Finite Order Domination in Graphs

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Abstract

The (previously studied) notions of secure domination and of weak Roman domination involve the construction of protection strategies in a simple graph $G = (V, E)$, by utilising the minimum number of guards needed at vertices in V to protect G in different scenarios (these minimum numbers are called the secure [weak Roman] domination parameters for the graph). In this paper these notions are generalised in the sense that safe configurations in G are not merely sought after one move, but rather after each of $k \geq 1$ moves. Some general properties of these generalised domination parameters are established, after which the parameter values are found for certain simple graph structures (such as paths, cycles, multipartite graphs and products of complete graphs, cycles and paths).

Keywords: Secure & weak Roman Domination, Higher Order Domination, Graph Protection.

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1 Introduction

A *guard function* for a graph $G = (V, E)$ is a mapping $f : V \mapsto \{0, 1, 2, \ldots\}$ such that $f(v)$ denotes the number of *guards* stationed at a vertex $v \in V$. A guard function partitions the vertex set of G into subsets $V_i = \{v : f(v) = i\}$, $i = 0, 1, 2, \ldots$ and we (imprecisely) write $f = (V_0, V_1, V_2, \ldots)$. A guard

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function is called *safe* if each $v \in V_0$ is adjacent to some $u \in V \backslash V_0$ (i.e. if $V \backslash V_0$ is a dominating set of G). The *weight* of a guard function is denoted

$$w(f) = \sum_{v \in V} f(v).$$

The following four kinds of safe guard functions have been studied in the literature:

1. A *dominating function* (DF) is a safe guard function $f = (V_0, V_1)$. Note that $f = (V_0, V_1)$ is a DF if and only if V_1 is a dominating set of G. The minimum weight of a DF is the well known lower domination number,
$$\gamma(G) = \min_{\text{DFs}} |V_1|.$$
See, for example, [5] for known results on this parameter.

2. Prompted by a strategy employed by Roman emperor Constantine, first Stewart [7] and then Cockayne, *et al.* [1], extended the notion of a DF to include so-called Roman domination. A *Roman dominating function* (RDF) is a safe guard function $f = (V_0, V_1, V_2)$ such that each $v \in V_0$ is adjacent to some $u \in V_2$. The minimum weight of an RDF is denoted
$$\gamma_R(G) = \min_{\text{RDFs}} \left(|V_1| + 2|V_2|\right),$$
which is called the *Roman domination number* of G.

3. As a result of using possibly too many guards in an RDF, Henning & Hedetniemi [6] suggested relaxing the definition somewhat to arrive at the notion of so-called weak Roman domination. A *weak Roman dominating function* (WRDF) is a safe guard function $f = (V_0, V_1, V_2)$ with the property that each $v \in V_0$ is adjacent to some $u \in V_1 \cup V_2$ such that
$$g(s) = \begin{cases} 1, & \text{if } s = v \\ f(u) - 1, & \text{if } s = u \\ f(s), & \text{if } s \in V \backslash \{u, v\} \end{cases}$$
is also a safe guard function. In this case we write $g = \text{move}(f, u \to v)$ to mean that g is the safe guard function obtained from the safe guard function f by moving a guard from vertex u to an adjacent vertex v, and leaving all other guards unchanged. The minimum weight of a WRDF is denoted
$$\gamma_r(G) = \min_{\text{WRDFs}} \left(|V_1| + 2|V_2|\right),$$
which is called the *weak Roman domination number* of G.

4. Finally, the definition of weak Roman domination was broadened yet further by Cockayne, *et al.* [3] to include the notion of secure domination. A *secure dominating function* (SDF) is a safe guard function $f = (V_0, V_1)$ with the property that each $v \in V_0$ is adjacent to some $u \in V_1$ such that

$$\begin{aligned} g(s) &= \text{move}(f, u \to v) \\ &= \begin{cases} 1, & \text{if } s = v \\ 0, & \text{if } s = u \\ f(s), & \text{if } s \in V \backslash \{u, v\} \end{cases} \end{aligned}$$

is also a safe guard function. The minimum weight of an SDF is denoted

$$\gamma_s(G) = \min_{\text{SDFs}} |V_1|,$$

which is called the *secure domination number* of G.

