Vertex-Magic Cubic Graphs

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Abstract

Let G_1 and G_2 be any two 2-regular graphs, each with n vertices. Let G be any cubic graph obtained from G_1 and G_2 by adding n edges, each of which joins a vertex in G_1 to a vertex in G_2. We show that G has a myriad of vertex-magic total labelings, with at least three different magic constants. This class of cubic graphs includes all generalized Peterson graphs.

1 Introduction

Suppose G is a graph with vertex set V and edge set E. Let $\mathrm{adj}(v)$, $v \in V$ be the set of all vertices adjacent to v. We will call λ a total numbering of G if λ is a map from $V \cup E$ into the integers $\{1, 2, \ldots, |V| + |E|\}$. We define the λ-weight of vertex x to be

$$\mathrm{wt}(x) = \lambda(x) + \sum_{y \in \mathrm{adj}(x)} \lambda(xy).$$

We call λ a total labeling if it is also a bijection, and we say that λ is vertex-magic if there is a constant h so that for every x, $\mathrm{wt}(x) = h$. A graph with such a labeling is called a vertex-magic graph.

There has been a flurry of recent activity on this new and exciting topic, including [1], [2], and [4], to name just a few. In [3], it is shown that $K_{m,m}$ is a vertex-magic graph, but if $n > m + 1$, then $K_{m,n}$ is not vertex-magic. This, together with other results and examples, led MacDougall to conjecture that any regular graph other than K_2 or $2K_3$ must be vertex-magic. We began our investigation by looking for a counterexample among the large class of 3-regular graphs described in the abstract, but instead proved that every graph in this class has a very large number of vertex-magic total labelings.

We should also mention that our result now shows that all generalized Peterson graphs are vertex-magic. Previously, it had been shown [5] that $P(m,r)$ is vertex-magic if m and r are relatively prime. One of the open problems stated in [5] is to find the complete list of possible magic constants for all generalized Peterson graphs. In this short note, we make a slight dent in this problem, as we have found three distinct magic constants for each of our graphs. We suspect there are many more.

In section 2 we construct total numberings, each of which can be "lifted" to three different labelings with three different magic constants. The listing is defined in section 3. We conclude by briefly describing the spectrum problem.

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2 Numberings

For the rest of this paper, G will refer to a cubic graph obtained from two 2-regular graphs G_1 and G_2 by adding n edges, each of which joins a vertex in G_1 to a vertex in G_2. Let V_i be the vertex set of G_i and let E_i be the edge set of G_i. Let E_s be the set of new edges, or spokes, added in order to obtain G. Thus, $E_s = E(G) \setminus (E(G_1) \cup E(G_2))$. In this section we will define five bijections.

$$f_i : V_i \to \{1,2,\dots,n\}, i = 1,2,$$
$$g_i : E_i \to \{1,2,\dots,n\}, i = 1,2,$$
$$c : E_s \to \{1,2,\dots,n\}.$$

Taken together, these bijections form a map

$$\mu : V \cup E \to \{1,2,\dots,n\},$$

which is obviously not injective and therefore not a total labeling. However, it is a total numbering of G, and we will pick μ so that the μ-weight of any vertex is $2n + 2$. Somewhat surprisingly, we can begin by picking c arbitrarily.

Next we will define g_1 in terms of c. Without loss of generality, we can assume that G_1 is the union of disjoint cycles. Let α be any one of those cycles, and write $\alpha = (v_1, v_2, \dots, v_a)$, where $v_i \in V_1$. We define $v_{a+1} = v_1$ and $v_0 = v_a$ in order to simplify our formulas and in particular, we can write all the edges of α as (v_i, v_{i+1}), where $1 \le i \le a$.

Define $g_1(v_i, v_{i+1}) = (n+1) - c(v_i, u_i)$, if $1 \le i \le a$, where u_i is the only vertex from V_2 that is incident with v_i.

Next define $f_1(v_i) = (n+1) - g_1(v_{i-1}, v_i)$, if $1 \le i \le a$. Thus we have:

$$\mu(v_i) = [f_1(v_i) + g_1(v_{i-1}, v_i)] + [g_1(v_i, v_{i+1}) + c(v_i, u_i)] = (n+1) + (n+1).$$

We complete the definition of g_1 and f_1 on the other cycles in exactly the same way. Note that g_1 inherits the bijective property from c, and f_1 inherits the bijective property from g_1. We also pick g_2 and f_2 in the same way, and again we can arrange the μ-weight at each vertex to be $2n+2$.

3 Labelings

The purpose of this section is to construct a vertex-magic total labeling λ for G with magic constant h. Roughly speaking, we want to view μ as the units digit of λ when written in base n.

We define λ by the following:

$\lambda(v) = \mu(v) + (m_i)n$, if $v \in V_i, i = 1, 2$.
$\lambda(u, v) = \mu(u, v) + (r_i)n$, if $u, v \in V_i, i = 1, 2$.
$\lambda(u, v) = \mu(u, v) + (s)n$, if $v \in V_1$ and $u \in V_2$.
where m_i, r_i, and s are integers that we will pick shortly.

Since $h = \text{wt}(v)$ for any $v \in V_i$, we have

$$h = (2n+2) + (m_i)n + (2r_i)n + (s)n.$$

In particular, $(m_1) + (2r_1) = (m_2) + (2r_2)$. Since we want λ to be a total labeling, we will assume $\{m_1, m_2, r_1, r_2, s\} = \{0, 1, 2, 3, 4\}$. In fact this guarantees that λ is a total labeling. It is without loss of generality that we may assume that $r_1 < r_2$. It is an easy exercise to see that these conditions leave us with exactly the following solutions:

Solution #1: $r_1 = 0, r_2 = 1, m_1 = 4, m_2 = 2, s = 3$.
In this case $h = 2 + 9n$.
Solution #2: $r_1 = 3, r_2 = 4, m_1 = 2, m_2 = 0, s = 1$.
In this case $h = 2 + 11n$.
Solution #3: $r_1 = 1, r_2 = 3, m_1 = 4, m_2 = 0, s = 2$.
In this case $h = 2 + 10n$.

Thus, we have proved:

Theorem 3.1 *Let G_1 and G_2 be any two 2-regular graphs, each with n vertices. Let G be any cubic graph obtained from G_1 and G_2 by adding n edges, each of which joins a vertex in G_1 to a vertex in G_2. Then G has many vertex magic total labelings for each of the magic constants $2 + 9n$, $2 + 10n$, and $2 + 11n$.*

4 Concluding Remarks

The set of possible magic constants of a graph is called its spectrum. One can rule out all but a small range of numbers using an arithmetic argument. Indeed, if one adds together the labels at each vertex, one gets:

$$| V | h = \sum_{v_i \in V} \lambda(v_i) + 2 \sum_{(v_i, v_j) \in E} \lambda(v_i, v_j).$$

The right hand side is minimized (respectively maximized) if the edges are labeled with the smallest (respectively biggest) possible numbers. We will say that the values not ruled out by this argument are admissible. Computer searches have found that for regular graphs, almost all admissible values can be realized as the magic constant of some labeling. A notable exception is K_4, which cannot realize the admissible values 19,22, or 25. Since K_4 is a 3-regular graph with $2n$ vertices ($n = 2$), it is surprising that there is no labeling with $h = 10n + 2$ as in our theorem. We would welcome a bigger list of exceptions as this may help to shed light on MacDougall's Conjecture.

References

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