8-Cycle Decompositions of the Cartesian Product of Two Complete Graphs

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Abstract

We establish necessary and sufficient conditions on m and n for $K_m \times K_n$, the Cartesian product of two complete graphs, to be decomposable into cycles of length 8. We also provide a complete classification of the leaves which are possible with maximum packings of complete graphs with 8-cycles.

1 Introduction

All graphs considered in this paper are finite and have no loops or multiple edges. The *Cartesian product* of two graphs, G_1 and G_2, is the graph $G_1 \times G_2$ having vertex set $V(G_1) \times V(G_2)$ and in which vertex (u_1, u_2) is adjacent to (v_1, v_2) if and only if either $u_1 = v_1$ and u_2 is adjacent to v_2 in G_2, or $u_2 = v_2$ and u_1 is adjacent to v_1 in G_1.

A *cycle* is a 2-regular connected graph (or subgraph of a graph). A *t-cycle* is a cycle containing exactly t edges, and is denoted by C_t. A *t-cycle decomposition* of a graph G consists of a set of t-cycles of G which partition the edge set of G. A *pure cycle* in $K_m \times K_n$ is a cycle whose edges are all contained within one copy of K_m or one copy of K_n.

In [5], it was shown that the necessary conditions for a decomposition of $K_m \times K_n$ into cycles of length 4 are sufficient. In [3] it was established that the necessary conditions for a 6-cycle decomposition of $K_m \times K_n$ are also sufficient. In this paper we extend the study of cycle decompositions of graph products by proving that the conditions which are necessary for a decomposition of $K_m \times K_n$ into cycles of length 8 are sufficient. Noting that $K_m \times K_n$ is isomorphic to the line graph of $K_{m,n}$, we also determine the necessary and sufficient conditions for $L(K_{m,n})$ to be 8-cycle decomposable.

In order to consider whether it is possible to decompose Cartesian products of complete graphs into cycles of length 8, we first must have an awareness of which complete graphs are 8-cycle decomposable. In cases where the complete graph K_n does not decompose into 8-cycles, we will need to know what leaves can be obtained from various packings of K_n, where a *packing* of K_n with 8-cycles consists of a set of edge-disjoint 8-cycles and the corresponding *leave* consists of the set of edges not used by the cycles of the packing. In particular, we establish the complete set of leaves which are realizable from maximum packings of complete graphs with 8-cycles, thereby extending earlier findings of Grinstead [4].

In establishing our results, we will make use of the following result of Sotteau [7]:

Theorem 1 *The graph $K_{m,n}$ is t-cycle decomposable if and only if $t \geq 4$, $m \equiv n \equiv t \equiv 0$ (mod 2), $t \leq 2m$, $t \leq 2n$, and t divides mn.*

In particular, the following corollary of this theorem will be frequently referenced:

Corollary 1 *The graph $K_{m,n}$ is 8-cycle decomposable if and only if $m \equiv n \equiv 0$ (mod 2), $m \geq 4$, $n \geq 4$, and 8 divides mn.*

2 Leaves for 8-Cycle Packings of K_n

It is known that K_n is completely decomposable into cycles of length 8 if and only if $n \equiv 1 \pmod{16}$ [1, 6]. Grinstead [4] has provided an outline of how to establish the size of a minimum leave when 8-cycles are removed from K_n, for $n \not\equiv 1 \pmod{16}$. Continuing from Grinstead, we now establish the complete set of possible smallest leaves, a summary of which is presented in Table 1.

2.1 When n is even

When n is even, the leave which remains after removing a maximum number of 8-cycles from K_n must be a subgraph for which each vertex has odd degree. Thus there must be at least $\frac{n}{2}$ edges remaining. For $n \equiv 0$ or 2 (mod 8), with $n \geq 8$, Grinstead showed that there are precisely $\frac{n}{2}$ edges in the leave. Thus, for $n \equiv 0$ or 2 (mod 8), the leave is a 1-factor when $n \geq 8$ (of course, if $n < 8$, then the leave is K_n itself).

For $n \equiv 4$ or 6 (mod 8), with $n > 8$, the number of edges in the leave is $\frac{n+8}{2}$, which is four more than would be in a 1-factor. There are 22 such potential leaves, which have been illustrated in [2, page 229]. Leaves X_9 and X_{20} (as denoted in [2]) require at least 14 vertices, while each of the other twenty require twelve or fewer vertices.

$n \pmod{16}$	Leave in K_n
0	1-factor
1	$\emptyset$
2	1-factor
3	C_3
4	4 edges more than a 1-factor
5	10 edges
6	4 edges more than a 1-factor
7	C_5
8	1-factor
9	C_4
10	1-factor
11	7 edges
12	4 edges more than a 1-factor
13	6 edges
14	4 edges more than a 1-factor
15	9 edges

Table 1: A summary of the leaves when the maximum number of 8-cycles are removed from K_n, $n \geq 8$.

To show that each of these 22 potential leaves is, in fact, attainable, let $n = 8k + \ell$, where $\ell \in \{4, 6\}$ and $k \geq 1$. If $k \geq 2$, then K_{n-12} decomposes into 8-cycles with a 1-factor leave. So, for $k \geq 2$, consider the vertex set of K_n as $V = A \cup B$, the disjoint union of a set A having cardinality $n - 12$ and a set B having cardinality 12. The subgraph induced by A can be decomposed into 8-cycles with a 1-factor leave and, from Corollary 1, the edges in the complete bipartite graph with bipartition (A, B) can be completely decomposed into 8-cycles. To finish our decomposition of K_n we now consider just the subgraph induced by B; that is, we consider K_{12}.

So, for $k \geq 2$ and for leaves other than X_9 or X_{20}, it suffices to show that the leave is attainable in K_{12}. We tested both K_{12} and K_{14} and determined that each of the twenty leaves on twelve or fewer vertices is indeed attainable.

For leaves X_9 and X_{20}, and for $k \geq 3$, choose $V = A \cup B$ where A contains $n - 20$ vertices and B has cardinality 20. Then, as before, the subgraph of K_n induced by A will have a 1-factor leave when packed with a maximum number of 8-cycles. The subgraph induced by B is now K_{20}, for which we verified that leaves X_9 and X_{20} are both attainable. We also verified that they can be obtained in K_{14} and K_{22}.

2.2 When $n \equiv 3$, 7, 11, or 15 (mod 16)

For the given values of n, the possible configurations of the edges remaining after taking a maximum 8-cycle packing of K_n are very important when considering the decompositions of the Cartesian products. Not only is the leave itself important, but the manner in which the leave is obtained in K_n is also crucial. In many instances we will wish to reclaim 8-cycles that were previously removed in the maximum packing to carry out the decomposition of the Cartesian products. We also wish to show that a complete bipartite graph isomorphic to $K_{4,4}$ can also be reclaimed in several cases. To show that this is possible, we use the map outlined by Grinstead [4] and establish the leaves in K_n in an inductive fashion.

The inductive loop begins with $n \equiv 11 \pmod{16}$, where we have an 8-cycle packing with a C_7 leave on seven vertices, $\frac{n-11}{8}$ disjoint sets of eight vertices in which the 8-cycles of the packing induce $K_{4,4}$, and four additional vertices. We examine K_{11} itself to establish the initial leaves possible, so in essence we begin with $n = 16k + 11$, where $k = 0$. Number the vertices in K_{11} from 1 to 11 and extract the following 8-cycles:

$$(3,7,10,6,11,8,5,9) \quad (1,10,4,7,2,5,3,11) \quad (1,8,3,10,2,11,4,9)$$
$$(1,6,4,8,2,9,11,7) \quad (1,4,3,2,6,8,10,5) \quad (1,2,4,5,7,9,6,3).$$

Note that K_{11} has 55 edges, so after removing these six 8-cycles there are seven edges remaining. The packing of K_{11} presented produces the C_7 leave $(5,6,7,8,9,10,11)$, which is the leave we use to start the inductive loop. So we begin with eleven vertices, seven of which take part in a 7-cycle and the other four have no edges. By taking different packings, we can obtain alternate 7-edge leaves in K_{11} consisting of combinations of 3- and 4-cycles. These alternate configurations, which we confirmed were attainable in any K_n with $n \equiv 11 \pmod{16}$, are presented in Figure 1.

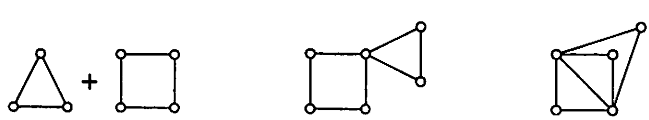


Figure 1: The non-C_7 7-edge leave configurations possible for K_n, $n \equiv 11$ (mod 16).

To proceed to the next step we add four vertices, numbered 12 through 15. This takes us to K_n, $n \equiv 15 \pmod{16}$, where the four new vertices have a copy of K_4 as well as edges to all of the starting vertices. The edges between the new vertices and vertices 1 through 4 form a complete bipartite graph isomorphic to $K_{4,4}$ with bipartition $(\{1,2,3,4\},\{12,13,14,15\})$, which is 8-cycle decomposable by Corollary 1. Between the four new vertices and each of the original sets of eight vertices, we obtain the 8-cycles of a $K_{4,8}$ as per Corollary 1. We now establish the leave in $n \equiv 15 \pmod{16}$ by examining the edges contained on vertices 12 through 15, between vertices $\{12, 13, 14, 15\}$ and $\{5, 6, 7, 8, 9, 10, 11\}$, and the 7-cycle $(5, 6, 7, 8, 9, 10, 11)$. From these edges we remove the four 8-cycles

$$(12, 8, 14, 11, 5, 15, 13, 10) \quad (12, 7, 14, 6, 13, 11, 15, 9)$$
$$(12, 5, 13, 9, 14, 10, 15, 6) \quad (12, 14, 5, 6, 7, 13, 8, 15)$$

and are then left with the 9-cycle $(7, 8, 9, 10, 11, 12, 13, 14, 15)$. Figure 2 illustrates the leave construction in this step with a generic starting point of $n \equiv 11 \pmod{16}$, with the vertices numbered appropriately. The groups of eight that are labelled in the diagram represent sets of vertices on which the 8-cycles of the packing yield copies of $K_{4,4}$, which there are none of in this first pass through the inductive loop; these are present in subsequent passes when 8-cycles are collected. The vertices that hold the 9-cycle leave are indicated, and vertices which form the bipartition of a $K_{4,4}$ are shaded in the diagram. In Table 2 we present a complete list of other 9-edge leave configurations, each of which we confirmed to be attainable in any K_n with $n \equiv 15 \pmod{16}$.

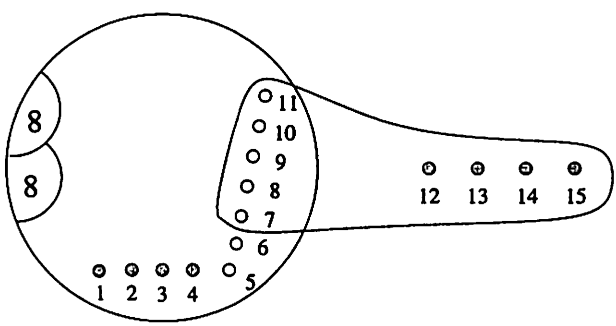


Figure 2: The first step in the inductive loop: obtaining a C_9 leave for $n \equiv 15 \pmod{16}$.