In [1, 3, 6] it was shown that, for any connected graph G,

$$\gamma(G) \leq \gamma_r(G) \leq \begin{cases} \gamma_R(G) \leq 2\gamma(G) \\ \gamma_s(G). \end{cases} \tag{1}$$

A number of interesting properties of these four parameters have been established in [1, 3, 6], and these parameters have also been determined for simple graph classes, such as complete graphs, paths, cycles and complete multipartite graphs. In [2, 3] bounds for these parameters are established for more complex graph structures, such as grid graphs, products of cycles, products of complete graphs and claw–free graphs. Finally, the general lower bound

$$\gamma_s(G) \geq \frac{n(2\Delta - 2t + 5)}{(\Delta + 1)^2 - (t-1)(t-2)} \tag{2}$$

was proved in [2] for any K_t–free graph G of order n and maximum degree Δ.

2 Foolproof generalisations

There is a fundamental difference between dominating functions and Roman dominating functions on the one hand, and weak Roman dominating functions and secure dominating functions on the other: the first two involve static configurations of guards on the vertices of G, while the second two involve moving a guard from one vertex to an adjecent vertex and are therefore dynamic. We shall only consider dynamic configurations of guards. The definitions of both weak roman domination and secure domination are "smart" in the following sense: they require that, for any unoccupied vertex v experiencing a problem, there is a guard at an adjacent vertex u such that moving the guard from u to v results in a safe guard function. It is the strategist's task to determine which move to make in order to resolve the problem at vertex v. It is therefore possible to define foolproof versions of these dynamic configuration cases (where a strategist is *not* required):

(5) A *foolproof weak Roman dominating function* (FWRDF) is a safe guard function $f = (V_0, V_1, V_2)$ such that, *for each* $u \in V_1 \cup V_2$ in the (open) neighbourhood of any $v \in V_0$, the function

$$\begin{aligned} g(s) &= \text{move}(f, u \to v) \\ &= \begin{cases} 1, & \text{if } s = v \\ f(u) - 1, & \text{if } s = u \\ f(s), & \text{if } s \in V \backslash \{u, v\} \end{cases} \end{aligned}$$

is also a safe guard function. The minimum weight of an FWRDF is denoted

$$\gamma_r^*(G) = \min_{\text{FWRDFs}} (|V_1| + 2|V_2|),$$

which is called the *foolproof weak Roman domination number* of G.

(6) A *foolproof secure dominating function* (FSDF) is a safe guard function $f = (V_0, V_1)$ such that, *for each* $u \in V_1$ in the (open) neighbourhood of any $v \in V_0$, the function

$$\begin{aligned} g(s) &= \text{move}(f, u \to v) \\ &= \begin{cases} 1, & \text{if } s = v \\ 0, & \text{if } s = u \\ f(s), & \text{if } s \in V \backslash \{u, v\} \end{cases} \end{aligned}$$

is also a safe guard function. The minimum weight of an FSDF is denoted

$$\gamma_s^*(G) = \min_{\text{FSDFs}} |V_1|,$$

which is called the *foolproof secure domination number* of G.

When referring to the previously studied versions of these definitions (as opposed to the new definition versions given above), we shall use the term *smart* secure [weak Roman] domination instead of *foolproof* secure [weak Roman] domination in order to distinguish between the two kinds of dynamic configurations. Note that, for any graph G,

$$\gamma_r(G) \leq \gamma_r^*(G) \quad \text{and} \quad \gamma_s(G) \leq \gamma_s^*(G), \tag{3}$$

since foolproof secure [weak Roman] domination requires a more robust configuration than does its smart counterpart. We also have the following (less trivial) result.

Proposition 1 *For any graph G, $\gamma_r^*(G) \le \gamma_R(G)$.*

Proof: Suppose $f_R = (V_0, V_1, V_2)$ is an RDF for the graph G. Then f_R is certainly a safe guard function for G, and any move of the form $g_r = \text{move}(f_R, u \to v)$ clearly results in a safe guard function g_r in the case where $u \in V_2$. Furthermore, if $u \in V_1$, then g_r is also a safe guard function, since in this case the sole possible reason for including u in V_1 in the first place, was that u should dominate itself. But after the move u is dominated by v. ■

The following generalisation of the bounds in (1) is therefore possible, by utilisation of (3) and Proposition 1.

