# Further Investigations on Balanced Arrays of Strength Six

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**Abstract**

In this paper we obtain some necessary existence conditions for bi-level balanced arrays of strength six by using some classical inequalities and by expressing the moments of the weights of the columns of such arrays in terms of its parameters. We present some illustrative examples to compare these results with the earlier known results.

## 1. Introduction and Preliminaries.

For ease of reference we present here the definition of a balanced array (B-array).

Definition 1.1. A matrix $T$ with $m$ rows, $N$ columns, and with $s$ symbols (say; 0, 1, 2, .., s-1) is called a B-array of strength t if in every $(t \times N;\ t \leq m)$ submatrix $T^*$ of $T$ and for every $(t \times 1)$ vector of $T^*$, we have the following condition satisfied: $\lambda(\underline{\alpha}; T^*) = \lambda(P(\underline{\alpha}), T^*)$ where $P(\underline{\alpha})$ is the vector obtained by permuting the elements of $\underline{\alpha}$, and $\lambda(\underline{\alpha}; T^*)$ represents the frequency with which $\underline{\alpha}$ appears in $T^*$.

Remark: In statistical design of experiments the rows (also called 'constraints') correspond to factors, the columns (called also 'runs') correspond to treatment combinations, and $s$ symbols to the levels at which each factor appears in the experiment.

In this paper we restrict ourselves to B-arrays with $t = 6$, and $s = 2$ (elements 0 and 1). It is quite clear that $w(\underline{\alpha}) = w[P(\underline{\alpha})]$ where $w(\underline{\alpha})$, called the weight of $\underline{\alpha}$, is the number of 1's in $\underline{\alpha}$. For this special case, we can say that every $(6 \times 1)$ vector $\underline{\alpha}$ of weight $i (0 \leq i \leq 6)$ appears a constant number (say, $\mu_i$) times. We call $\underline{\mu}'=(\mu_0, \mu_1, .., \mu_6)$ the index set of array $T$, and $T$ is denoted by $(m, N, s = 2,\ t = 6,\ \underline{\mu}')$ where clearly $N = \sum_{i=0}^{6} \binom{6}{i} \mu_i$. B-arrays with different strengths have been extensively used in the construction of optimal fractional factorial designs, and are related to numerous areas of combinatorial mathematics. For example, B-arrays with $t = 6$ and $s = 2$ give rise to, under certain conditions, fractional factorial designs of resolution seven. If $\mu_i = \mu$ (for each $i$), then B-arrays are reduced to orthogonal arrays (O-arrays), and the incidence matrix of a balanced incomplete block design (BCBD) is merely a B-array with $t = 2$ with the restriction that each column of $T$ has the same weight. To learn more about the applications and importance of these configurations in statistics and combinatorics, the interested reader may consult the references (by no means an exhaustive list) at the end of this paper, and also further references given therein.

To construct B-arrays with $t = 6$, $s = 2$, $\underline{\mu}'=(\mu_0, \mu_1, .., \mu_6)$, and $m > t$ is a difficult and nontrivial problem. The problem of obtaining an upper bound on m for a given $\underline{\mu}'$ is very important in combinatorics and statistical design of experiments. Such problems for O-arrays and B-arrays (with different values of $t$) have been investigated, among others, by Bose and/or Bush [1, 2], Chopra and/or Dios [3, 4, 5], Rafter and Seiden[9], Rao [10], Saha, et al. [11], Yamamoto, et al. [14], etc.

## 2. Main Results with Discussion

The following results can be easily established.

Lemma 2.1. A B-array $T(m = 6,\ N,\ t = 6,\ s = 2;\ \underline{\mu}')$ always exists.

Lemma 2.2. A B-array $T$ with index set $\underline{\mu}'=(\mu_0, \mu_1, .., \mu_6)$ is also of strength $t'$ where $0 \leq t' \leq 6$.

