N(v)$ and $d(v_{1,2}, u) = 3$, it follows that v is not a boundary vertex of u.

Case 2. $u \in V(G_1) - \{v_{11}\}$, *say* $u = v_{1,2}$. Then $d(v, u) = 1$. Since $v_{2,1} \in N(v)$ and $d(v_{2,1}, u) = 2$, it follows that v is not a boundary vertex of u.

Case 3. $u \in V(G_2)$ *or* $u \in V(G_4)$, *say* $u \in V(G_2)$. If $u = v_{2,1}$, then $d(u, v) = 1$. Since $v_{1,2} \in N(v)$ and $d(v_{1,2}, u) = 2$, it follows that v is not a boundary vertex of u. Thus we may assume that $u \in V(G_2) - \{v_{2,1}\}$, say $u = v_{2,2}$. So $d(u, v) = 2$. Since $v_{4,1} \in N(v)$ and $d(v_{4,1}, u) = 3$, it follows that v is not a boundary vertex of u.

Thus v is not a boundary vertex of any vertex in $V(G) - (V(G_3) - \{v_{3,1}\})$, implying that $b(v) \le r$. Therefore, $b(v) = r$. ■

Let r and n be integers with $0 \le r \le n-1$ such that at least one of these integers is even. If r is even, then we write $r = 2k$; while if r is odd, we write $r = 2k+1$. We define a graph $G_{r,n}$. Let $V(G_{r,n}) = \{v_1, v_2, \cdots, v_n\}$. If r is even, then

$$E(G_{r,n}) = \{v_i v_j : |i-j| \le k \text{ or } |i-j| \ge n-k\}.$$

If r is odd, then

$$E(G_{r,n}) = E(G_{r-1,n}) \cup \{v_i v_j : |i-j| = n/2\}.$$

Then $G_{r,n}$ is an r-regular graph of order n. Furthermore, each graph $G_{r,n}$ that is not complete has the property that for every two adjacent vertices u and v, $N[u] \neq N[v]$.

Although Proposition 4.6 states that an r-boundary regular graph exists for every positive integer r, no conditions on the order of the graph are stipulated. This brings up the following question.

Problem 4.7 *For which pairs r, n of positive integers, does there exist an r-boundary regular graph of order n?*

The following result is well-known.

Theorem C *There exists an r-regular graph of order n if and only if r and n are integers such that $0 \leq r \leq n-1$ and at least one of r and n is even.*

We give a partial answer to Problem 4.7. The following well-known result will be useful.

Theorem D *If G is a graph of order n such that $\deg v \geq (n-1)/2$ for every vertex v of G, then G is connected and, in fact, $\operatorname{diam}(G) \leq 2$.*

Theorem 4.8 *For every pair r, n of positive integers such that $n \geq 2r+1$ and at least one of r and n is even, there exists an r-boundary regular graph of order n.*

Proof. Let $G = G_{n-r-1,n}$. Thus G is an $(n-r-1)$-regular graph of order n. Since $n-r-1 \geq (n-1)/2$, it follows by Theorem D that $\operatorname{diam}(G) = 2$.

Let $v \in V(G)$. Thus $\deg_G v = n-r-1$. Hence there are exactly r vertices u of G such that $d(u,v) = 2$. Since $\operatorname{diam}(G) = 2$, it follows that $d(u,w) \leq 2$ for every $w \in N(v)$. Therefore, v is a boundary vertex of u and $b(v) \geq r$.

Next let $x \in N(v)$. We show that v is not a boundary vertex of x. Since $N[x] \neq N[v]$, there exists a vertex y such that $xy \notin E(G)$ and $vy \in E(G)$. Hence $d(x,v) = 1$ and $d(x,y) = 2$. Since v is not a boundary vertex of x, it follows that $b(v) = r$. ∎

In addition, for $n \geq 2$, the complete graph K_n is $(n-1)$-boundary regular of order n. Thus K_2 is 1-boundary regular of order 2. We have already noted that there is a 1-boundary regular graph of even order $n \geq 4$. That there is no 1-boundary regular graph of any odd order is verified next.

Theorem 4.9 *There is no nontrivial connected graph of odd order that is* 1*-boundary regular.*

Proof. Assume, to the contrary, that there exists a connected 1-boundary regular graph G of odd order $n = 2k + 1$. Since every vertex of G has boundary degree 1, there are at most $k - 1$ pairs $\{x, x'\}$ of vertices of G such that each of x and x' is a boundary vertex of the other. Hence there is at least one pair $\{x, y\}$ of vertices such that x is a boundary vertex of y, but y is not a boundary vertex of x.