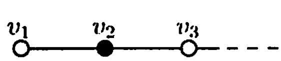
Corollary 1 *For any graph G,*

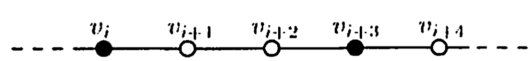
$$\gamma(G) \le \gamma_r(G) \le \begin{cases} \gamma_r^*(G) \le \gamma_R(G) \le 2\gamma(G) \\ \gamma_s(G) \le \gamma_s^*(G). \end{cases} \quad (4)$$

It is, of course, possible to establish values for the dynamic configuration parameters in the foolproof case for simple graph classes, as has been done for the smart cases in [1, 3, 6]. We demonstrate this for the case of foolproof secure domination of paths.

Theorem 1 *For any path P_n, $\gamma_s^*(P_n) = \lceil \frac{n}{2} \rceil$.*

Proof: It is first shown, by contradiction, that $\gamma_s^*(P_n) \ge \lceil \frac{n}{2} \rceil$. Suppose $f = (V_0, V_1)$ is an FSDF for the path $P : v_1 \cdots v_n$ with $|V_1| < \lceil \frac{n}{2} \rceil$. Then at least one of the following situations occur (without loss of generality, by choice of the vertex labelling):

(i) 

(ii) 

where dark vertices denote elements of V_1. Suppose (i) holds. Then $g = \text{move}(f, v_2 \to v_3)$ is not a safe guard function, since $v_1 \in V_0$ is not adjacent to any $u \in V_1$. Now suppose (ii) holds. Then $g = \text{move}(f, v_{i+3} \to v_{i+4})$ is not a safe guard function, since $v_{i+2} \in V_0$ is not adjacent to any $u \in V_1$. These contradictions show that

$$\gamma_s^*(P_n) \ge \left\lceil \frac{n}{2} \right\rceil. \quad (5)$$

To see that

$$\gamma_s^*(P_n) \leq \left\lceil \frac{n}{2} \right\rceil, \tag{6}$$

observe that $f_{\text{odd}} = (\tilde{V}_0, \tilde{V}_1)$, with $\tilde{V}_1 = \{v_i : i \equiv 1 \pmod 2\}$ and $\tilde{V}_0 = V(P_n) \backslash \tilde{V}_1$ is an FSDF for P_n if n is odd. Otherwise $f_{\text{even}} = (\bar{V}_0, \bar{V}_1)$ with $\bar{V}_1 = \{v_i : i \equiv 2, 3 \pmod 4\}$ and $\bar{V}_0 = V(P_n) \backslash \bar{V}_1$ is an FSDF for P_n if n is even. The desired result follows by a combination of (5) and (6). ■

Similar results to the one above for other graph classes or for the parameter γ_r^* are possible at this point, but we refrain from proving such results until we have established a more general setting for smart and foolproof domination.

3 Higher order generalisations

It is natural to generalise the notions of smart [foolproof] secure and weak Roman domination so that safe configurations are guaranteed after each of $k \geq 1$ moves to a sequence of problem vertices (henceforth informally called a *problem sequence*) instead of considering only one problem vertex in the graph at a time. The following four definitions achieve just such a generalisation.

(7) A *smart k-weak Roman dominating function* (k-SWRDF) is a safe guard function $f^{(0)} = (V_0^{(0)}, V_1^{(0)}, V_2^{(0)})$ with the property that, for any sequence of vertices $v_0, v_1, \ldots v_{k-1}$, there exists a sequence of vertices $u_i \in V_1^{(i)} \cup V_2^{(i)}$ in the neighbourhoods of v_i such that the functions $f^{(i+1)}(s) = \text{move}(f^{(i)}, u_i \rightarrow v_i)$ are also safe guard functions for all $i = 0, \ldots, k-1$. The minimum weight of a k-SWRDF is denoted

$$\gamma_{r,k}(G) = \min_{k\text{-SWRDFs}} \left(|V_1^{(0)}| + 2|V_2^{(0)}| \right),$$

which is called the *smart k-weak Roman domination number* of G.

(8) Similarly, a *foolproof k-weak Roman dominating function* (k-FWRDF) is a safe guard function $f^{(0)} = (V_0^{(0)}, V_1^{(0)}, V_2^{(0)})$ with the property that, for any sequence of vertices $v_0, v_1, \ldots, v_{k-1}$, the functions $f^{(i+1)}(s) = \text{move}(f^{(i)}, u_i \rightarrow v_i)$ are also safe guard functions for any sequence of vertices $u_i \in V_1^{(i)} \cup V_2^{(i)}$ in the neighbourhoods of v_i and all $i = 0, \ldots, k-1$. The minimum weight of a k-FWRDF is denoted

$$\gamma_{r,k}^*(G) = \min_{k\text{-FWRDFs}} \left(|V_1^{(0)}| + 2|V_2^{(0)}| \right),$$

which is called the *foolproof k-weak Roman domination number* of G.

