Sufficient conditions for super-edge-connected graphs depending on the clique number

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Abstract

Let $\delta(G)$ and $\lambda(G)$ be the minimum degree and edge-connectivity of a graph G, respectively. A graph G is maximally edge-connected, if $\lambda(G) = \delta(G)$ and super-edge-connected, if every minimum edge cut consists of edges adjacent to a vertex of minimum degree.

In this paper sufficient conditions for super-edge-connected graphs depending on the clique number and the minimum degree are presented. These results show that some known sufficient conditions for maximally edge-connected graphs even lead to super-edge-connected graphs.

Keywords: *super-edge-connectivity, edge-connectivity, clique number, minimum degree*

We consider finite, undirected, and simple graphs G with the vertex set $V(G)$ and the edge set $E(G)$. If $v \in V(G)$ is a vertex of a graph G, then let $d(v)$ its *degree*, and denote by $\delta(G) = \delta$ its *minimum degree*. For two disjoint sets $X, Y \subset V(G)$ let (X, Y) be the set of edges from X to Y. If $X \subseteq V(G)$, then $G[X]$ is the subgraph induced by X.

An *edge-cut* of a connected graph G is the set of edges whose removal disconnects the graph G. The *edge-connectivity* $\lambda(G) = \lambda$ of a graph G is defined as the minimum cardinality of an edge-cut over all edge-cuts of G. The inequality $\lambda(G) \leq \delta(G)$ is immediate. A graph G is *maximally*

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edge-connected, if $\lambda(G) = \delta(G)$. A graph G is called *super-edge-connected* or *super-λ*, if every minimum edge cut is trivial, that means, that every minimum edge cut consists of edges adjacent to a vertex of minimum degree. Clearly, if G is super-λ, then G is maximally edge-connected.

The *clique number* of a graph G is the maximum order among the complete subgraphs of G. For other graph theory terminology we follow Chartrand and Lesniak [2].

Sufficient conditions for graphs to be super-λ were given by several authors, for example: Kelmans [8], Lesniak [9], Boesch and Tindell [1], Fàbrega and Fiol [5], [6], Fiol [7], Soneoka [10], and Volkmann [14].

In 1995, Dankelmann and Volkmann [3] gave the following condition for maximally edge-connected graphs with no clique of order $p+1$, which was first proved by Volkmann [12] for p-partite graphs.

Theorem 1 (Dankelmann, Volkmann [3]). Let G be a graph of order n with minimum degree δ and edge-connectivity λ. If G contains no clique of order $p+1$ and

$$n \leq 2\left\lfloor \frac{p\delta}{p-1} \right\rfloor - 1,$$

then $\lambda = \delta$.

We will show that graphs which fulfill the conditions of Theorem 1, in the most cases are even super-λ. We start with a more or less known lemma, however, for reason of completeness, we give its short proof.

Lemma 2. Let G be a graph of edge-connectivity λ. If G is not super-λ, then there exist two disjoint sets $X, Y \subset V(G)$ with $X \cup Y = V(G)$ and $|(X,Y)| = \lambda$ such that $|X|, |Y| \geq \max\{2, \delta(G)\}$.

Proof. Since the graph G is not super-λ, there exist two disjoint sets $X, Y \subset V(G)$ with $X \cup Y = V(G)$ and $|(X,Y)| = \lambda$ such that $|X|, |Y| \geq 2$. Consequently, the lemma is proved for $\delta(G) = \delta \leq 2$. Now let $\delta \geq 3$ and suppose, without loss of generality, that $2 \leq |X| \leq \delta - 1$. Then we obtain the contradiction

$$\begin{aligned} |X|\delta \leq \sum_{x \in X} d(x) &\leq |X|(|X|-1) + \lambda \\ &\leq (\delta-1)(|X|-1) + \delta. \ \square \end{aligned}$$

Our main lemma is based on the following inequality. Its proof is

Proposition 3. If $2a = m + \eta$ such that $m \geq 4$ is an integer and $\frac{1}{m-2} < \eta < 1$, then

$$a + \sqrt{a^2 - 2a} > \lfloor 2a \rfloor - 1.$$

Lemma 4. Let G be a graph with no complete subgraph of order $p+1$, minimum degree $\delta \geq 3$ and edge-connectivity λ. If G is not super-λ, then there exist two disjoint subsets $X, Y \subset V(G)$ with $X \cup Y = V(G)$ and $|(X,Y)| = \lambda$ such that

$$\begin{aligned} |X|, |Y| &\geq \left\lfloor \frac{p\delta}{p-1} \right\rfloor - 1, \text{ if } \delta = p \text{ or } \delta = k(p-1), \\ |X|, |Y| &\geq \left\lfloor \frac{p\delta}{p-1} \right\rfloor, \text{ otherwise.} \end{aligned}$$

Proof. In view of Lemma 2, there exist two disjoint sets $X, Y \subset V(G)$ with $X \cup Y = V(G)$ and $|(X,Y)| = \lambda$ such that $|X|, |Y| \geq \delta$. In the case $p \geq \delta + 2$, it is a simple matter to show that $|X|, |Y| \geq \delta \geq \lfloor \delta p/(p-1) \rfloor$, and in the case $\delta \leq p \leq \delta + 1$, we observe that $|X|, |Y| = \delta \geq \lfloor \delta p/(p-1) \rfloor - 1$. Thus, it remains to prove Lemma 4 for $p \leq \delta - 1$.