We proceed now to $n \equiv 3 \pmod{16}$ by adding four vertices numbered 16 through 19. Along with the six edges of the K_4 induced by the new vertices, we also have the 9-edge leave from the previous step, as well as the edges between the four new vertices and the first 15 vertices to work with. Note that within this construction the edges between vertices 16 through 19 and vertices 1 through 6 constitute a complete bipartite graph isomorphic to $K_{4,6}$ with bipartition $(\{16, 17, 18, 19\}, \{1, 2, 3, 4, 5, 6\})$, which is 8-cycle

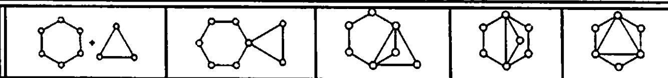
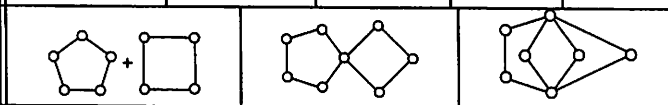
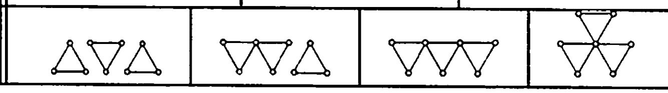
Cycle Combinations	Possible Leave Configurations
$C_6 + C_3$	
$C_5 + C_4$	
$C_3 + C_3 + C_3$	

Table 2: All possible non-C_9 9-edge leaves for K_n, $n \equiv 15 \pmod{16}$.

decomposable by Corollary 1. We also use Corollary 1 to decompose the copies of $K_{4,8}$ induced by vertices 16 through 19 and the original sets of eight vertices. After removing these edges we are left with the C_9 leave, the newly introduced K_4, and the edges between vertices 16 through 19 and vertices 7 through 15. This totals 51 edges, which we can decompose into six 8-cycles and one 3-cycle. By removing an appropriate packing of 8-cycles we can obtain the 3-cycle $(11, 16, 17)$. Note that, with only three edges, the only possible leave is a 3-cycle. The packing which accomplishes this consists of the following 8-cycles:

$$(18, 19, 15, 7, 8, 9, 10, 11) \quad (17, 13, 18, 14, 16, 19, 11, 12)$$
$$(17, 15, 18, 12, 16, 13, 19, 14) \quad (17, 18, 16, 15, 14, 13, 12, 19)$$

along with a complete bipartite graph isomorphic to $K_{4,4}$ with bipartition $(\{16, 17, 18, 19\}, \{7, 8, 9, 10\})$, to which we apply Corollary 1. This addition of vertices and generation of the leave is illustrated in Figure 3. The newly constructed copy of $K_{4,4}$ is highlighted by shading of the vertices which form the bipartition, and the vertices containing leaves from both steps are also indicated.

The next step involves determining a 5-edge leave in K_n, where $n \equiv 7 \pmod{16}$, for which a 5-cycle is the only possible configuration. We enter this step by first adding vertices 20 through 23. To establish the leave we consider the edges of the K_4 induced by the four new vertices, the C_3 leave from the previous step, and the edges between vertices 20 through 23 and four other vertices, which can be arbitrarily chosen, as long as they do not include the seven vertices in $\{11, 16, 17, 20, 21, 22, 23\}$ already under consideration. Without loss of generality, we choose these vertices to be $\{1, 2, 3, 4\}$. From these edges we extract the following four 8-cycles:

$$(17, 11, 22, 3, 20, 1, 21, 23) \quad (17, 20, 4, 21, 16, 23, 1, 22)$$
$$(11, 16, 20, 22, 4, 23, 2, 21) \quad (17, 16, 22, 2, 20, 23, 3, 21).$$

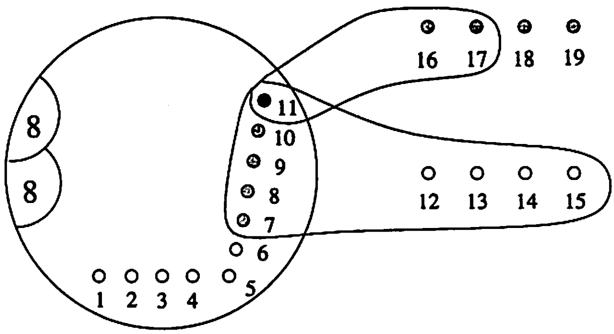


Figure 3: The second step in the inductive loop: obtaining a C_3 leave for $n \equiv 3 \pmod{16}$.

With this packing the leave is the 5-cycle $(11, 20, 21, 22, 23)$. The edges between vertices $\{20, 21, 22, 23\}$ and the remaining numbered vertices constitute a complete bipartite graph isomorphic to $K_{4,12}$ with bipartition $(\{20, 21, 22, 23\}, \{5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 18, 19\})$, which can be decomposed into 8-cycles by Corollary 1. To finish this step, the edges between the four new vertices and each of the original sets of eight vertices form copies of $K_{4,8}$, which can be decomposed using Corollary 1. This step of the inductive cycle is shown in Figure 4, where the vertices for all of the leaves up to this point are circled, and the vertices of the $K_{4,4}$ established in the previous step are shaded.

Figure 5 depicts the final step of the inductive loop. Since, with K_{11}, we started with zero groups of 8-cycles and eleven vertices, seven of which contained a C_7 and the other four had no edges, we wish to show in this step that a configuration similar to that of the starting point is attainable. Hence we add four new vertices and assign them numbers from 24 through 27. With this addition we examine K_n, $n \equiv 11 \pmod{16}$, and as a result have edges from these vertices to the first twenty-three vertices, along with the 5-cycle $(11, 20, 21, 22, 23)$ from the previous step, and the edges between these new vertices and the original sets of eight vertices. We use the edges of the K_4, the 5-cycle and the edges between the vertices with these edges to determine the leave. From these edges we remove the 8-cycles

$$(24, 21, 25, 26, 11, 27, 22, 23) \quad (24, 26, 27, 20, 11, 23, 25, 22)$$
$$(24, 25, 27, 23, 26, 22, 21, 20)$$

for which the leave is the 7-cycle $(24, 27, 21, 26, 20, 25, 11)$. The other edges present are decomposed by applying Corollary 1 to the complete bipartite graph isomorphic to $K_{4,18}$ with bipartition $(\{24, 25, 26, 27\}, \{1, 2, 3, 4, 5, 6,$

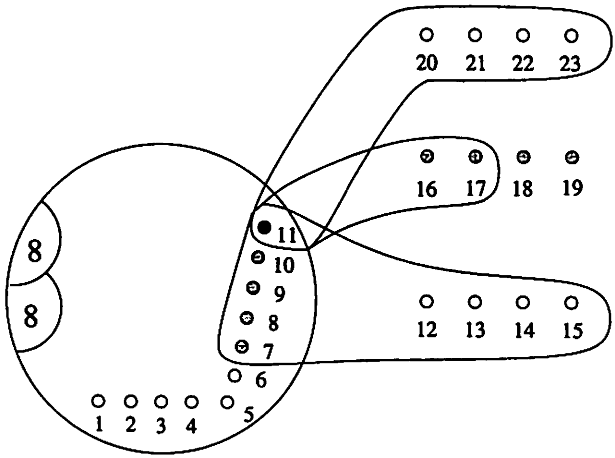


Figure 4: The third step in the inductive loop: obtaining a C_5 leave for $n \equiv 7 \pmod{16}$.

7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 19}), as well as to the copies of $K_{4,8}$ that are induced by the four new vertices and the original sets of eight vertices.

We have now obtained a maximum 8-cycle packing of K_n with $n \equiv 11 \pmod{16}$. In this packing, the original disjoint sets of eight vertices (each containing a copy of $K_{4,4}$) are still present, and are now joined by two additional disjoint sets of eight vertices, having vertex sets $\{1, 2, 3, 4, 12, 13, 14, 15\}$ and $\{7, 8, 9, 10, 16, 17, 18, 19\}$. We also have the 7-cycle (24, 27, 21, 26, 20, 25, 11), and a set of four vertices, namely $\{5, 6, 22, 23\}$, which are disjoint from the 7-cycle as well as each of the sets of eight vertices on which we have copies of $K_{4,4}$. Hence the inductive loop is complete.

2.3 When $n \equiv 1, 5, 9$, or $13 \pmod{16}$

To establish the smallest possible leaves in K_n for $n \equiv 1, 5, 9$, or $13 \pmod{16}$, we use an inductive loop similar to that used in Section 2.2. For any given step in either loop, the method used to advance to the next step is the same, and so the diagrams presenting individual steps in the loop for K_n, $n \equiv 11, 15, 3$, or $7 \pmod{16}$ are illustrative of the steps in this loop.

We begin with $n = 1$, for which it is apparent that we are able to obtain an 8-cycle decomposition of K_n such that on each of $\frac{n-1}{8}$ disjoint sets of eight vertices the decomposition induces a $K_{4,4}$. The graph also contains a single vertex, disjoint from each of these sets of eight vertices; we label this vertex with number 1.

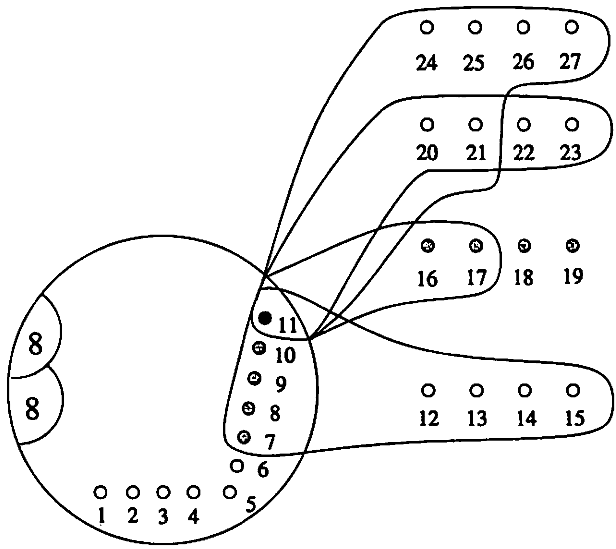


Figure 5: The fourth and final step in the inductive loop: obtaining a C_7 leave for $n \equiv 11 \pmod{16}$.

