The Covering Numbers $g_3^{(4)}(v)$

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ABSTRACT. The number $g_3^{(4)}(v)$ represents the minimum cardinality of a pairwise balanced design on v elements in which the largest block size is four and every pair occurs exactly 3 times. We give a survey of the results for this quantity.

1 Introduction

Since this survey is designed for a general audience, we present it in a leisurely form and include preliminary material that will be familiar to many readers. We start with a v-set (the elements may be numbered $1, 2, \ldots, v$), and consider perfect λ-coverings (or pairwise balanced designs) in which the largest block size is 4 and each pair occurs exactly λ times. There will be a large number of such λ-coverings. Clearly the λ-covering of maximal cardinality will include one block of size 4 and all other blocks of size 2; so this maximal cardinality is $\lambda\binom{v}{2} - 5$. We are interested in determining the minimal cardinality, denoted by $g_\lambda^{(4)}(v)$, of such a λ-covering.

Example. Take $\lambda = 1$, $v = 7$.

The possible λ-coverings are as follows:

$$1234, 1567, 25, 26, 27, 35, 36, 37, 45, 46, 47;$$
$$1234, 156, 257, 367, 17, 26, 35, 45, 46, 47;$$
$$1234, 156, 257, 9 \text{ pairs};$$
$$1234, 156, 12 \text{ pairs};$$
$$1234, 15 \text{ pairs}.$$

Clearly, the second of these coverings provides the result $g_1^{(4)}(7) = 10$.

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2 Historical Remarks

The quantity $g_1^{(4)}(v)$ was determined by Stanton and Stinson [8], except for the values $v = 17, 18, 19$. Later papers determined $g_1^{(4)}(17) = 30$ or 31; $g_1^{(4)}(18) = 31$ or 32 or 33; $g_1^{(4)}(19) = 35$. See references [2], [4], [5], [6], [7]. Recently [1] Grannell, Griggs and Stanton settled the case $g_2^{(4)}(v)$.

In this survey, I want to present the results for $\lambda = 3$, which are quite different from those for $\lambda = 1$ and $\lambda = 2$.

We start with the reminder that a Balanced Incomplete Block Design (BIBD) is a system (v, b, r, k, λ) in which

$$\begin{aligned} v &= \text{number of elements} \\ b &= \text{number of blocks} \\ r &= \text{frequency of any element} \\ k &= \text{size of any block} \\ \lambda &= \text{frequency of any pair.} \end{aligned}$$

By counting elements and by counting pairs, it immediately follows that

$$bk = rv, \quad \lambda(v-1) = r(k-1).$$

Many years ago, Hanani proved that, for $k = 4$, any set of parameters that satisfies these 2 conditions corresponds to a BIBD that exists. Consequently, if we set $k = 4$, $\lambda = 3$, then there is a BIBD with parameters

$$(v, \frac{v(v-1)}{4}, v-1, 4, 3).$$

So this design exists if $v = 4n$ or $v = 4n+1$. Consequently, it immediately follows that

$$\begin{aligned} g_3^{(4)}(4n) &= n(4n-1), \\ g_3^{(4)}(4n+1) &= n(4n+1). \end{aligned}$$

In both these cases, all blocks have size 4. So we have only to consider the cases $v = 4n+3$, $v = 4n+2$.

Before tackling these 2 remaining cases, we establish a very useful bound.

Let $g = g_\lambda^{(4)}(v)$ and suppose that the minimal pairwise balanced design contains g_i blocks of length i $(i = 2, 3, 4)$. Then

$$g = g_2 + g_3 + g_4.$$

Suppose further that the g block lengths are k_i, where $k_i = 2$ or 3 or 4. Thus, there are g_2 blocks of length $k_i = 2$, g_3 blocks of length $k_i = 3$, g_4 blocks of length $k_i = 4$.

Then $\sum_{i=1}^{g}(k_i-3)(k_i-4) = 2g_2 + 0g_3 + 0g_4 = 2g_2$.