Let $v_1, v_2 \in V(G)$ such that v_2 is a boundary vertex of v_1, but v_1 is not a boundary vertex of v_2. Since every vertex v has an eccentric vertex, which is necessarily a boundary vertex of v, it follows that v_2 is the unique eccentric vertex of v_1 but v_1 is not an eccentric vertex of v_2. Then $e(v_1) < e(v_2)$. Continuing in this manner, we arrive at a sequence $v_1, v_2, \ldots, v_\ell$ of $\ell \geq 3$ distinct vertices of G such that

$$e(v_1) < e(v_2) < \ldots < e(v_\ell)$$

and such that the unique eccentric vertex of v_ℓ is v_j, where $1 \leq j \leq \ell - 1$. We cannot have $2 \leq j \leq \ell - 1$, for otherwise, $b(v_j) \geq 2$. Thus v_1 is the unique eccentric vertex of v_ℓ. However, then,

$$e(v_1) < e(v_2) < \ldots < e(v_\ell) < e(v_1),$$

which is a contradiction. ∎

It can also be shown that there is no 3-boundary regular graph of order 6. Whether there exists a 3-boundary regular graph of order 7 is not known. Indeed, we have the following open question.

Problem 4.10 *Does there exist a pair r, n of odd integers such that there is an r-boundary degree regular connected graph of order n?*

5 Boundary Vertices Revisited

We conclude this paper by looking at boundary vertices from another point of view. First, we present the following result.

Proposition 5.1 *Let G be a connected graph containing vertices u and v such that v is a boundary vertex of u. If u lies in an $x - v$ geodesic for some vertex x of G, then v is also a boundary vertex of x.*

Proof. Since v is a boundary vertex of u, it follows that $d(w, u) \leq d(v, u)$ for all $w \in N(v)$. Because u lies on an $x - v$ geodesic, $d(v, x) = d(v, u) +$

$d(u,x)$. Thus, if $w \in N(v)$, then $d(w,x) \leq d(w,u) + d(u,x) \leq d(v,u) + d(u,x) = d(v,x)$. Therefore, v is a boundary vertex of x. ∎

For a vertex v in a nontrivial connected graph G, a $u-v$ geodesic P is called a *ray* at v if u lies in no $x-v$ geodesic for all $x \in V(G) - \{u\}$. If P is a $u-v$ ray at v, then u is called a *ray vertex of* v. In other words, if u is a ray vertex of v and lies on an $x-v$ geodesic, then $x = u$. We denote the set of ray vertices at v by $R(v)$. Since all eccentric vertices of v are in $R(v)$, it follows that $R(v) \neq \emptyset$.

Proposition 5.2 *Let v be a vertex in a nontrivial connected graph G. Then every vertex of G lies on some ray at v.*

Proof. Let $R(v) = \{v_1, v_2, \cdots, v_k\}$. Assume, to the contrary, that there exists a vertex $w \in V(G)$ such that w lies on no $v-v_i$ geodesic in G for all i $(1 \leq i \leq k)$. Thus $w \notin \{v\} \cup R(v)$. So there exists a vertex x such that w lies in some $v-x$ geodesic in G. Among all such vertices x, let x^* be one whose distance from v is maximum. We claim that $x^* \in R(v)$; for if $x^* \notin R(v)$, then there exists $x' \in V(G)$ such that x^* lies in some $v-x'$ geodesic Q in G. Let P be a $v-x^*$ geodesic of G containing w. Also, let Q_1 be the $v-x^*$ subpath of Q and Q_2 the x^*-x' subpath of Q. Thus Q_1 is a $v-x^*$ geodesic in G. Then the $v-x'$ path P' obtained from P and Q_2 is a $v-x'$ geodesic containing w. Since $d(v,x') > d(v,x^*)$, this contradicts the maximality of x^*. Thus $x^* \in R(v)$, as claimed. ∎

Corollary 5.3 *Let v be a vertex in a nontrivial connected graph G. If v is a boundary vertex of G, then v is a boundary vertex of at least one ray vertex of v.*

Proof. Since v is a boundary vertex of G, it follows that v is a boundary vertex of some vertex u in G. If $u \in R(v)$, we have the desired result. Thus we may assume that $u \notin R(v)$. By Proposition 5.2, there exists $v' \in R(v)$ such that u lies in some $v-v'$ geodesic in G. It then follows from Proposition 5.1 that v is a boundary vertex of v' in $R(v)$. ∎

We are now in a position to give an alternative description of the set of boundary vertices of a vertex.

Proposition 5.4 *Let G be a nontrivial connected graph and let $v \in V(G)$. Then $R(v)$ is the set of boundary vertices of v.*

Proof. Certainly, every boundary vertex of G is a ray vertex of G. Thus it suffices to show that every ray vertex of G is a boundary vertex of G. Assume, to the contrary, that there exists a vertex $u \in R(v)$ such that u is not a boundary vertex of v. Then there exists $w \in N(u)$ such that

$d(w, v) > d(u, v)$. Since w and u are adjacent, $d(w, v) = d(u, v) + 1$. This implies that u lies in some $v - w$ geodesic in G, which is a contradiction. ∎

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