We now add four vertices, numbered 2, 3, 4, and 5 to proceed to the next step where $n \equiv 5 \pmod{16}$. The edges between the four new vertices and each of the original disjoint sets of eight vertices induce copies of $K_{4,8}$, which can be decomposed into 8-cycles using Corollary 1. The six edges of the K_4 induced by the new vertices along with the four edges between vertex 1 and the four new vertices is actually a complete graph on five vertices, K_5, which is taken as the leave. Note that in subsequent passes through the inductive loop when $n > 5$, different 10-edge leaves become realizable as more vertices become available. Table 3 lists the potential 10-edge leaves for K_n, $n \equiv 5 \pmod{16}$. Most of the leaves lie on more than five vertices, and so they cannot be realized in K_5. Therefore K_{21} was checked and it was confirmed that the leave configurations were all attainable. To see that a leave attainable in K_n where $n > 21$ is also attainable for any $n \equiv 5 \pmod{16}$, $n > 5$, consider $n = 16k + 5$. Note that

$$n = 16k + 5 = 16(k-1) + 21.$$

Create two groups of vertices, one with 20 vertices and the other with $16(k-1) + 1$. So we have divided our complete graph into two complete graphs, K_{20} and $K_{16(k-1)+1}$ with edges between each pair of vertices of these two graphs. The complete graph $K_{16(k-1)+1}$ completely decomposes

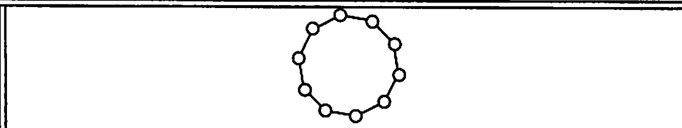
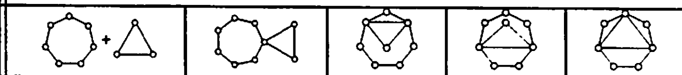
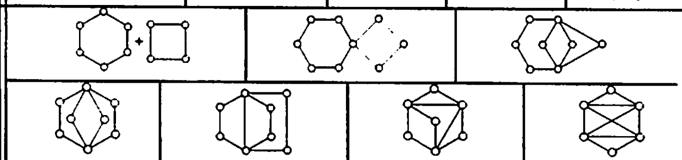
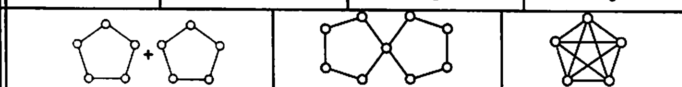
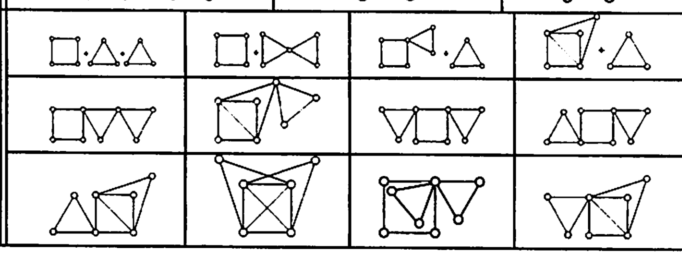
Cycle Combinations	Possible Leave Configurations
C_{10}	
$C_7 + C_3$	
$C_6 + C_4$	
$C_5 + C_5$	
$C_4 + C_3 + C_3$	

Table 3: All possible 10-edge leaves for K_n, $n \equiv 5 \pmod{16}$.

into 8-cycles. Now choose an arbitrary vertex in the group of $16(k-1)+1$, and call it v. The edges between the remaining $16(k-1)$ vertices and the other set of 20 form a complete bipartite graph isomorphic to $K_{20,16(k-1)}$ and can be decomposed into copies of C_8 by Corollary 1. We have now decomposed all of the edges of K_n except for the edges in K_{20} and the edges between K_{20} and vertex v, which is just K_{21}. Hence if a leave is realizable in K_{21}, then it is realizable in K_n, $n \equiv 5 \pmod{16}$, for any n such that $n > 5$.

The next step in the loop, $n \equiv 9 \pmod{16}$, is reached by adding vertices numbered 6, 7, 8, and 9. Corollary 1 is used to decompose the copies of $K_{4,8}$ induced by vertices 6 through 9 and the original sets of eight vertices. Note that the vertices added induce the complete graph K_4, and along with the K_5 leave from the previous step as well as the edges between the vertices of these graphs we obtain K_9, the complete graph on nine vertices. The graph of K_9 has 36 edges, and so a leave with four edges should remain after removing a maximum 8-cycle packing. From this K_9 we remove the following four 8-cycles

$$(1,3,6,4,7,5,2,8) \quad (1,2,9,7,3,8,4,5)$$
$$(1,6,5,3,9,4,2,7) \quad (1,4,3,2,6,8,5,9)$$

and we are left with the 4-cycle leave $(6,7,8,9)$. Note that, since there are only four edges remaining after a maximum 8-cycle packing, then a 4-cycle is the only configuration of leave possible for such a packing.

Proceeding to $n \equiv 13 \pmod{16}$ we add vertices numbered 10, 11, 12, and 13. The edges between the four new vertices and vertices 2 through 5 induce a complete bipartite graph isomorphic to $K_{4,4}$ with bipartition $(\{2,3,4,5\},\{10,11,12,13\})$ which is 8-cycle decomposable by Corollary 1. In addition, the edges between the four new vertices and the original disjoint sets of eight vertices induce copies of $K_{4,8}$ can also be decomposed using Corollary 1. The edges remaining are used to establish the leave, and include the C_4 leave from the previous step, the graph of K_4 induced by the four new vertices, the edges between vertices 6 through 9 and 10 through 13, as well as the four edges from vertex 1 to vertices 10 through 13. This totals 30 edges, and so by removing the three 8-cycles

$$(1,11,7,10,6,12,8,13) \quad (1,10,9,11,6,13,7,12) \quad (10,13,11,8,7,6,9,12)$$

we obtain the 6-cycle leave $(10,11,12,13,9,8)$. Two additional 6-edge leave configurations are also possible with a maximum 8-cycle packing, one consisting of two disjoint copies of C_3 and the other two non-disjoint 3-cycles that share one vertex (a bowtie). Both were verified as attainable leaves in K_n for all $n \equiv 13 \pmod{16}$.

To complete the first cycle of the inductive loop we add four vertices numbered 14, 15, 16, and 17. In the final step we must completely decompose these edges into 8-cycles, and in doing so obtain two additional disjoint sets of eight vertices on which we have copies of $K_{4,4}$ and one additional vertex, which were the conditions of the initial step of the loop. The edges between vertices 14 through 17 and 6 through 9 induce a complete bipartite graph isomorphic to $K_{4,4}$ with bipartition $(\{6,7,8,9\},\{14,15,16,17\})$. In addition, another copy of $K_{4,4}$ with bipartition $(\{2,3,4,5\},\{14,15,16,17\})$ can be identified, and both can be decomposed into 8-cycles by Corollary 1. The edges between the four new vertices and the original disjoint sets of eight vertices induce copies of $K_{4,8}$ which can also be decomposed using Corollary 1. The edges remaining are those of the new K_4, the C_6 leave of the previous step, the edges between vertices 14 through 17 and 10 through 13, and the four edges from vertex 1 to the four new vertices. This totals 32 edges, which completely decompose into the following four 8-cycles:

$$(14,10,17,12,16,11,15,1) \quad (14,12,15,13,16,1,17,11)$$
$$(14,17,16,15,10,8,9,13) \quad (14,15,17,13,12,11,10,16).$$

Note that with this decomposition, we obtain two disjoint sets of eight vertices that take part in copies of $K_{4,4}$ generated in the loop, namely the sets $\{2,3,4,5,10,11,12,13\}$ and $\{6,7,8,9,14,15,16,17\}$. The original disjoint sets of eight vertices that we started with are still present, and we have one additional vertex which does not take part in any copies of $K_{4,4}$, namely vertex number 1. Since the starting conditions are satisfied (namely having disjoint sets of eight vertices which induce copies of $K_{4,4}$, plus one additional vertex), the induction is complete.

2.4 Obtaining a complete leave in K_n, n odd

Before addressing the 8-cycle decompositions of the Cartesian product of two complete graphs, we wish to first establish the ability to obtain a complete graph K_m as the leave when an 8-cycle packing is extracted from a larger complete graph K_n. That is, we claim that by taking an appropriate 8-cycle packing of K_n we can obtain a leave that is a complete graph on m vertices, where $m \equiv n \pmod{16}$ and $1 \leq m \leq 15$. Note that this is not a maximum packing as was used to establish the leaves of K_n, $n \equiv 1, 2, 3, \cdots, 15 \pmod{16}$ in Sections 2.2 and 2.3.

Lemma 1 *For odd $n \geq 9$, there exists an 8-cycle packing of K_n with a K_m leave, where $m \equiv n \pmod{16}$ and $1 \leq m \leq 15$. Moreover, $\frac{n-m}{8}$ copies of $K_{4,4}$, vertex-disjoint from each other as well as from the K_m leave, can be identified in the 8-cycles of the packing.*

Proof. Let $m \equiv n \pmod{16}$ such that $1 \leq m \leq 15$.

If $m = 1$, then partition the vertices of K_n into two sets, A and C, with $|A| = n - 1$ and $|C| = 1$.

If $m = 3$, then partition the vertices of K_n into three sets, A, B, and C, having cardinality $n - 19$, 18, and 1, respectively. The bipartite graph with bipartition (A, B) can be decomposed into 8-cycles by Corollary 1. $B \cup C$ forms a K_{19}. By numbering the vertices of $B \cup C$ from 1 to 19, and then removing the following seventeen 8-cycles, we are left with a $K_{4,4}$ with bipartition $(\{1,2,3,4\},\{5,6,7,8\})$, another $K_{4,4}$ with bipartition $(\{9,10,11,12\},\{13,14,15,16\})$, and a K_3 with vertex set $\{17,18,19\}$.

(12,17,15,13,19,14,16,18) (11,17,14,13,16,19,15,18)
(9,12,10,18,14,15,16,17) (7,14,8,11,10,17,13,18)
(6,15,7,17,8,18,9,19) (5,16,8,10,9,11,12,19)
(5,14,6,16,7,12,8,15) (4,18,6,13,8,7,11,19)
(3,17,6,12,5,13,7,19) (2,17,5,11,6,9,8,19)
(1,18,5,8,6,7,10,19) (1,16,3,18,2,15,4,17)
(1,14,4,16,2,13,3,15) (1,12,3,14,2,11,4,13)
(1,10,4,12,2,9,3,11) (1,4,3,2,10,5,7,9)
(1,2,4,9,5,6,10,3).

All remaining edges in K_n are now found within $A \cup C$.

For $m > 3$, we also consider three sets, A, B, and C, having cardinality $n - m$, $m - 1$, and 1, respectively. The bipartite graph with bipartition (A, B) can be decomposed into 8-cycles by Corollary 1 and the subgraph induced by $B \cup C$ forms the K_m which we will have as our leave. The remaining 8-cycles are entirely contained within $A \cup C$.

Now, for any value of m, let $|A| = 16k$. If $k > 1$, then we can subdivide A into k sets of 16 vertices each, say $A_1, \ldots, A_k$. Then on $A_i \cup C$ we place an 8-cycle decomposition of K_{17}, for $1 \leq i \leq k$. For each pair A_i and A_j with $1 \leq i < j \leq k$, the bipartite graph with bipartition (A_i, A_j) can be decomposed as sixteen edge-disjoint copies of $K_{4,4}$. It is easily seen that $2k$ vertex-disjoint copies of $K_{4,4}$ can be identified.

If $k = 1$, then $A \cup C$ is K_{17}. Number the vertices from 1 to 17, with vertex 17 representing the single vertex in C. The removal of the following thirteen 8-cycles then yields a $K_{4,4}$ with bipartition $(\{1,2,3,4\}, \{5,6,7,8\})$, and another with bipartition $(\{9,10,11,12\},\{13,14,15,16\})$, both of which are 8-cycle decomposable by Corollary 1.