By reason of symmetry, it is enough to prove the desired bounds for the set X. Since the subgraph $G[X]$ contains no complete subgraph of order $p+1$, the well-known Theorem of Turán [11] (see also [13], p. 212) yields

$$2|E(G[X])| \leq \frac{p-1}{p}|X|^2$$

and hence, with $x = |X|$, we deduce that

$$\delta \geq \lambda = |(X,Y)| = \sum_{v \in X} d(v) - 2|E(G[X])| \geq \delta x - \frac{p-1}{p}x^2.$$

Consequently, it follows that

$$x^2 - \frac{p\delta}{p-1}x + \frac{p\delta}{p-1} \geq 0.$$

The roots of the corresponding quadratic equation are

$$x_1 = \frac{p\delta}{2(p-1)} + \sqrt{\left(\frac{p\delta}{2(p-1)}\right)^2 - \frac{p\delta}{p-1}}$$

and

$$x_2 = \frac{p\delta}{2(p-1)} - \sqrt{\left(\frac{p\delta}{2(p-1)}\right)^2 - \frac{p\delta}{p-1}}.$$

Because of $p \le \delta - 1$, we observe that $x_2 < \delta$, and so, in view of Lemma 2, we conclude that $x \ge x_1$. This implies

$$\begin{aligned} x &\ge \frac{p\delta}{2(p-1)} + \sqrt{\left(\frac{p\delta}{2(p-1)} - 1\right)^2 - 1} \\ &> \frac{p\delta}{2(p-1)} + \frac{p\delta}{2(p-1)} - 2 \\ &= \frac{p\delta}{p-1} - 2. \end{aligned}$$

Hence, $|X| \ge \lfloor p\delta/(p-1) \rfloor - 1$, and the lemma is proved for $\delta = p$ and $\delta = k(p-1)$.

Next we consider the case $\delta = k(p-1)+r$ with $k \in \mathbf{N}$ and $1 \le r \le p-2$. We define

$$2a = \frac{p\delta}{p-1} = pk + \frac{rp}{p-1} = pk + r + \frac{r}{p-1} = m + \eta,$$

with $\eta = r/(p-1)$. Since $k = r = 1$ is not possible, it is straightforward to verify that $m = pk + r \ge 4$ and

$$1 > \eta \ge \frac{1}{p-1} > \frac{1}{p} \ge \frac{1}{m-2}.$$

Using Proposition 3 with $m = pk + r$ and $\eta = r/(p-1)$ and the fact that $|X| \ge x_1$, we obtain $|X| \ge \lfloor p\delta/(p-1) \rfloor$. This completes the proof of Lemma 4. □

Remark 5. For $\delta = 1$, the bounds in Lemma 4 are not of interest. The cycle C_4 of length 4 shows that the bound in Lemma 4 is not valid for $\delta = p = 2$, and in the case $\delta = 2$ and $p \ge 3$, the lower bounds for $|X|$ and $|Y|$ in Lemma 4 would be 2, as in Lemma 2.

Corollary 6. Let G be a graph of order n with minimum degree $\delta \ge 3$ and edge-connectivity λ. If G contains no clique of order $p+1$ and

$$\begin{aligned} n &\le 2\left\lfloor \frac{p\delta}{p-1} \right\rfloor - 3, \text{ if } \delta = p \text{ or } \delta = k(p-1), \\ n &\le 2\left\lfloor \frac{p\delta}{p-1} \right\rfloor - 1, \text{ otherwise,} \end{aligned}$$

then G is super-λ.

Corollary 7 (Fiol [7]). Let G be a bipartite graph of order n with minimum degree $\delta \geq 3$ and edge-connectivity λ. If

$$\delta \geq \left\lfloor \frac{n+2}{4} \right\rfloor + 1,$$

then G is super-λ.

Using Lemma 4, one can prove analogously to Theorem 3 in [4], the following degree sequence condition for graphs to be super-λ.

Corollary 8. Let G be a graph of order n with no complete subgraph of order $p+1$, with degree sequence $d_1 \geq d_2 \geq \ldots \geq d_n = \delta \geq 3$, and edge-connectivity λ.

1. Case. Let $\delta = p$ or $\delta = t(p-1)$ with $t \in \mathbb{N}$. If $n \leq 2\lfloor \delta p/(p-1) \rfloor - 3$ or $n \geq 2\lfloor \delta p/(p-1) \rfloor - 2$ and

$$\sum_{i=1}^{k} d_i + \sum_{i=1}^{(2p-1)k} d_{n+1-i} \geq k(p-1)n + 2\delta + 1$$

for $k = 1$ when $\delta = p$ and for some k with $1 \leq k \leq \lfloor \delta/(p-1) \rfloor - 1$ when $\delta = t(p-1)$, then G is super-λ.

2. Case. Let $\delta \neq p$ and $\delta \neq t(p-1)$ with $t \in \mathbb{N}$. If $n \leq 2\lfloor \delta p/(p-1) \rfloor - 1$ or $n \geq 2\lfloor \delta p/(p-1) \rfloor$ and

$$\sum_{i=1}^{k} d_i + \sum_{i=1}^{(2p-1)k} d_{n+1-i} \geq k(p-1)n + 2\delta + 1$$

for some k with $1 \leq k \leq \lfloor \delta/(p-1) \rfloor$, then G is super-λ.

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