But $\sum_{i=1}^{g}(k_i-3)(k_i-4) = 2k_i(k_i-1) - 6\sum k_i + 12\sum 1$.

Now $\sum k_i(k_i-1) = \lambda v(v-1)$, $\sum 1 = g$.

Also $6\sum k_i = 6\sum r_i$, where the r_i are the frequencies of the v elements in the PBD. Clearly,

$$r_i \geq \left\lceil \frac{\lambda(v-1)}{3} \right\rceil,$$

and so we may write

$$r_i = \left\lceil \frac{\lambda(v-1)}{3} \right\rceil + E_i,$$

where E_i is an integer ≥ 0.

By substitution, we obtain

$$2g_2 = \lambda v(v-1) - 6\sum\left\{\left\lceil \frac{\lambda(v-1)}{3} \right\rceil + E_i\right\} + 12g.$$

Solving for $12g$, we obtain Formula (1).

$$12g = v\left\{6\left\lceil \frac{\lambda(v-1)}{3} \right\rceil - \lambda(v-1)\right\} + 6\sum E_i + 2g_2. \tag{1}$$

3 The Case $v = 4n+3$

Apply Formula (1) to the case $\lambda = 3$, $v = 4n+3$. Then

$$\begin{aligned} 12g &= (4n+3)\{3(4n+2)\} + 6\sum E_i + 2g_2, \\ &= 48n^2 + 60n + 18 + 6\sum E_i + 2g_2. \end{aligned}$$

Then $g \geq 4n^2 + 5n + 2$.

We can readily remove the possibility of equality. If $q = 4n^2 + 5n + 2$, we have

$$\begin{aligned} g_2 + g_3 + g_4 &= 4n^2 + 5n + 2, \\ g_2 + 3g_3 + 6g_4 &= 3(4n+3)(2n) = 24n^2 + 30n + 9. \end{aligned}$$

It follows that $5g_2 + 3g_3 = 3$, and so $g_2 = 0$, $g_3 = 1$. Thus there is one block of length 3, $4n^2 + 5n + 1$ blocks of length 4.

Now consider an element occurring once in the triple, γ times in the quadruples. It must occur with $3(4n+2)$ other elements and so

$$2 + 3\gamma = 3(4n+2).$$

This is clearly impossible, and so

$$g \geq 4n^2 + 5n + 3.$$

4 A Pseudo-Solution for $v = 4n + 3$

First, we introduce a useful piece of notation. We agree that literal symbols shall be invariant but that numerical symbols shall cycle according to a specified modules. Thus, for example, $[a, 1, 2, 3]_5$ will indicate the set of 5 blocks obtained by cycling modulo 5, that is, the blocks

$$(a, 1, 2, 3), (a, 2, 3, 4), (a, 3, 4, 5), (a, 4, 5, 1), (a, 5, 1, 2).$$

It is easy to obtain solutions in $4n^2+5n+4$ blocks, and we illustrate the procedure.

For $v = 7$, take blocks ab, ab, ab, $[a, 1, 2, 3]_5$, $[b, 1, 2, 4]_5$. Since the numerical differences in these 2 cycles are $(1, 2, 1)$ and $(1, 2, 2)$, we see that every pair occurs exactly 3 times.

Similarly, for $v = 11$, take blocks ab, ab, ab, $[a, 1, 4, 5]_9$, $[b, 1, 3, 6]_9$, and $[1, 2, 3, 5]_9$. Here the numerical differences are $(3, 4, 1)$, $(2, 4, 3)$, and $(1, 2, 4, 1, 3, 2)$. So again every pair occurs 3 times.

The case $v = 15$ is particularly interesting. We get blocks ab, ab, ab, $[a, 1, 4, 5]_{13}$, $[b, 1, 3, 8]_{13}$, and $[1, 2, 4, 10]_{13}$, $[1, 2, 4, 10]_{13}$. The 2 sets of numerical triples have differences $(3, 4, 1)$ and $(2, 6, 5)$, and so generate the families Steiner Triple System on 13 points. The 26 quadruples provide 2 copies of $PG(2, 3)$ and thus form the BIBD $(13, 26, 8, 4, 2)$.