(9) A *smart k-secure dominating function* (k-SSDF) is a safe guard function $f^{(0)} = (V_0^{(0)}, V_1^{(0)})$ with the property that, for any sequence of vertices $v_0, v_1, \ldots, v_{k-1}$, there exists a sequence of vertices $u_i \in V_1^{(i)}$ such that the functions $f^{(i+1)}(s) = \text{move}(f^{(i)}, u_i \to v_i)$ are also safe guard functions for all $i = 0, \ldots, k-1$. The minimum weight of a k-SSDF is denoted

$$\gamma_{s,k}(G) = \min_{k\text{-SSDFs}} |V_1^{(0)}|,$$

which is called the *smart k-secure domination number* of G.

(10) Similarly, a *foolproof k-secure dominating function* (k-FSDF) is a safe guard function $f^{(0)} = (V_0^{(0)}, V_1^{(0)})$ with the property that, for any sequence of vertices v_i ($i = 0, \ldots, k-1$), the functions $f^{(i+1)}(s) = \text{move}(f^{(i)}, u_i \to v_i)$ are also safe guard functions for any sequence of vertices $u_i \in V_1^{(i)}$ in the neighbourhoods of v_i and all $i = 0, \ldots, k-1$. The minimum weight of a k-SSDF is denoted

$$\gamma^*_{s,k}(G) = \min_{k\text{-FSDFs}} |V_1^{(0)}|,$$

which is called the *foolproof k-secure domination number* of G.

We therefore have the special cases $\gamma_r = \gamma_{r,1}$, $\gamma^*_r = \gamma^*_{r,1}$, $\gamma_s = \gamma_{s,1}$ and $\gamma^*_s = \gamma^*_{s,1}$. The case $k = 0$ corresponds to the situation where no moves are allowed. So, for convenience, let $\gamma_{r,0} = \gamma^*_{r,0} = \gamma_{s,0} = \gamma^*_{s,0} = \gamma$. Furthermore, the following relationships between the smart and foolproof versions of the newly defined parameters trivially hold, as a generalisation of (3).

Proposition 2 *For any graph G and any $k \in \mathbb{N}$, $\gamma_{r,k}(G) \leq \gamma^*_{r,k}(G)$ and $\gamma_{s,k}(G) \leq \gamma^*_{s,k}(G)$.*

The following growth relationships of the parameters with respect to increasing values of k hold.

Proposition 3 *For any graph G and any $k \in \mathbb{N}_0$,*

(a) $\gamma_{r,k}(G) \leq \gamma_{r,k+1}(G)$,

(b) $\gamma^*_{r,k}(G) \leq \gamma^*_{r,k+1}(G)$,

(c) $\gamma_{s,k}(G) \leq \gamma_{s,k+1}(G)$,

(d) $\gamma^*_{s,k}(G) \leq \gamma^*_{s,k+1}(G)$.

Proof: (a) Any $(k+1)$–WRDF with minimum weight $\gamma_{r,k+1}(G)$ for G is also a k–WRDF for G, and the weight of this last dominating function is bounded from below by $\gamma_{r,k}(G)$. The proofs of parts (b)–(d) are similar. ■

It is easy to see that the following result is true.

Lemma 1 *For any graph G and any edge $e \in E(G)$, $\gamma_{r,k}(G) \leq \gamma_{r,k}(G-e)$ and $\gamma_{s,k}(G) \leq \gamma_{s,k}(G-e)$, for all $k \in \mathbb{N}_0$.*

This lemma may be used repeatedly to prove the following two results.

Proposition 4 *If the vertex set of a graph G is partitioned into two subsets S_1 and S_2, then, for all $k \in \mathbb{N}_0$,*

(a) $\gamma_{r,k}(G) \leq \gamma_{r,k}(\langle S_1\rangle) + \gamma_{r,k}(\langle S_2\rangle)$,

(b) $\gamma_{s,k}(G) \leq \gamma_{s,k}(\langle S_1\rangle) + \gamma_{s,k}(\langle S_2\rangle)$.