(8,12,9,17,13,16,14,15) (7,12,10,11,8,16,17,15)
(6,14,13,15,16,7,11,17) (5,14,8,10,9,11,12,17)
(4,15,6,13,8,7,14,17) (3,16,6,12,5,13,7,17)
(2,15,5,11,6,9,8,17) (1,16,5,8,6,7,10,17)
(1,14,4,16,2,13,3,15) (1,12,3,14,2,11,4,13)
(1,10,4,12,2,9,3,11) (1,4,3,2,10,5,7,9)
(1,2,4,9,5,6,10,3).

□

3 Decomposing $K_m \times K_n$ into 8-cycles

3.1 Necessary Conditions

Since $K_m \times K_n$ is isomorphic to $K_n \times K_m$, we will consider each $\{m, n\}$ set only once.

Lemma 2 *Given that $K_m \times K_n$ is 8-cycle decomposable, then either*

1. $m \equiv 0 \pmod 4$ *and* $n \equiv 0 \pmod 2$,
2. $m, n \equiv 1 \pmod{16}$,
3. $m \equiv 3 \pmod{16}$ *and* $n \equiv 15 \pmod{16}$,
4. $m \equiv 5 \pmod{16}$ *and* $n \equiv 13 \pmod{16}$,
5. $m \equiv 7 \pmod{16}$ *and* $n \equiv 11 \pmod{16}$, *or*
6. $m, n \equiv 9 \pmod{16}$.

Proof. The graph $K_m \times K_n$ has mn vertices, each having degree $m+n-2$. Hence, $K_m \times K_n$ has $\frac{(mn)(m+n-2)}{2}$ edges. Given that $K_m \times K_n$ is 8-cycle decomposable the number of edges in the graph must be divisible by 8. Also, note that each vertex in the graph must have even degree. Hence, $m \equiv n \pmod 2$ and $16|(mn)(m+n-2)$; these conditions are met in each of the above six cases and only in these cases. □

3.2 Sufficient Conditions

We now consider each of the stated necessary conditions in turn to establish sufficiency.

Lemma 3 *If $m \equiv 0$ (mod 4) and $n \equiv 0$ (mod 2) then $K_m \times K_n$ is 8-cycle decomposable.*

Proof. Let $m \equiv 0 \pmod 4$ and consider two cases for n:

Case 1 ($n \equiv 0$ (mod 4)). Let $m = 4k$ and $n = 4\ell$, where $k \geq 0$ and $\ell \geq 0$. Then the graph $K_m \times K_n$ has 4ℓ columns of $4k$ vertices each and $4k$ rows of 4ℓ vertices each. To decompose this graph we first divide the columns and rows into groups of four creating $k\ell$ blocks of size 4×4.

Figure 6 illustrates the vertices of $K_4 \times K_4$ as well as several 8-cycles which collectively yield an 8-cycle decomposition of $K_4 \times K_4$. Note that each 4×4 block can be decomposed using this decomposition. Then the only edges left in $K_m \times K_n$ are those between the vertices of different blocks. Now consider any pair of 4×4 blocks which share a row (resp. column) of $K_m \times K_n$. For each row (resp. column) in the blocks let A be the set of four vertices on the row (resp. column) in one of the two blocks, and let B be the set of four vertices on the same row (resp. column) but in the other block. Then the complete bipartite graph with bipartition (A,B) is isomorphic to $K_{4,4}$, and is 8-cycle decomposable by Corollary 1. Therefore by considering all combinations of A and B we can complete the decomposition of $K_m \times K_n$ into 8-cycles.

Case 2 ($n \equiv 2$ (mod 4)). Let $m = 4k$ and $n = 4\ell + 2$, where $k \geq 0$ and $\ell \geq 0$. In this case the graph of $K_m \times K_n$ is such that we have $4\ell + 2$ columns of vertices isomorphic to K_m and $4k$ rows each containing a graph isomorphic to K_n.

Consider first the case where $\ell = 0$. Hence we are examining the graph of $K_m \times K_2$. As in Case 1 we divide the rows into groups of four, thereby creating k blocks of size 4×2. In Figure 7 an 8-cycle decomposition of $K_4 \times K_2$ is presented, which can be used to decompose the edges within

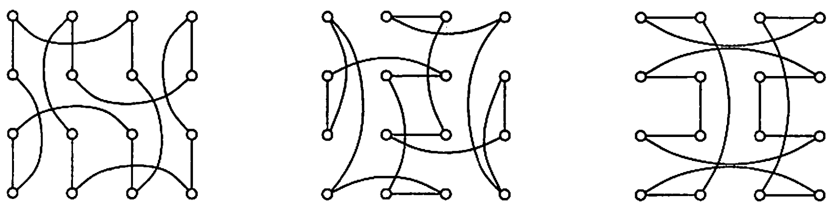


Figure 6: An 8-Cycle Decomposition of $K_4 \times K_4$.

the blocks of size 4×2. In addition, using Corollary 1 we know that an 8-cycle decomposition of the edges between any pair of 4×2 blocks is possible, since the edges between each pair of such blocks form two disjoint copies of $K_{4,4}$.

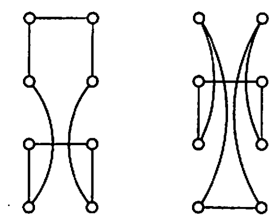


Figure 7: An 8-Cycle Decomposition of $K_4 \times K_2$.

Consider now $\ell > 0$. In this case, divide the rows into k groups of four and the columns into $\ell - 1$ groups of four and one group of six. This produces $k(\ell - 1)$ blocks of size 4×4 and k blocks of size 4×6. The edges in and between the blocks of size 4×4 can be decomposed in the same manner as Case 1. The edges within the 4×6 blocks can be decomposed using an 8-cycle decomposition of $K_4 \times K_6$, such as is obtained by the unique 8-cycle decompositions of each of the subgraphs of $K_4 \times K_6$ presented in Figure 8.

As before, the edges between pairs of 4×6 blocks as well as between pairs of 4×4 and 4×6 blocks form several complete bipartite graphs, each of which is 8-cycle decomposable by Corollary 1. □

Lemma 4 *If $m, n \equiv 1$ (mod 16) then $K_m \times K_n$ is 8-cycle decomposable.*

Proof. Since $m, n \equiv 1 \pmod{16}$, both K_m and K_n are 8-cycle decomposable [1, 6]. Hence, the Cartesian product $K_m \times K_n$ can be completely decomposed into pure horizontal and pure vertical 8-cycles. □

Lemma 5 *If $m \equiv 3$ (mod 16) and $n \equiv 15$ (mod 16) then $K_m \times K_n$ is 8-cycle decomposable.*

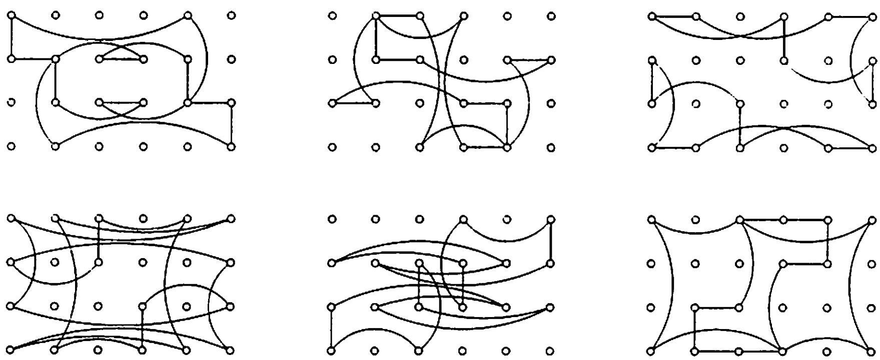


Figure 8: An 8-Cycle Decomposition of $K_4 \times K_6$.

Proof. Let $m = 16k + 3$ and $n = 16\ell + 15$. The graph in which $m > n$ is decomposed in a slightly different manner than the graph in which $m < n$, but for both cases we first use the maximum number of pure 8-cycles in each row and column. We therefore now have a 3-cycle in each column and a 9-cycle (or some other configuration containing 9 edges from Table 2) in each row. However, for several of these packings, we will rely on the constructions presented in Section 2 to allow us to reclaim pure 8-cycles from the packings. The edges of these reclaimed cycles will then be combined with the edges of the leaves in each row and column in a manner that will permit an 8-cycle decomposition of $K_m \times K_n$.

If $m < n$ then $k \leq \ell$. We can then divide the graph of $K_m \times K_n$ into the following sections: a large square block (with dimensions $16k \times 16k$), a block of size 3×15, two blocks of size $3 \times 16k$ and $16k \times 15$ and one further block of size $(16k+3) \times 16(\ell - k)$. Smaller 8×8 blocks are then constructed along the main diagonal of the larger $16k \times 16k$ block, and $2(\ell - k)$ blocks of size 3×8 are formed within the $(16k+3) \times 16(\ell - k)$ block, as illustrated in Figure 9.

In Section 2.2 it was established that an 8-cycle decomposition of K_n with $n \equiv 3 \pmod{16}$ whereby the 3-cycle leave shares its vertices with one of the 8-cycles is possible. Specifically, these 3- and 8-cycles were $(11, 16, 17)$ and (17,13,18,14,16,19,11,12), respectively. So in each of the first $16k$ columns, we now take the 3-cycle leave and reclaim the corresponding 8-cycle, placing the combined eleven edges on the eight vertices in the column of the corresponding 8×8 block. In each of the first $16k$ rows, we take the graph shown in Figure 10 as the leave, placing its nine edges on the eight vertices in the row of the corresponding 8×8 block. The ability to obtain this leave for K_n where $n \equiv 15 \pmod{16}$ was justified

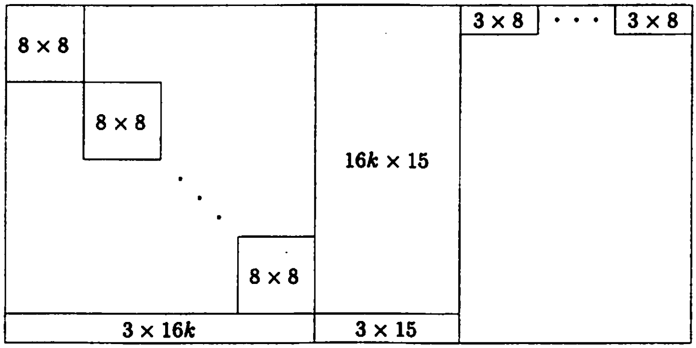


Figure 9: Division of the grid in the case where $k \leq \ell$ for Lemma 5.

earlier in Section 2.2. Now consider one of the 8×8 blocks, and number the vertices 1 to 64, starting with the upper-left vertex and proceeding down the first column so that consecutive vertices in any given row differ by eight. An 8-cycle decomposition of the edges in the 8×8 blocks appears in Appendix 5.1.1. Note also that the placement of the eleven edges within each column of the 8×8 block is not the same as for all of the other columns, nor is the placement of edges within each row the same as within the other rows, and so the placement of the 160 edges within the block is also prescribed by the edges listed in Appendix 5.1.1.