This type of pattern continues. For $v = 19$, we have 3 pairs ab, 17 quadruples $[a, 1, 5, 6]_{19}$, 17 quadruples $[b, 1, 4, 8]_{17}$, 51 quadruples $[1, 2, 8, 9]_{17}$, $[1, 3, 7, 9]_{17}$, and $[1, 4, 6, 9]_{17}$. The differences from the numerical triples are $(4, 5, 1)$ and $(3, 7, 4)$. The differences from $[1, 2, 8, 9]$ can be represented by a quadrilateral with sides $1, 6, 1, 8$, and diagonals of 7 and 7; those from $1, 3, 7, 9$ by a quadrilateral with sides $2, 4, 2, 8$, and diagonals of 6 and 6; and those from $1, 4, 6, 9$ by a quadrilateral with sides $3, 5, 3, 8$, and diagonals 5 and 5.

These PBDs on $4n^2 + 5n + 4$ points are intimately connected with the BIBDs having a triplicated block studied by D.A. Preece [3].

5 The Solution is $4n^2 + 5n + 3$ Blocks

As indicated in the last section, a solution in $4n^2 + 5n + 4$ blocks is readily obtained. We first show that, for $v = 7$, this is the correct value.

Suppose if possible that $g_3^{(4)}(7) = 12$. then

$$g_2 + g_3 + g_4 = 12,$$
$$g_2 + 3g_3 + 6g_4 = 63.$$

Then $5g_2 + 3g_3 = 6(12) - 63 = 9$, where $g_3 = 3$, $g_4 = 9$. Let an element occur β times in the triples, γ times in the quadruples. Then $2\beta + 3\gamma = 18$

and so (β, γ) is either $(0,6)$ or $(3,4)$. It follows that there are 3 elements of type $(3,4)$ and thus we have 3 triples *abc*, *abc*, *abc*. But then a, b, c, need 12 quadruples and there are only 9. So $g_3^{(4)}(7) > 12$.

We have seen that 13 blocks suffice and, indeed, the solution is unique. If we proceed as above, we at once have $5g_2 + 3g_3 = 6(13) - 63 = 15$. The possibilities are $g_2 = 3$, $g_3 = 0$, $g_4 = 10$, or $g_2 = 0$, $g_3 = 5$, $g_4 = 8$.

If $g_3 = 5$, $g_4 = 8$, and we proceed as before, then $(\beta, \gamma) = (0,6)$ or $(3,4)$. It follows that there are 5 elements of type $(3,4)$, 2 elements of type $(0,6)$. These last 2 elements occur only in quadruples, and so produce a pair that occurs 4 times. Hence the only possibility is $g_2 = 3$, $g_4 = 10$.

In the last section we gave a solution in 13 blocks, and it turns out that this is the only exception to the result that

$$g_3^{(4)}(4n+3) = 4n^2 + 5n + 3.$$

We illustrate the construction for $n > 1$. For $n = 2$, $v = 11$, take blocks as follows.

abc, *abc*, *abc*,
2 blocks $[1,3,5,7]_8$
24 blocks $[a,1,4,5]_8, [b,1,2,4]_8, [c,1,2,4]_8$.

The 2 blocks produce the 2-difference and the 4-difference. The other blocks produce the differences $3,4,4,1$; $1,3,2$; $1,3,2$. Since each difference occurs 3 times, all pairs occur 3 times, as required.

Similarly, for $v = 15$, $n = 3$, we take blocks

abc, *abc*, *abc*,
$[1,4,7,10]_{12}$ (3 blocks)
$[a,1,3,5]_{12}, [b,1,3,6]_{12}, [c,1,4,5]_{12}$
$[1,2,6,7]_{12}$

There are other solutions.