Proposition 5 *If H is a spanning subgraph of G, then $\gamma_{r,k}(G) \leq \gamma_{r,k}(H)$ and $\gamma_{s,k}(G) \leq \gamma_{s,k}(H)$ for all $k \in \mathbb{N}_0$.*

Note that, in general, it is not possible to establish results similar to those of Lemma 1 and Propositions 4 and 5 for the parameters $\gamma^*_{r,k}(G)$ and $\gamma^*_{s,k}(G)$. For example, removing an edge may increase or decrease the value of $\gamma^*_{s,1}$. This may be seen by observing that $\gamma^*_{s,1}(P_4) = 2$, while $\gamma^*_{s,1}(P_4 - e) = \gamma^*_{s,1}(P_1) + \gamma^*_{s,1}(P_3) = 3$. On the other hand $\gamma^*_{s,2}(P_4) = 3$, yet $\gamma^*_{s,2}(P_4 - e) = \gamma^*_{s,2}(P_2) + \gamma^*_{s,2}(P_2) = 2$.

It is easily seen that the following result is true.

Proposition 6 *If G is an order n graph such that, for some subset of vertices $S = \{v_1, \ldots, v_m\} \subseteq V(G)$, the graph $G - S$ possesses a perfect matching, then, for all $k \in \mathbb{N}_0$,*

$$\gamma_{s,k}(G) \leq \frac{n-m}{2} + m = \frac{m+n}{2}.$$

Finally, we conclude this section with a summary of relationships between the various parameters considered in this section, as a generalisation of (4).

Theorem 2 *The relationships*

$$\left.\begin{array}{ccccc} \gamma(G) & \leq & \gamma_{r,k}(G) & \leq & \gamma_{s,k}(G) \\ & & \text{IA} & & \text{IA} \\ \gamma(G) & \leq & \gamma^*_{r,k}(G) & \leq & \gamma^*_{s,k}(G) \end{array}\right\} \tag{7}$$

hold for all $k \in \mathbb{N}$.

4 Parameters for special graphs

In this section we consider a number of simple graph classes and find values for the four new finite order domination parameters considered in this paper.

4.1 Paths

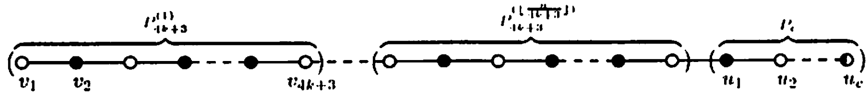
For paths we establish three of the four finite order parameter values precicely, as summarised in the following theorem.

Theorem 3 *For any path P_n, $n \geq 2$,*

(a) $\gamma_{r,k}(P_n) = \gamma_{s,k}(P_n) = \left\lceil \dfrac{2k+1}{4k+3} n \right\rceil$, for all $k \in \mathbb{N}_0$,

(b) $\gamma^*_{s,k}(P_n) = \begin{cases} \left\lceil \frac{k+1}{k+3} n \right\rceil & \text{if } 2 \leq k \leq n-2 \\ n-1 & \text{if } k \geq n-1. \end{cases}$

Proof: (a) It is first shown that $\gamma_{s,k}(P_n) \leq \lceil \frac{2k+1}{4k+3} n \rceil$ for the path P_n : $v_1 \cdots v_n$. Partition the path P_n into $\lfloor \frac{n}{4k+3} \rfloor$ subpaths $P^{(\ell)}_{4k+3}$ of length $4k+3$ ($\ell = 1, \ldots, \lfloor \frac{n}{4k+3} \rfloor$) and one (possibly empty) subpath P_c : $u_1 u_2 \cdots u_c$ of length $c \equiv n \pmod{4k+3}$, and consider the function $f = (V_0^{(0)}, V_1^{(0)})$, where $V_1^{(0)} = \{v_i : i \pmod{4k+3} \equiv 0 \pmod 2,\ v_i \in V(P^{\ell}_{4k+3})\} \cup \{u_j \in V(P_c) : j \equiv 1 \pmod 2\}$ and $V_0^{(0)} = V(P_n) \backslash V_1^{(0)}$. We shall show that f is a k-SSDF for P_n, by showing that for any sequence of k problem vertices there exist $m \leq k$ moves that renders safe guard functions $f^{(i)}$ for P_n ($i = 0, \ldots, k$).