Remark: It is not difficult to see that, considered as an array of strength $t'$, the $j$th element $A_{jt'}$ of the parameter-vector of the array is given by $A_{jt'} = \sum_{i=0}^{t-t'} \binom{t-t'}{i} \mu_{i+j}, j = 0, 1, 2, .., t'$ with the convention that $\binom{a}{b} = 1$ if $a = b = 0$. It is quite clear that $A_{jt'}$ elements are merely linear combinations of $\mu_i$'s. Furthermore, if $t' = t = 6$, then $A_{jt'}$ coincide with $\mu_j$ (the given set of parameters). The next result only makes use of the elements $A_{t't'}$ where $0 \leq t' \leq 6$.

Lemma 2.3. Consider a B-array $T$ with $m$ rows, $t = 6$, and $s = 2$. Let $x_j (j = 0, 1, 2, .., m)$ denote the frequency of the columns of weight $j$ in $T$.

Then the following results are true:

$$\sum_{j=0}^{m} j^k x_j = N \qquad \text{if } k = 0 \tag{2.1}$$

$$\sum_{j=0}^{m} j^k x_j = \sum_{r=1}^{k} a_r m_r A_{r,r} \qquad \text{for } 1 \le k \le 6$$

where $m_r$ stands for $m(m-1)(m-2)...(m-r+1)$, and $a_r$ are known positive integers which appear while deriving the above results.
Remark: The results of Lemma 2.3 express the moment about 0 of the weights of the columns of $T$ as a polynomial function in $m, \mu_{0,}\mu_{1,}.., \mu_6$. For computational ease, we provide next the values of $a_r$ for various values of $k$: $(k=1;\ a_1=1)$, $(k=2;\ a_1=a_2=1)$, $(k=3;\ a_1=a_3=1,\ a_2=3)$, $(k=4;\ a_1=a_4=1,\ a_2=7,\ a_3=6)$, $(k=5;\ a_1=a_5=1,\ a_2=15,\ a_3=25,\ a_4=10)$, and $(k=6;\ a_1=a_6=1,\ a_2=31,\ a_3=90,\ a_4=65,\ a_5=15)$. Next we quote, without proof, the following result from Chopra and Dios [5] for later use.
Result: Consider a B-array $T$ of strength six, with $m$ rows, and index set $\underline{\mu}'$. Then the following is true:

$$\begin{aligned}
&[B_0B_2 - B_1^2]\left[\begin{array}{c} B_0^5B_6 - 6B_0^4B_5B_1 + 15B_0^3B_4B_1^2 - 20B_0^2B_3B_1^3 \\ +15B_0B_2B_1^4 - 5B_1^6 \end{array}\right] \\
&\ge [B_0^3B_4 - 4B_0^2B_3B_1 + 6B_0B_2B_1^2 - 3B_1^4]^2 \\
&+ [B_0B_2 - B_1^2]\,[B_0^2B_3 - 3B_0B_2B_1 + 2B_1^3]^2
\end{aligned} \tag{2.2}$$

where $B_k$ stands for $\sum_{j=0}^{m} j^k x_j$. Next, we state the well-known Cauchy-Schwarz's classical inequalities.
Result: For sequences of non-negative real numbers $(a_1, a_2, ..., a_n)$, $(b_1 b_2, ..., b_n)$, $(c_1, c_2, ..., c_n)$, and $(d_1, d_2, ..., d_n)$ the following inequalities hold:

$$(a)\ \left(\sum_{k=1}^{n} a_k b_k\right)^2 \le \sum a_k^2 \sum b_k^2 \tag{2.3}$$

$$(b)\ \left(\sum a_k b_k c_k\right)^4 \le \sum a_k^4 \sum b_k^4 \left(\sum c_k^2\right)^2$$

$$(c)\ \left(\sum a_k b_k c_k\right)^3 \le \sum a_k^3 \sum b_k^3 \sum c_k^3$$

$$(d)\ \left(\sum a_k b_k c_k d_k\right)^4 \le \sum a_k^4 \sum b_k^4 \sum c_k^4 \sum d_k^4$$

Next, we obtain a set of necessary conditions for the existence of B-arrays by using (2.1) and (2.3).