As a final illustration, we take $n = 4$, $v = 19$. One solution, of many available, is

abc, *abc*, *abc*,
$[1,5,9,13]_{16}$ (4 blocks)
$[a,1,3,7]_{16}, [b,1,4,5]_{16}, [c,1,3,6]_{16}$
$[1,3,6,12]_{16}, [1,2,8,9]_{16}$.

The general pattern for $v = 4n + 3$ is to have 3 blocks *abc*, n blocks of 4 numbers, $4n$ quads with each of a and b and c, $4n(n-2)$ numerical quads, thus producing the total

$$3 + n + 3(4n) + (n-2)4n = 4n^2 + 5n + 3 \text{ blocks}$$

As n increases, the number of solutions goes up very rapidly. Here is one solution for $n = 5$, $v = 23$.

abc, abc, abc
$[1, 6, 11, 16]_{20}$ 5 blocks
$[a, 1, 2, 4]_{20}, [b, 1, 3, 7]_{20}, [c, 1, 7, 14]_{20},$
$[1, 2, 10, 11]_{20}, [1, 4, 8, 16]_{20}, [1, 3, 6, 17]_{20}$

6 The Case $v = 4n + 2$

We apply the bound from Section 2 and find

$$12g = (4n+2)\{6(4n+1) - 3(4n+1)\} + 6\sum E_i + 2g_2.$$

Thence, $g \geq 4n^2 + 3n + 1$.

Suppose, if possible, that $g_2+g_3+g_4 = 4n^2+3n+1$. Since $g_2+3g_3+6g_4 = 4n^2 + 18n + 3$, we deduce that $5g_2 + 3g_3 = 3$.

Hence $g_3 = 1$, $g_4 = 4n^2 + 3n$. But, if an element appears in the single triple, it must appear with $2 + 3\gamma$ elements (it appears in γ quads). Then $2 + 3\gamma = 3(4n + 1)$. This is impossible; hence

$$g \geq 4n^2 + 3n + 2.$$

Now let $g_2 + g_3 + g_4 = 4n^2 + 3n + 2$. Then $5g_2 + 3g_3 = 9$. Hence $g_2 = 0$, $g_3 = 3$, $g_4 = 4n^2 + 3n - 1$. In general, this will occur but, as might be expected, the case $v = 6$ ($n = 1$) is an exception. If $g_3^{(4)}(6) = 9$, then we let β and γ be the frequencies of an element in triples and quads. Since $2\beta + 3\gamma = 15$, we see that $(\beta, \gamma) = (0, 5)$ or $(3, 3)$ or $(6, 1)$. But $g_3 = 3$ and so there are 3 blocks abc; this requires 9 quads to contain the elements a, b, c.

If $g_3^{(4)}(6) = 10$, then $5g_2 + 3g_3 = 15$ and $g_2 = 3$, $g_4 = 8$, or $g_3 = 5$, $g_4 = 5$. The first case requires blocks ab, ab, ab, and this would neccesitate 10 quads. In the second case, $2\beta + 3\gamma = 15$ and $(\beta, \gamma) = (0, 5)$ or $(3, 3)$; so there are 5 elements of type $(3, 3)$, one element of type $(0, 5)$. This structure is easily achieved by using blocks $[1, 2, 3]_5$, $[1, 2, 4, b]_5$. So $g_3^{(4)}(6) = 10$. Note that this is just the solution for $v = 7$ with element a deleted.

Now we illustrate the behaviour of blocks for $n > 1$. In this case $g_3 = 3$, $g_4 = 4n^2 + 3n - 1$.

For $n = 2$ ($v = 10$), we have 3 triples, 21 quads. The bound is easily met by taking 3 triples abc along with quads aF, bF, cF, where F is the Fano geometry.

For $n = 3$ ($v = 14$), there must be 44 quads. In this case, we easily construct the covering by taking 3 triples abc, along with 44 quads $[a, 1, 4, 5]_{11}$, $[b, 1, 5, 6]_{11}$, $[c, 1, 3, 5]_{11}$, $[1, 2, 4, 7]_{11}$. There are, of course, other solutions. Another is $[a, 1, 2, 4]_{11}$, $[b, 1, 3, 7]_{11}$, $[c, 1, 3, 6]_{11}$, $[1, 2, 5, 6]_{11}$.