We only have to consider the case where the whole sequence of problem vertices occurs in one subpath $P^{(\ell)}_{4k+3}$, because if there exists a move sequence that renders safe guard functions $f^{(i)}$ for $P^{(\ell)}_{4k+3}$ ($i = 0, \ldots, k$), given any sequence of k problem vertices within the subpath, then there also exists a move sequence that renders safe configurations for fewer problem vertices within the subpath. Since either P_c or $P_c - v$ possesses a perfect matching, it follows by Proposition 6 that $\gamma_{s,k}(P_c) = \lceil c/2 \rceil$ and hence there exists a move sequence wholly within P_c that renders safe configurations for P_c, given any sequence of $k \geq 1$ problem vertices in P_c. Therefore consider, without loss of generality, a sequence of k problem vertices $v_{i_j} \in V(P^{(1)}_{4k+3})$, $j = 1, \ldots, k$. We consider two main cases:

<u>**Case A:**</u> $v_{i_j} \notin \{v_{4k+1}, v_{4k+2}, v_{4k+3}\}$ for all $j = 1, \ldots, k$. In this case

$$\mathcal{M}_1 = \bigcup_{i=1}^{2k} \langle v_{2i-1}, v_{2i} \rangle$$

is a perfect matching of the subpath $\langle v_1, \ldots, v_{4k} \rangle$. Therefore

$$\gamma_{s,k}(P_{4k+3}^{(1)}) \le 2k + 1,$$

by utilisation of Propositions 6 and 4(b) and Theorem 2.

<u>**Case B:**</u> $v_{i_j} \in \{v_{4k+1}, v_{4k+2}, v_{4k+3}\}$ for *some* $j \in \{1, \ldots, k\}$. In this case we distinguish between two further subcases:

<u>**Subcase B(i):**</u> $v_{i_j} \notin \{v_1, v_2, v_3\}$ for all $j = 1, \ldots, k$. In this subcase

$$\mathcal{M}_2 = \bigcup_{i=2}^{2k+1} \langle v_{2i}, v_{2i+1} \rangle$$

is a perfect matching of the subpath $\langle v_4, \ldots, v_{4k+3} \rangle$. Hence we have, by a similar argument as in Case A, that $\gamma_{s,k}(P_{4k+3}^{(1)}) \le 2k + 1$.

<u>**Subcase B(ii):**</u> $v_{i_j} \in \{v_1, v_2, v_3\}$ for *some* $j \in \{1, \ldots, k\}$. In this subcase there are at most $k - 2$ problem vertices in the subpath $\langle v_4, \ldots, v_{4k} \rangle$. But then it follows, by the pigeonhole principle, that there are at least 4 consecutively labelled vertices that are not problem vertices: suppose they are $v_{2\ell}, v_{2\ell+1}, v_{2\ell+2}, v_{2\ell+3}$ (the case where the first of these labels is odd, is similar). Then

$$\mathcal{M}_3 = \bigcup_{i=1}^{\ell} \langle v_{2i-1}, v_{2i} \rangle \quad \text{and} \quad \mathcal{M}_4 = \bigcup_{i=\ell+2}^{2k+1} \langle v_{2i}, v_{2i+1} \rangle$$

are perfect matchings of the subpaths $\hat{P} := \langle v_1, \ldots, v_{2\ell} \rangle$ and $\tilde{P} := \langle v_{2\ell+4}, \ldots, v_{4k+3} \rangle$ respectively, and we have, again by Theorem 2 and Propostions 6 and 4(b), that $\gamma_{s,k}(\hat{P} \cup \tilde{P}) \le 2k$. Hence $\gamma_{s,k}(P_{4k+3}^{(1)}) \le 2k + 1$, because $\gamma(\langle v_{2\ell+1}, v_{2\ell+2}, v_{2\ell+3} \rangle) = 1$.