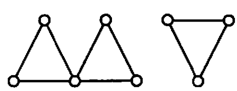


Figure 10: The leave required for the decomposition of the 8×8 blocks.

Next we examine the decomposition of the edges remaining in the 3×15 block, which has 3-cycle leaves on the columns and 9-cycle (or 9-edge) leaves on the rows. Note that by Lemma 1, we could have obtained a K_{15} leave in each of these rows by employing partial (instead of maximum) packings. By doing so, we find that we can obtain a complete graph of $K_3 \times K_{15}$ on the 3×15 block, which can then be decomposed into 8-cycles as is done in Appendix 5.1.2, where we again enumerate the vertices column by column, beginning with the upper-left vertex.

Now if $k < \ell$, then the only edges that remain are the 3-cycles on each column of the $(16k+3) \times 16(\ell - k)$ block. To finish the decomposition, we arbitrarily choose the first three rows to contain the 3-cycles for each column and then group the columns into sets of eight, thereby generating $2(\ell - k)$

blocks of size 3 × 8. On the rows of each such block we reclaim previously extracted pure 8-cycles, by refering to Section 2.2, and then carry out an 8-cycle decomposition of the edges within the block. This decomposition is illustrated in Figure 11.

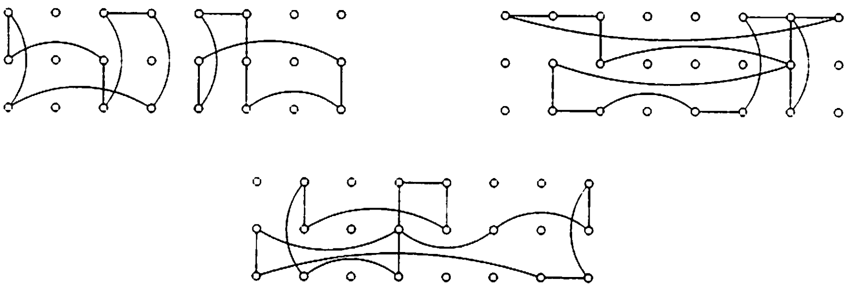


Figure 11: An 8-Cycle Decomposition of the 3 × 8 block.

If $m > n$ divide the graph into sections as illustrated in Figure 12, so that we obtain one $16\ell \times 16\ell$ block (in which we will form 2ℓ blocks of size 8×8), one block of size 3×15, two auxiliary blocks having sizes $3 \times 16\ell$ and $16\ell \times 15$, and one other block of size $16(k-\ell) \times (16\ell+15)$ in which we will form $2(k-\ell)$ blocks of size 8×9. As was done before, we pack each row and column with 8-cycles so that each of the 2ℓ blocks of size 8×8 has the decomposition provided in Appendix 5.1.1, and so that the 3×15 block contains a $K_3 \times K_{15}$ for which a decomposition appears in Appendix 5.1.2.

In each of the bottom $16(k-\ell)$ rows, we use a maximum packing having a 9-cycle leave, which we place on the first nine vertices of the row. The task now will be to focus on each of the 8×9 blocks. In each column of each 8×9 block, let A be the set of the first four vertices, and let B be the set containing the last four vertices. Now on each column we reclaim two 8-cycles which have the configuration of a $K_{4,4}$, with bipartition (A,B). These disjoint copies of $K_{4,4}$ appear on the disjoint sets of eight vertices that were discussed in Section 2.2 and which were illustrated in Figure 3. A list of 8-cycles that produces a decomposition of this 8×9 block is presented in Appendix 5.1.3, thereby completing the proof. □

Lemma 6 *If* $m \equiv 5 \pmod{16}$ *and* $n \equiv 13 \pmod{16}$ *then* $K_m \times K_n$ *is 8-cycle decomposable.*

Proof. Let $m = 16k + 5$ and $n = 16\ell + 13$.

Consider first $K_5 \times K_{16\ell+13}$. We divide the graph in a manner similar to the division in the proof of Lemma 5. Hence divide the columns into

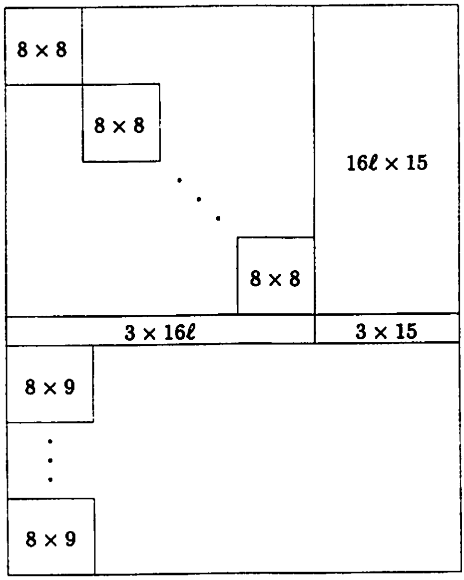


Figure 12: Division of the grid in the case where $k > \ell$ for Lemma 5.

one set of thirteen and 2ℓ sets of eight, thereby producing 2ℓ blocks of size 5×8 and one block of size 5×13. Using Lemma 1, we can employ a partial packing on each row so that we have a leave consisting of K_{13} (placed within the 5×13 block) as well as 2ℓ disjoint 8-cycles (each placed within a 5×8 block). To decompose the $K_5 \times K_{13}$ that is contained within the 5×13 block, we use Appendix 5.2.3. To decompose the edges within each 5×8 block, we use Appendix 5.2.2.

Now consider $m = 16k + 5$ and $n = 16\ell + 13$, with $m > 5$. The graph in which $m < n$ is decomposed in a slightly different manner than the graph in which $m > n$, but in each case we begin by extracting a maximum pure 8-cycle packing of each row and column, except in five rows and thirteen columns, where we use a partial packing to obtain a complete leave of K_{13} for each row and K_5 for each column. From Section 2.3, we know that we have 6 edges left in each of the $m - 5$ rows and 10 edges left in each of the $n - 13$ columns in which we extracted a maximum pure 8-cycle packing.

If $m < n$ then $k \leq \ell$. Divide the columns into three sets, one of size thirteen, one of size $16k$, and one of size $16(\ell - k)$ and divide the rows into one set of size five and one of size $16k$. The $m \times n$ grid now consists of a large square $16k \times 16k$ block, a 5×13 block, a block of size $(16k + 5) \times 16(\ell - k)$, and two other auxiliary blocks (having sizes $16k \times 13$ and $5 \times 16k$) not directly used in the remaining decomposition.

We select the 5 rows having a K_{13} leave, and the 13 columns having a K_5 leave, to form the 5×13 block, so that this block contains a copy of $K_5 \times K_{13}$, which we have previously seen to be decomposable into 8-cycles.

To decompose the edges within the square $16k \times 16k$ block we form $2k$ blocks of size 8×8 oriented along the main diagonal. For each of the $16k$ columns, it is possible (as indicated in Section 2.3) to have removed the edges of a maximum 8-cycle packing in a manner such that we now have the 10-edge leave shown in Figure 13 occurring on the eight vertices of the corresponding 8×8 block. For the rows we let the 6-edge leave be a 6-cycle. An 8-cycle decomposition of K_n, $n \equiv 13 \pmod{16}$ whereby the 6-cycle leave shares six of the eight vertices of one of the 8-cycles appears in Section 2.3. Hence we are now able to reclaim one such 8-cycle per row, placing it and the 6-cycle both on the eight vertices of a row of an 8×8 block. Thus, for the 8×8 blocks we have a copy of the 10-edge leave shown in Figure 13 on each column, and a combination of a 6-cycle and an 8-cycle on each row. A list of 8-cycles used in a decomposition of such an 8×8 block is presented in Appendix 5.2.4.

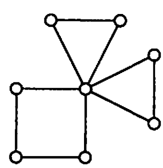


Figure 13: The leave used in the decomposition of the 8×8 blocks.

To finish the case where $m < n$ requires the decomposition of the 10-edge leave in each column of the $(16k + 5) \times 16(\ell - k)$ block. This can be done by first using K_5 as the 10-edge leave and then taking this leave to lie on the first five vertices of each column. In a manner similar to the case when $m = 5$, we now reclaim $2(\ell - k)$ disjoint 8-cycles in sets of eight vertices in each of the first five rows. This effectively forms $2(\ell - k)$ blocks of size 5×8 within the $(16k+5) \times 16(\ell - k)$ block, each having a copy of K_5 on each column and an 8-cycle on each row, which is 8-cycle decomposable as indicated in Appendix 5.2.2.

Now suppose $m > n$, in which case $k > \ell$. Begin the decomposition by separating the columns into two sets, having sizes 16ℓ and 13, and collect the rows into three sets, having sizes 16ℓ, 5, and $16(k - \ell)$. This divides the graph into smaller grids of size $16\ell \times 16\ell$, 5×13, and $16(k - \ell) \times (16\ell + 13)$, along with two auxiliary blocks of size $16\ell \times 13$ and $5 \times 16\ell$. The edges within the $16\ell \times 16\ell$ and 5×13 blocks can be decomposed in the manner described in the previous cases, leaving only the edges in the $16(k-\ell) \times (16\ell+13)$ block to be decomposed. To accomplish this task we form within this block $2(k-\ell)$ blocks of size 8×6, and take the 6-cycle leave on each row to lie on the first six vertices of the row. Noting that there are no edges on the six columns of each block at this stage of the decomposition, we use Lemma 1 to reclaim $2(k-\ell)$ disjoint previously removed 8-cycles on each column and then decompose the resulting subgraph using Appendix 5.2.1, thereby completing the proof. □

Lemma 7 *If $m \equiv 7 \pmod{16}$ and $n \equiv 11 \pmod{16}$ then $K_m \times K_n$ is 8-cycle decomposable.*

Proof. Let $m = 16k+7$ and $n = 16\ell+11$. We begin by arbitrarily choosing seven rows and eleven columns and in these take a partial 8-cycle packing so that we obtain a copy of K_{11} as the leave on the rows, and a K_7 as the leave on the columns. By taking a maximum packing of the remaining rows and columns of $K_m \times K_n$ with pure horizontal and vertical 8-cycles, we are left with a 7-edge leave on each row and one of two possible leaves on each column: a copy of K_7 on each column if $m = 7$, or a C_5 if $m > 7$. The graph of $K_7 \times K_n$ is decomposed in a different manner from $K_m \times K_n$, $m > 7$.

We examine the case where $m = 7$ and $n = 16\ell + 11$ first. Divide the grid into a number of blocks, one of size 7×11 and 2ℓ blocks of size 7×8, so that we have a $K_7 \times K_{11}$ on the 7×11 block, a decomposition of which is provided in Appendix 5.3.1. In each of the 7×8 blocks, we have only the edges of eight vertical copies of K_7. Thus we now refer to Lemma 1 to reclaim, in each row, 4ℓ 8-cycles in pairs such that we have 2ℓ disjoint copies of $K_{4,4}$. On each row of each 7×8 block, now place the edges of one such $K_{4,4}$ so that it has bipartition (A, B) where A is the set of the first four vertices of the row and B is the set of the remaining four vertices of the row. To now decompose the edges of each 7×8 block, refer to Appendix 5.3.5.