Theorem 2.1. Consider a B-array $T$ with $\underline{\mu}'=(\mu_0,\mu_1,..,\mu_6)$ and with $m$ rows. Let $x_j$ be the number of columns of weight $j$ in $T$. Then the following results are true:

$$(a)\ B_5^2 \le B_6B_4 \text{ where } B_k = \sum j^k x_j \tag{2.4}$$

$$(b)\ B_4^2 \le B_5B_3$$

$$(c)\ B_4^3 \le B_6B_5B_1$$

$$(d)\ B_4^4 \le B_6B_5B_3B_2$$

$$(e)\ B_3^2 \le B_5B_1$$

$$(f)\ B_3^3 \le \min\left[B_5B_2^2,\ B_6B_2B_1,\ B_5B_4\right]$$

$$(g)\ B_3^4 \le B_0B_6B_5B_1$$

$$(h)\ B_2^3 \le B_0^2B_6$$

$$(i)\ B_2^4 \le B_0^2B_5B_3$$

Proof outline: All the above results can be easily obtained by using (2.3) with appropriate values of $a_k$, $b_k$, $c_k$, and $d_k$. For example, to obtain (2.4a) we set $a_j = j^3\sqrt{x_j}$, and $b_j = j^2\sqrt{x_j}$ in (2.3a) and the result follows. To obtain (2.4d), we use (2.3d) with the following substitutions: $a_j = j^{\frac{6}{4}}\sqrt[4]{x_j}$, $b_j = j^{\frac{5}{4}}\sqrt[4]{x_j}$, $c_j = j^{\frac{3}{4}}\sqrt[4]{x_j}$, and $d_j = j^{\frac{2}{4}}\sqrt[4]{x_j}$ and the result (2.4d) follows.

Note: Clearly every inequality in (2.4) is a polynomial function of $m$ only for a given $\underline{\mu}'$. It is thus easy to check every condition in (2.4) by using a computer program. If one of these conditions is contradicted for $m = k+1$ (say), then clearly an upper bound on the number of constraints is $k$. We must point out, however, that an array $T$ with parameters satisfying all the conditions in (2.4) may or may not exist. Next we provide some illustrative examples to compare the results obtained by using (2.4) with those obtained by using (2.2).

Example 2.1. Consider a B-array $T$ with $\underline{\mu}'$=(2, 3, 3, 3, 3, 3, 2) with $m = 10$. Clearly $N = 190$, and we find L.H.S. = 1.059097 $\times 10^{20}$ while R.H.S. = 2.094517 $\times 10^{20}$ which is a contradiction if we use (2.2). If we use (2.4a), we obtain the contradiction for the same array at $m = 19$. Therefore $m \le 9$ from (2.2), and $m \le 18$ using (2.4a). Thus we obtain a sharper upper bound by using (2.2).

Example 2.2. Take $\underline{\mu}'$=(4, 4, 3, 2, 3, 4, 4). No contradiction was obtained by using (2.2) and all values of $m$ through 30 (for $m = 30$, the L.H.S. = 7.02604E + 24, R.H.S. = 6.206146E + 23). Taking $m = 22$ and using

(2.4a), we obtain a contradiction since L.H.S. = 3.386455E + 15 and R.H.S. = 3.382959E + 15. Thus we can say that $m \leq 21$ is a smaller upper bound as compared to the one obtained from (2.2).

Example 2.3. Consider an array $T$ with $\underline{\mu}'$=(1, 2, 1, 1, 4, 3, 2). For all $m$ in the range $6 \leq m \leq 50$ we observed (2.2) to be always satisfied (e.g., using $m = 50$, L.H.S. = 1.988602E + 25 and R.H.S. = 3.370953E + 24). Thus our upper bound on $m$, according to (2.2), exceeds 50. If we use (2.4b) with $m = 11$, we get a contradiction since L.H.S. = 7.372528E + 10, and R.H.S. - 7.339436E + 10 implying that L.H.S. $\not<$ R.H.S. Thus it gives $m \leq 10$, a smaller upper bound.

From the above it is quite clear that no single condition provides us, so far, with uniformly better results on the number of constraints for all arrays (our speculation that (2.2) could perhaps be such a condition is obviously not true).

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