Consequently we have, in all cases, that

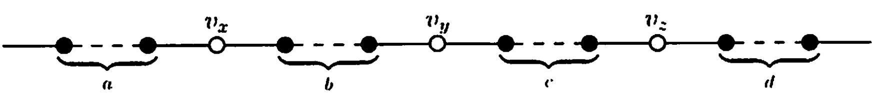
$$(\gamma_{r,k}(P_n) \le)\ \gamma_{s,k}(P_n) \le \left\lfloor \frac{n}{4k+3} \right\rfloor (2k+1) + \left\lceil \frac{c}{2} \right\rceil \le \left\lceil \frac{2k+1}{4k+3} n \right\rceil. \quad (8)$$

(The last inequality can be proved by first showing that $\lceil \frac{c}{2} \rceil = \lceil \frac{2k+1}{4k+3} c \rceil$ if $c < 4k + 3$.) To prove that

$$(\gamma_{s,k}(P_n) \ge)\ \gamma_{r,k}(P_n) \ge \left\lceil \frac{2k+1}{4k+3} n \right\rceil, \quad (9)$$

suppose, to the contrary, that $\gamma_{r,k}(P_n) \le \lceil \frac{(2k+1)n}{4k+3} \rceil - 1 = \lfloor \frac{n}{4k+3} \rfloor (2k+1) + \lceil \frac{c}{2} \rceil - 1$. Then there will be a subpath $P^{(i)}_{4k+3}$, $i \in \{1, \dots, \lfloor \frac{n}{4k+3} \rfloor\}$, containing at most $2k$ vertices from $V_1^{(0)} \cup V_2^{(0)}$, or else P_c will contain at most $\lceil \frac{c}{2} \rceil - 1$ vertices from $V_1^{(0)} \cup V_2^{(0)}$. We consider the former possibility first. Suppose, without loss of generality, that $P^{(1)}_{4k+3}$ contains at most $2k$ vertices from $V_1^{(0)} \cup V_2^{(0)}$. Consider the set of problem vertices $I = \{v_{4\ell-1} : \ell = 1, \dots, k\}$. Because this is an independent set, $I \subseteq V_1^{(k)} \cup V_2^{(k)}$. Furthermore, because $f^{(k)}$ must be a safe guard function, $J = \{v_{4\ell+1} : l = 0, \dots, k\}$ must be dominated by vertices in $V_1^{(k)} \cup V_2^{(k)}$. But no vertex in I is adjacent to vertices in J. Therefore $|(V_1^{(k)} \cup V_2^{(0)}) \cap V(P^{(1)}_{4k+3})| \ge 2k+1$, which is a contradiction. Finally, if P_c contains only $\lceil \frac{c}{2} \rceil - 1$ vertices from $V_1^{(0)} \cup V_2^{(0)}$, then we get a similar contradiction by considering the problem vertex sequence $v_{4\ell-1}$, $\ell = 1, \dots, \lceil \frac{c-2}{4} \rceil$, $c \ge 3$. The desired result for $\gamma_{r,k}(P_n)$ and $\gamma_{s,k}(P_n)$ therefore follows by a combination of (8) and (9).

(b) Consider the case $k \le n-2$. It is shown first, by contradiction, that every subpath $P^{(i)}_{k+3} : v_{i \pmod n} \cdots v_{i+k+2 \pmod n}$ of length $k+3$ within $P_n : v_0 v_1 \cdots v_{n-1}$ contains at least $k+1$ vertices from $V_1^{(0)}$ for any k-FSDF $f^{(0)} = (V_0^{(0)}, V_1^{(0)})$. Suppose, to the contrary, that there exists such a subpath $P^{(i)}_{k+3}$ of P_n containing only k vertices from $V_1^{(0)}$ (and hence 3 vertices from $V_0^{(0)}$). There is exactly one possible case:



Here dark vertices denote elements of $V_1^{(0)}$ and $a, b, c, d \ge 0$, with $a+b+c+d = k$. The sequences of moves

$$f^{(j+1)} = \text{move}(f^{(j)}, v_{z-1-j} \to v_{z-j}), \quad j = 0, \dots, c-1$$

and

$$f^{(c+\ell+1)} = \text{move}(f^{(c+\ell)}, v_{x+\ell+1} \to v_{x+\ell}), \quad \ell = 0, \dots, b-1$$

render unsafe configurations in P_n after $b+c$ moves, because $v_y \in V_0^{(b)}$ is not adjacent to any $u \in V_1^{(b)}$.