In the case remaining, $m = 16k + 7$, $m > 7$ and $n = 16\ell + 11$, we first consider the graph where $m < n$. Divide the graph into several blocks: one of size $16k \times 16k$, one of size 7×11, one of size $(16k + 7) \times 16(\ell - k)$, and two auxiliary blocks of sizes $16k \times 11$ and $7 \times 16k$, so that within the 7×11 block there is a copy of $K_7 \times K_{11}$, which we can decompose into 8-cycles using Appendix 5.3.1. Within the $16k \times 16k$ block, form $2k$ blocks of size 8×8 along its main diagonal, with each 8×8 block having a 7-cycle in each row and a 5-cycle in each column. The placement of these 5- and 7-cycles, and the subsequent 8-cycle decomposition of the 8×8 block is provided in Appendix 5.3.2.

All that remains for this case is to decompose the edges on the columns in the $(16k+7) \times 16(\ell-k)$ block. For this we arbitrarily choose a set of five rows, and divide the columns into groups of eight, thereby forming $2(\ell-k)$ 5×8 blocks. Since no edges remain on the five rows of each of these blocks, we must reclaim previously extracted C_8's for these rows, using Lemma 1 to do so. A decomposition of the 5×8 blocks is presented in Appendix 5.3.3. Note also that the edges listed in Appendix 5.3.3 prescribe the placement of each 8-cycle and each 5-cycle in its respective row or column.

To complete the proof, we have left to examine the case where $m > n$. In this case we begin in the same manner; that is take a partial 8-cycle packing of seven rows and eleven columns to obtain K_{11} and K_7 leaves on these rows and columns, respectively. On the remaining rows and columns we take a maximum packing, leaving a C_7 as the leave on the rows, and a C_5 on the columns. We then divide the graph into regions composed of a $16\ell\times 16\ell$ grid, a 7×11 grid, a $16(k-\ell)\times(16\ell+11)$ grid, as well as two auxiliary regions of size $16\ell\times 11$ and $7\times 16\ell$. We have already addressed the decomposition of the first two, leaving only the third block, in which we form $2(k-\ell)$ blocks of size 8×7. We take the 7-edge leave on each of the rows of these blocks to be a 7-cycle. Since no edges remain on the columns we reclaim previously removed 8-cycles, again using Lemma 1, reclaiming one 8-cycle within each column of each 8×7 block. We now decompose each 8×7 block using Appendix 5.3.4. □

Lemma 8 *If $m,n\equiv 9 \pmod{16}$ then $K_m\times K_n$ is 8-cycle decomposable.*

Proof. We assume that $m=16k+9\le n=16\ell+9$, and so $k\le\ell$. Divide the graph into blocks: one having size 9×9, one of size $16k\times 16k$, two auxiliary blocks having sizes $16k\times 9$ and $9\times 16k$, and if $m\ne n$, also a block of size $(16k+9)\times 16(\ell-k)$.

By using Lemma 1, we can obtain a K_9 leave in each of the nine rows and nine columns in which the 9×9 grid appears. The resulting $K_9\times K_9$ subgraph can be decomposed into 8-cycles by using Appendix 5.4.1.

In each of the rows and columns of the $16k\times 16k$ block, we take a maximum 8-cycle packing, producing a 4-cycle leave in each row and column. Now form $4k$ blocks of size 4×4 along the main diagonal of the $16k\times 16k$ block. It is within these 4×4 blocks that we position each 4-cycle leave, so that each 4×4 block can now be decomposed using Figure 14.

Now, if $m\ne n$, in the $(16k+9)\times 16(\ell-k)$ block form $2(\ell-k)$ blocks of size 4×8 so that each such block is on a common set of four rows, and these four rows are a subset of the nine rows of the 9×9 grid. We may now use Lemma 1 to reclaim one 8-cycle per row of each 4×8 grid. By also ensuring that the vertical 4-cycles in the $(16k+9)\times 16(\ell-k)$ block are contained within the 4×8 blocks, we can now complete the proof by using Figure 15 to decompose each 4×8 block. □

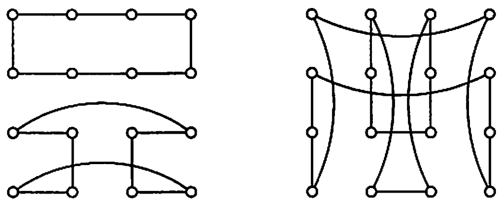


Figure 14: An 8-Cycle Decomposition of the 4 × 4 block.

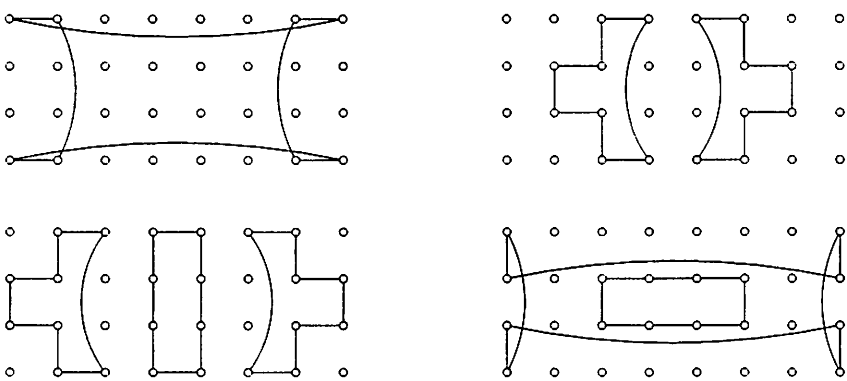


Figure 15: An 8-Cycle Decomposition of the 4 × 8 block.

3.3 Conclusion

Combining the results of Lemmata 2 through 8 we conclude:

Theorem 2 *The graph $K_m \times K_n$ is 8-cycle decomposable if and only if either*

1. $m \equiv 0$ *(mod 4) and* $n \equiv 0$ *(mod 2),*
2. $m, n \equiv 1$ *(mod 16),*
3. $m \equiv 3$ *(mod 16) and* $n \equiv 15$ *(mod 16),*
4. $m \equiv 5$ *(mod 16) and* $n \equiv 13$ *(mod 16),*
5. $m \equiv 7$ *(mod 16) and* $n \equiv 11$ *(mod 16), or*
6. $m, n \equiv 9$ *(mod 16).*

4 Acknowledgement

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5 Appendix

5.1 8-cycle decompositions required for Lemma 5

5.1.1 8-cycle decomposition of the 8 × 8 block

(30, 32, 64, 62, 58, 34, 37, 38)	(27, 28, 36, 40, 37, 53, 51, 35)	(26, 29, 37, 33, 49, 55, 56, 50)
(22, 54, 55, 51, 27, 29, 61, 62)	(19, 43, 47, 41, 44, 60, 64, 59)	(18, 21, 23, 63, 59, 57, 25, 26)
(18, 20, 28, 31, 29, 53, 54, 50)	(9, 16, 40, 38, 35, 39, 31, 25)	(9, 12, 10, 42, 44, 45, 61, 57)
(9, 11, 59, 62, 54, 56, 16, 15)	(8, 32, 31, 55, 23, 17, 41, 48)	(8, 24, 22, 38, 34, 26, 58, 64)
(6, 14, 15, 12, 52, 49, 41, 46)	(5, 7, 15, 47, 46, 14, 13, 21)	(4, 52, 56, 40, 39, 55, 63, 60)
(3, 19, 22, 30, 28, 29, 45, 43)	(2, 10, 11, 19, 21, 24, 48, 42)	(2, 5, 13, 12, 4, 44, 47, 7)
(1, 33, 36, 20, 19, 24, 17, 49)	(1, 2, 3, 6, 5, 8, 4, 7)	

5.1.2 8-cycle decomposition of $K_3 \times K_{15}$

(35, 38, 44, 45, 39, 36, 42, 41)	(28, 29, 41, 40, 34, 37, 31, 43)	(27, 39, 37, 40, 28, 30, 42, 45)
(25, 40, 31, 32, 41, 38, 37, 43)	(25, 27, 42, 33, 45, 36, 35, 34)	(23, 41, 26, 38, 29, 32, 35, 44)
(21, 42, 24, 39, 33, 27, 30, 45)	(21, 36, 34, 22, 43, 40, 42, 39)	(20, 35, 29, 23, 38, 32, 44, 41)
(20, 26, 35, 23, 24, 36, 30, 29)	(19, 37, 28, 25, 26, 29, 44, 43)	(16, 37, 25, 22, 31, 28, 34, 43)
(15, 36, 18, 27, 24, 33, 30, 39)	(15, 30, 24, 21, 27, 26, 32, 33)	(15, 24, 18, 30, 21, 33, 36, 27)
(14, 41, 17, 38, 20, 23, 26, 44)	(13, 34, 16, 25, 19, 40, 22, 37)	(13, 28, 22, 23, 17, 18, 33, 31)
(13, 22, 16, 28, 19, 34, 31, 25)	(9, 42, 18, 16, 19, 22, 24, 45)	(8, 41, 11, 35, 14, 32, 20, 44)
(8, 32, 11, 38, 14, 29, 17, 35)	(7, 40, 16, 13, 15, 18, 45, 43)	(6, 39, 9, 36, 12, 42, 15, 45)
(5, 38, 8, 29, 11, 26, 17, 44)	(5, 26, 8, 23, 11, 20, 17, 32)	(4, 37, 7, 34, 10, 40, 13, 43)
(3, 42, 6, 33, 9, 30, 12, 45)	(3, 36, 6, 27, 12, 21, 18, 39)	(3, 30, 6, 21, 9, 24, 12, 33)
(3, 24, 6, 15, 12, 18, 9, 27)	(3, 18, 6, 12, 10, 31, 19, 21)	(2, 41, 5, 20, 8, 17, 11, 44)
(2, 35, 5, 14, 11, 12, 39, 38)	(2, 29, 5, 11, 8, 14, 23, 32)	(2, 23, 5, 8, 9, 15, 14, 26)
(2, 17, 5, 6, 3, 15, 21, 20)	(1, 40, 4, 31, 7, 28, 10, 43)	(1, 34, 4, 25, 7, 22, 10, 37)
(1, 28, 4, 19, 13, 10, 16, 31)	(1, 22, 4, 13, 7, 19, 10, 25)	(1, 16, 7, 8, 2, 14, 20, 19)
(1, 10, 7, 4, 16, 17, 14, 13)	(1, 4, 10, 11, 2, 3, 9, 7)	(1, 2, 5, 4, 6, 9, 12, 3)