For $n = 4$ ($v = 18$), we require 5 sets of quads taken modulo 15. So the solution has the form

abc (thrice)

15 quads a, x, x, x

15 quads b, x, x, x

15 quads c, x, x, x

2 quad systems x, x, x, x

where all elements x cycle mod 15.

The case $v = 22$ is particularly interesting. We can use a BIBD to replace three of the needed quad systems. Thus we may take

abc (thrice)

19 quads $[a, 1, 4, 5]_{19}$

19 quads $[b, 1, 3, 10]_{19}$

19 quads $[c, 1, 6, 12]_{19}$

57 quads forming a BIBD $(19, 57, 12, 4, 2)$.

The first 57 quads have differences $3, 4, 1$; $2, 9, 7$; $5, 8, 6$; and so form a cyclic STS(19).

In general, for $v = 4n + 2$, we have 3 triples abc, a with a triple cycled mod $4n - 1$, b with a triple cycled mod $4n - 1$, c with a triple cycled mod $4n - 1$, $n - 2$ quads systems cycled mod $4n - 1$. This gives the total number of blocks

$$3 + (4n - 1)(3 + n - 2) = 4n^2 + 3n + 2.$$

7 Remarks and Summary

Of course one can often make use of the simplification illustrated in the last section. For example, if $v = 34$, we require 3 triples abc and 9 quad systems, as indicated at the end of the last section. But there is a BIBD $(31, 155, 20, 4, 2)$, and this is equivalent to 5 of the required quad systems. So we can take this BIBD along with 3 triples abc, quads $[a, 1, 9, 10]_{31}$, $[b, 1, 5, 11]_{31}$, $[c, 1, 6, 13]_{31}$, $[1, 3, 14, 17]_{31}$.

It is useful to sum up the results mod 12, as in [8].

v	$g_3^{(4)}(v)$	Exceptions
$12t$	$36t^2 - 3t$	
$12t+1$	$36t^2+3t$	
$12t+2$	$36t^2+9t+2$	
$12t+3$	$36t^2+15t+3$	
$12t+4$	$36t^2+21t+3$	
$12t+5$	$36t^2+27t+3$	
$12t+6$	$36t^2+33t+9$	$g^{(4)}(6)=10$
$12t+7$	$36t^2+39t+12$	$g_3^{(4)}(7)=13$
$12t+8$	$36t^2+45t+14$	
$12t+9$	$36t^2+51t+18$	
$12t+10$	$36t^2+57t+24$	
$12t+11$	$36t^2+63t+29$	

References

[1] M.J. Grannell, T.S. Griggs, and R.G. Stanton, The Covering Numbers $g_2^{(4)}(v)$, to appear.

[2] M. Grütmüller, I.T. Roberts, and R.G. Stanton, An Improved Lower Bound for $g^{(4)}(18)$, to appear.

[3] D.A. Preece, Personal communication at the 34th Southeastern Conference on Combinatorics, Graph Theory, and Computing, March 2003.

[4] R.G. Stanton, A Lower Bound for $g^{(4)}(18)$, *Congressus Numerantium* **146** (2000), 153–156.

[5] R.G. Stanton, An Improved Upper Bound on $g^{(4)}(18)$, *Congressus Numerantium* **142** (2000), 29–32.

[6] R.G. Stanton, An Improved Lower Bound for $g^{(4)}(17)$, unpublished.

[7] R.G. Stanton, The Exact Covering of Pairs on Nineteen Points with Block Sizes Two, Three, and Four, *JCMCC* 4 (1988), 69–78.

[8] R.G. Stanton and D.R. Stinson, Perfect Pair-Coverings with Block Sizes Two, Three, and Four, *J. Combinatorics, Information, and System Sciences* **8** (1983), 21–25.