5.1.3 8-cycle decomposition of the 8 × 9 block

(65, 69, 68, 72, 67, 71, 66, 70)	(58, 62, 59, 67, 70, 68, 71, 63)	(52, 53, 61, 60, 63, 59, 64, 56)
(50, 55, 52, 60, 64, 72, 66, 58)	(49, 56, 51, 59, 61, 58, 64, 57)	(49, 53, 51, 54, 62, 57, 63, 55)
(43, 46, 54, 52, 44, 47, 55, 51)	(41, 48, 44, 45, 53, 50, 54, 49)	(36, 39, 47, 42, 50, 56, 48, 40)
(34, 37, 45, 41, 47, 43, 48, 42)	(33, 39, 35, 43, 45, 42, 46, 41)	(28, 32, 40, 35, 38, 46, 44, 36)
(28, 30, 38, 33, 40, 34, 39, 31)	(26, 32, 27, 35, 37, 36, 38, 34)	(20, 24, 32, 25, 33, 37, 29, 28)
(20, 21, 29, 26, 30, 27, 31, 23)	(18, 24, 19, 27, 29, 25, 31, 26)	(17, 23, 18, 21, 19, 22, 30, 25)
(11, 14, 22, 20, 12, 15, 23, 19)	(10, 16, 12, 13, 21, 17, 22, 18)	(9, 14, 10, 15, 11, 16, 24, 17)
(7, 15, 9, 16, 8, 72, 65, 71)	(4, 12, 14, 6, 70, 62, 60, 68)	(2, 10, 13, 11, 3, 67, 69, 66)
(1, 9, 13, 5, 69, 61, 57, 65)	(1, 7, 4, 5, 3, 6, 2, 8)	(1, 5, 2, 7, 3, 8, 4, 6)

5.2 8-cycle decompositions required for Lemma 6

5.2.1 8-cycle decomposition of the 8 × 6 block

(41, 42, 43, 44, 45, 46, 47, 48)	(26, 27, 28, 29, 37, 36, 35, 34)	(17, 18, 26, 25, 33, 40, 32, 24)
(12, 13, 14, 15, 23, 22, 21, 20)	(9, 16, 24, 23, 31, 32, 25, 17)	(7, 15, 16, 8, 48, 40, 39, 47)
(6, 14, 22, 30, 31, 39, 38, 46)	(5, 13, 21, 29, 30, 38, 37, 45)	(4, 12, 11, 19, 20, 28, 36, 44)
(3, 11, 10, 18, 19, 27, 35, 43)	(1, 9, 10, 2, 42, 34, 33, 41)	(1, 2, 3, 4, 5, 6, 7, 8)

5.2.2 8-cycle decomposition of the 5 × 8 block

(26, 28, 38, 36, 40, 37, 39, 29)	(22, 25, 24, 34, 31, 33, 35, 32)	(21, 24, 23, 33, 32, 34, 35, 25)
(17, 19, 24, 22, 23, 28, 29, 27)	(16, 19, 18, 17, 20, 30, 26, 21)	(16, 17, 37, 36, 39, 40, 38, 18)
(12, 14, 13, 33, 34, 39, 38, 37)	(7, 9, 14, 11, 13, 12, 15, 10)	(6, 10, 9, 29, 30, 28, 27, 26)
(6, 7, 22, 21, 23, 25, 10, 8)	(4, 14, 15, 11, 31, 16, 20, 19)	(2, 5, 3, 13, 15, 40, 30, 27)
(1, 11, 12, 7, 8, 9, 6, 36)	(1, 4, 3, 2, 32, 31, 35, 5)	(1, 2, 4, 5, 20, 18, 8, 3)

5.2.3 8-cycle decomposition of $K_5 \times K_{13}$

(57, 62, 63, 60, 61, 64, 58, 65)	(56, 61, 63, 59, 64, 60, 65, 62)	(55, 58, 61, 57, 64, 63, 65, 59)
(53, 63, 57, 60, 56, 65, 54, 64)	(53, 60, 54, 62, 64, 55, 65, 61)	(53, 56, 55, 54, 58, 60, 62, 59)
(53, 54, 57, 59, 56, 58, 62, 55)	(46, 52, 51, 49, 50, 48, 61, 59)	(45, 49, 46, 50, 52, 48, 47, 51)
(43, 52, 47, 49, 48, 51, 64, 56)	(43, 49, 44, 52, 45, 48, 46, 51)	(42, 50, 45, 47, 43, 48, 44, 51)
(41, 52, 42, 46, 47, 50, 63, 54)	(41, 49, 42, 44, 45, 46, 43, 50)	(40, 51, 41, 48, 42, 45, 58, 53)
(40, 46, 41, 47, 42, 43, 44, 50)	(36, 38, 37, 39, 65, 52, 49, 62)	(35, 48, 40, 45, 41, 42, 55, 61)
(35, 37, 36, 49, 40, 52, 39, 38)	(33, 38, 34, 60, 47, 40, 44, 46)	(31, 39, 33, 35, 36, 34, 47, 44)
(30, 37, 32, 36, 33, 34, 35, 39)	(29, 42, 40, 43, 41, 44, 57, 55)	(29, 36, 31, 38, 32, 58, 63, 37)
(28, 39, 29, 34, 30, 33, 59, 54)	(28, 35, 30, 32, 33, 31, 34, 37)	(28, 33, 29, 30, 31, 35, 32, 34)
(27, 40, 41, 28, 32, 31, 57, 53)	(27, 38, 30, 28, 31, 29, 32, 39)	(24, 50, 37, 27, 36, 30, 56, 63)
(22, 24, 23, 25, 64, 38, 29, 35)	(20, 25, 22, 21, 26, 24, 37, 33)	(19, 24, 20, 26, 39, 34, 21, 25)
(18, 22, 20, 23, 21, 24, 25, 26)	(17, 43, 30, 27, 31, 18, 57, 56)	(17, 24, 18, 21, 20, 19, 22, 26)
(16, 25, 18, 20, 17, 23, 19, 26)	(15, 28, 29, 16, 23, 22, 61, 54)	(12, 51, 38, 25, 15, 26, 65, 64)
(12, 13, 52, 26, 14, 27, 28, 38)	(11, 24, 16, 21, 17, 25, 51, 50)	(10, 49, 23, 15, 24, 14, 53, 62)
(10, 12, 25, 14, 23, 36, 39, 13)	(9, 48, 22, 17, 18, 23, 62, 61)	(8, 47, 21, 15, 22, 16, 55, 60)
(8, 21, 14, 22, 9, 35, 27, 34)	(8, 12, 9, 11, 10, 23, 26, 13)	(7, 46, 20, 16, 19, 21, 60, 59)
(6, 45, 32, 19, 15, 20, 59, 58)	(6, 19, 14, 20, 7, 33, 27, 32)	(6, 11, 8, 10, 9, 13, 7, 12)
(5, 44, 18, 15, 17, 19, 58, 57)	(5, 13, 6, 10, 7, 11, 37, 31)	(4, 30, 17, 14, 18, 19, 45, 43)
(3, 42, 16, 18, 5, 11, 63, 55)	(2, 41, 15, 16, 17, 4, 56, 54)	(2, 13, 4, 12, 5, 10, 36, 28)
(1, 40, 14, 16, 3, 13, 65, 53)	(1, 14, 15, 2, 11, 3, 29, 27)	(1, 12, 3, 7, 9, 4, 11, 13)
(1, 10, 4, 6, 9, 2, 12, 11)	(1, 8, 5, 7, 2, 10, 3, 9)	(1, 6, 3, 5, 2, 8, 4, 7)
(1, 4, 3, 2, 6, 8, 9, 5)	(1, 2, 4, 5, 6, 7, 8, 3)	

5.2.4 8-cycle decomposition of the 8×8 block

(35, 37, 40, 56, 48, 43, 51, 59)	(29, 30, 62, 54, 46, 47, 45, 61)	(26, 28, 44, 60, 57, 62, 38, 34)
(23, 31, 47, 43, 35, 27, 51, 55)	(21, 23, 39, 35, 51, 50, 42, 45)	(17, 19, 21, 29, 26, 31, 55, 49)
(16, 48, 24, 22, 21, 61, 57, 64)	(13, 53, 37, 39, 47, 15, 63, 61)	(13, 21, 24, 64, 56, 49, 53, 29)
(10, 14, 22, 46, 38, 37, 34, 42)	(10, 12, 52, 54, 49, 41, 57, 58)	(9, 41, 47, 42, 18, 58, 50, 49)
(7, 31, 25, 33, 36, 44, 47, 55)	(6, 54, 30, 46, 44, 20, 60, 62)	(6, 8, 32, 26, 10, 15, 14, 38)
(5, 37, 33, 9, 16, 56, 53, 45)	(5, 6, 30, 26, 25, 41, 48, 8)	(4, 36, 20, 17, 33, 49, 52, 60)
(3, 43, 27, 28, 36, 40, 64, 59)	(3, 19, 11, 16, 40, 24, 32, 27)	(2, 50, 18, 23, 7, 39, 63, 58)
(2, 7, 8, 4, 28, 20, 21, 18)	(1, 25, 17, 9, 10, 11, 59, 57)	(1, 3, 8, 2, 10, 13, 12, 4)

5.3 8-cycle decompositions required for Lemma 7

5.3.1 8-cycle decomposition of $K_7 \times K_{11}$

(61, 62, 76, 69, 66, 73, 77, 63)	(59, 66, 70, 69, 68, 67, 74, 73)	(56, 63, 59, 60, 67, 64, 68, 70)
(55, 56, 77, 70, 65, 72, 71, 76)	(54, 61, 57, 64, 69, 65, 68, 75)	(53, 56, 54, 55, 69, 62, 60, 74)
(51, 56, 52, 54, 53, 67, 65, 58)	(50, 57, 60, 58, 63, 70, 64, 71)	(49, 56, 50, 55, 52, 66, 67, 70)
(46, 47, 68, 54, 51, 55, 53, 60)	(44, 65, 51, 53, 46, 49, 77, 72)	(43, 64, 50, 54, 47, 75, 77, 71)
(42, 49, 44, 51, 50, 43, 57, 63)	(41, 69, 48, 62, 59, 61, 75, 76)	(39, 46, 43, 45, 73, 52, 50, 53)
(36, 43, 44, 45, 59, 58, 57, 71)	(35, 56, 42, 38, 52, 45, 49, 63)	(34, 62, 55, 41, 42, 39, 67, 69)
(34, 35, 42, 37, 72, 58, 44, 48)	(33, 61, 47, 44, 37, 65, 66, 68)	(32, 60, 39, 40, 38, 45, 46, 67)
(31, 59, 38, 39, 41, 48, 45, 66)	(30, 58, 37, 38, 31, 52, 51, 72)	(29, 64, 36, 41, 37, 39, 74, 71)
(27, 55, 34, 33, 54, 40, 41, 62)	(25, 39, 36, 40, 26, 68, 61, 60)	(24, 45, 31, 35, 32, 53, 52, 59)
(23, 44, 30, 51, 37, 40, 75, 72)	(22, 43, 29, 57, 36, 38, 66, 64)	(21, 70, 35, 30, 37, 36, 42, 77)
(20, 41, 34, 32, 33, 47, 48, 55)	(19, 54, 26, 28, 70, 42, 40, 68)	(19, 21, 56, 28, 24, 26, 47, 40)
(18, 53, 25, 32, 31, 34, 76, 74)	(17, 66, 24, 25, 27, 41, 38, 73)	(16, 65, 23, 30, 33, 31, 73, 72)
(15, 50, 36, 29, 34, 30, 65, 64)	(15, 36, 22, 50, 29, 35, 49, 43)	(14, 63, 28, 22, 29, 32, 74, 77)
(13, 69, 27, 22, 25, 28, 77, 76)	(12, 61, 26, 23, 28, 35, 33, 75)	(11, 60, 18, 46, 25, 26, 75, 74)
(10, 59, 17, 52, 24, 27, 76, 73)	(9, 58, 16, 51, 23, 25, 74, 72)	(8, 57, 15, 22, 23, 24, 73, 71)
(7, 70, 14, 49, 28, 21, 35, 77)	(7, 56, 14, 42, 21, 20, 62, 63)	(7, 21, 17, 20, 34, 27, 28, 42)
(6, 69, 20, 15, 8, 43, 48, 76)	(6, 55, 13, 48, 27, 23, 58, 62)	(6, 41, 13, 14, 35, 7, 49, 48)
(5, 68, 12, 33, 26, 22, 71, 75)	(5, 54, 12, 26, 19, 33, 40, 61)	(5, 40, 12, 19, 15, 21, 49, 47)
(5, 7, 6, 34, 13, 20, 27, 26)	(4, 67, 25, 18, 39, 32, 46, 74)	(4, 53, 11, 67, 18, 21, 63, 60)
(4, 39, 11, 32, 18, 20, 48, 46)	(4, 25, 11, 14, 8, 29, 30, 32)	(4, 6, 27, 13, 12, 14, 28, 7)
(3, 66, 10, 45, 17, 19, 75, 73)	(3, 52, 10, 31, 24, 22, 57, 59)	(3, 38, 24, 17, 18, 19, 47, 45)
(3, 24, 10, 38, 17, 15, 29, 31)	(2, 65, 9, 44, 16, 20, 76, 72)	(2, 51, 9, 30, 16, 19, 61, 58)
(2, 37, 23, 16, 18, 11, 46, 44)	(2, 23, 9, 37, 16, 17, 31, 30)	(2, 7, 3, 17, 10, 14, 21, 16)
(1, 64, 8, 13, 9, 16, 15, 71)	(1, 50, 8, 12, 11, 13, 62, 57)	(1, 36, 8, 11, 9, 12, 47, 43)
(1, 22, 8, 10, 12, 5, 33, 29)	(1, 8, 9, 10, 11, 4, 18, 15)	(1, 6, 3, 5, 2, 9, 14, 7)
(1, 4, 3, 2, 6, 20, 19, 5)	(1, 2, 4, 5, 6, 13, 10, 3)	

5.3.2 8-cycle decomposition of the 8 × 8 block

(39, 40, 48, 56, 64, 63, 55, 47)	(34, 38, 62, 58, 60, 44, 41, 42)	(29, 31, 39, 38, 30, 54, 46, 45)
(19, 21, 29, 25, 33, 41, 49, 51)	(14, 16, 24, 32, 40, 36, 44, 46)	(11, 19, 20, 36, 34, 26, 27, 35)
(6, 14, 12, 20, 23, 7, 63, 62)	(4, 12, 11, 15, 16, 64, 60, 52)	(3, 7, 5, 21, 24, 23, 31, 27)
(2, 3, 59, 51, 53, 45, 42, 58)	(1, 5, 13, 61, 53, 54, 50, 49)	(1, 2, 10, 18, 26, 25, 17, 9)

5.3.3 8-cycle decomposition of the 5 × 8 block

(31, 33, 38, 39, 34, 35, 40, 36)	(23, 28, 30, 29, 26, 27, 37, 38)	(21, 23, 25, 30, 35, 32, 31, 26)
(13, 18, 20, 25, 24, 29, 34, 33)	(7, 8, 28, 27, 17, 16, 11, 12)	(6, 9, 19, 20, 15, 14, 13, 11)
(5, 10, 15, 12, 17, 19, 39, 40)	(3, 4, 14, 24, 22, 21, 16, 18)	(1, 6, 7, 2, 22, 32, 37, 36)
(1, 2, 3, 8, 10, 9, 4, 5)		

5.3.4 8-cycle decomposition of the 8 × 7 block

(38, 39, 40, 48, 47, 55, 54, 46)	(35, 37, 45, 41, 42, 50, 51, 43)	(33, 36, 44, 46, 48, 56, 49, 41)
(25, 28, 52, 44, 43, 42, 34, 33)	(22, 23, 24, 32, 40, 37, 29, 30)	(18, 19, 51, 52, 53, 54, 30, 26)
(17, 20, 28, 27, 35, 34, 26, 25)	(9, 10, 12, 16, 24, 21, 19, 11)	(8, 16, 15, 23, 31, 39, 55, 56)
(7, 15, 14, 13, 21, 53, 45, 47)	(4, 12, 20, 22, 14, 6, 38, 36)	(3, 11, 13, 5, 29, 32, 31, 27)
(1, 9, 17, 18, 2, 10, 50, 49)	(1, 2, 3, 4, 5, 6, 7, 8)	

5.3.5 8-cycle decomposition of the 7 × 8 block

(2, 44, 16, 15, 29, 30, 23, 51)	(11, 53, 18, 16, 17, 45, 44, 9)	(4, 53, 50, 22, 43, 1, 3, 2)
(54, 53, 52, 24, 22, 36, 8, 50)	(47, 45, 46, 25, 22, 23, 44, 43)	(33, 30, 16, 37, 23, 25, 32, 29)
(19, 16, 51, 9, 10, 52, 50, 15)	(12, 9, 37, 36, 39, 38, 10, 8)	(5, 4, 39, 37, 38, 17, 52, 3)
(55, 52, 54, 26, 25, 53, 51, 50)	(48, 44, 46, 18, 39, 11, 8, 43)	(27, 24, 38, 41, 36, 40, 26, 22)
(20, 17, 31, 10, 11, 46, 43, 15)	(13, 10, 12, 11, 32, 30, 9, 8)	(56, 51, 55, 54, 5, 6, 1, 50)
(56, 53, 55, 13, 12, 54, 51, 52)	(49, 44, 47, 46, 4, 3, 45, 43)	(49, 46, 48, 47, 26, 23, 24, 45)
(42, 37, 40, 19, 18, 17, 15, 36)	(42, 39, 41, 37, 2, 1, 36, 38)	(35, 30, 34, 31, 32, 4, 1, 29)
(35, 32, 34, 33, 5, 2, 30, 31)	(28, 23, 27, 26, 33, 31, 29, 22)	(28, 25, 27, 55, 6, 3, 31, 24)
(28, 27, 41, 40, 39, 25, 24, 26)	(21, 16, 20, 19, 33, 32, 18, 15)	(21, 18, 20, 55, 56, 54, 19, 17)
(14, 9, 13, 48, 27, 34, 29, 8)	(14, 11, 13, 41, 20, 48, 45, 10)	(14, 13, 34, 20, 21, 19, 47, 12)
(7, 2, 6, 48, 49, 47, 5, 1)	(7, 4, 6, 41, 42, 40, 38, 3)	(7, 6, 34, 35, 33, 12, 40, 5)
(7, 49, 28, 35, 21, 42, 14, 56)	(7, 35, 14, 49, 21, 56, 28, 42)	

5.4 8-cycle decompositions required for Lemma 8

5.4.1 8-cycle decomposition of $K_9 \times K_9$

(74, 79, 77, 75, 76, 81, 78, 80)	(67, 69, 78, 79, 73, 81, 80, 76)	(65, 68, 71, 67, 70, 66, 72, 69)
(64, 65, 71, 69, 70, 79, 80, 73)	(61, 63, 81, 72, 67, 64, 71, 70)	(58, 62, 60, 61, 59, 77, 68, 67)
(57, 66, 68, 64, 70, 65, 74, 75)	(56, 61, 58, 63, 72, 64, 66, 65)	(55, 62, 57, 63, 59, 56, 74, 73)
(54, 63, 55, 58, 57, 59, 68, 72)	(51, 54, 53, 71, 62, 56, 60, 69)	(49, 53, 52, 79, 61, 55, 60, 58)
(48, 54, 50, 52, 51, 60, 78, 75)	(47, 52, 49, 51, 53, 50, 77, 74)	(46, 64, 55, 57, 56, 58, 76, 73)
(45, 54, 46, 53, 48, 52, 70, 72)	(43, 61, 52, 46, 51, 50, 68, 70)	(41, 45, 42, 44, 80, 62, 59, 50)
(39, 57, 48, 51, 42, 69, 66, 75)	(38, 65, 47, 48, 50, 49, 76, 74)	(37, 55, 46, 50, 47, 51, 78, 73)
(36, 63, 45, 39, 48, 49, 54, 81)	(35, 36, 45, 37, 46, 47, 53, 80)	(34, 52, 43, 41, 44, 53, 62, 61)
(32, 59, 41, 42, 43, 45, 81, 77)	(31, 67, 49, 40, 44, 43, 79, 76)	(31, 36, 32, 35, 62, 44, 45, 40)
(29, 56, 47, 38, 43, 40, 67, 65)	(29, 36, 30, 66, 48, 46, 49, 47)	(28, 64, 37, 44, 38, 41, 77, 73)
(28, 37, 39, 42, 40, 38, 56, 55)	(25, 43, 34, 32, 33, 36, 54, 52)	(24, 60, 42, 38, 39, 41, 68, 69)
(24, 25, 26, 71, 44, 35, 33, 51)	(23, 50, 32, 30, 34, 33, 60, 59)	(22, 49, 31, 35, 30, 39, 66, 67)
(21, 48, 30, 31, 34, 35, 71, 66)	(20, 38, 29, 35, 28, 36, 72, 65)	(19, 46, 28, 34, 29, 33, 69, 64)
(17, 62, 26, 22, 27, 72, 71, 80)	(16, 61, 25, 23, 27, 36, 34, 79)	(15, 42, 33, 28, 31, 29, 74, 78)
(14, 68, 32, 28, 30, 33, 78, 77)	(13, 58, 40, 22, 24, 42, 78, 76)	(13, 31, 22, 20, 29, 32, 41, 40)
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(9, 72, 18, 54, 27, 25, 79, 81)	(9, 36, 18, 45, 27, 20, 47, 54)	(8, 71, 17, 44, 26, 23, 77, 80)
(8, 53, 35, 26, 19, 27, 63, 62)	(8, 35, 17, 53, 26, 21, 39, 44)	(7, 70, 34, 25, 22, 21, 75, 79)
(7, 52, 16, 70, 25, 21, 57, 61)	(7, 34, 16, 25, 20, 19, 37, 43)	(6, 69, 15, 18, 17, 26, 24, 78)
(6, 51, 15, 17, 16, 18, 63, 60)	(6, 33, 24, 20, 11, 38, 37, 42)	(5, 68, 23, 24, 19, 22, 76, 77)
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(2, 9, 3, 21, 12, 14, 23, 20)	(1, 64, 10, 14, 5, 23, 19, 73)	(1, 46, 10, 13, 11, 14, 59, 55)
(1, 28, 10, 12, 13, 4, 40, 37)	(1, 10, 11, 12, 3, 7, 25, 19)	(1, 8, 5, 7, 2, 11, 18, 9)
(1, 6, 3, 5, 2, 8, 4, 7)	(1, 4, 3, 2, 6, 8, 9, 5)	(1, 2, 4, 5, 6, 7, 8, 3)

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