This contradiction shows that $|V(P^{(i)}_{k+3}) \cap V_1^{(0)}| \ge k+1$ for all $i = 0, 1, \dots, n-1$. In order to fulfil this property, it follows that $|V_1^{(0)}| \ge (k+1)\lfloor \frac{n}{k+3} \rfloor + r$, where $r \ge \frac{k+1}{k+3}c$, rendering the lower bound

$$\gamma^*_{s,k}(P_n) \ge (k+1) \left\lfloor \frac{n}{k+3} \right\rfloor + \left\lceil \frac{k+1}{k+3} c \right\rceil, \quad \text{if } k \le n-2, \qquad (10)$$

with $c \equiv n \pmod{k+3}$. To see that this bound is sharp, partition the path P_n into $\lfloor \frac{n}{k+3} \rfloor$ subpaths $P^{(j)}_{k+3} : v_{j(k+3)}, v_{j(k+3)+1}, \ldots, v_{j(k+3)+k+2}$ ($j = 0, \ldots, \lfloor \frac{n}{k+3} \rfloor - 1$) and one subpath $P_c : v_{\lfloor n/(k+3) \rfloor (k+3)}, \ldots, v_{n-1}$ of length $c \equiv n \pmod{k+3}$. Consider the safe guard function $f^{(0)} = (V_0^{(1)}, V_1^{(0)})$, where

$$P^{(j)}_{k+3} \cap V_1^{(0)} = \{v_i : i \equiv 1, 2, \ldots, k, k+1 \pmod{k+3}\}, \ j = 0, \ldots, \left\lfloor \frac{n}{k+3} \right\rfloor - 1,$$

where

$$P_c \cap V_1^{(0)} = \begin{cases} \{v_i : i \equiv 0, 1, \ldots, c-1 \pmod{k+3}\} & \text{if } 1 \leq c \leq \lfloor \frac{k+2}{2} \rfloor \\ \{v_i : i \equiv 1, 2, \ldots, c-1 \pmod{k+3}\} & \text{if } \lfloor \frac{k+2}{2} \rfloor < c \leq k+2 \end{cases}$$

and where $V_0^{(0)} = V(P_n) \backslash V_1^{(0)}$. Clearly $f^{(0)}$ is a k–FSDF for P_n, and hence

$$\begin{aligned} \gamma^*_{s,k}(P_n) &\leq w(f^{(0)}) & (11) \\ &= (k+1) \left\lfloor \frac{n}{k+3} \right\rfloor + (c+1) - \left\lceil \frac{2c+1}{k+3} \right\rceil \\ &= (k+1) \left\lfloor \frac{n}{k+3} \right\rfloor + \left\lceil \frac{(c+1)(k+3)}{k+3} - \frac{2c+1}{k+3} - \frac{k+2}{k+3} \right\rceil \\ &= (k+1) \left\lfloor \frac{n}{k+3} \right\rfloor + \left\lceil \frac{k+1}{k+3} c \right\rceil \quad \text{if } k \leq n-2. & (12) \end{aligned}$$

It follows by (10) and (12) that

$$\gamma^*_{s,k}(P_n) = (k+1) \left\lfloor \frac{n}{k+3} \right\rfloor + \left\lceil \frac{k+1}{k+3} c \right\rceil = \left\lceil \frac{k+1}{k+3} n \right\rceil, \quad \text{if } k \leq n-2,$$

where the last equality can be proved by rewriting c in terms of n and k.

Finally note that

$$\left\lceil \frac{k+1}{k+3} n \right\rceil = n - 1 \quad \text{if } k = n-2.$$

It follows by Proposition 3 that

$$\gamma^*_{s,k}(P_n) \geq \gamma^*_{s,n-2}(P_n) = n - 1 \tag{13}$$

for any $k \geq n-2$. But certainly

$$\gamma^*_{s,k}(P_n) \leq n - 1 \tag{14}$$

for all $k \in \mathbb{N}$. A combination of (13) and (14) yields the desired result $\gamma^*_{s,k}(P_n) = n - 1$ for all $k \geq n-2$. ■

Note that the corresponding case $k = 1$ in Theorem 3(b) was established in Theorem 1.

Finally, we have the following conjecture.

Conjecture 1 $\gamma^*_{r,k}(P_n) = \gamma^*_{s,k}(P_n)$ *for any* $n \in \mathbb{N}$.

4.2 Cycles

For cycles it is also possible to establish three of the finite order parameter values exactly.

Theorem 4 *For any cycle* C_n,

(a) $\gamma_{r,k}(C_n) = \gamma_{s,k}(C_n) = \left\lceil \frac{2k+1}{4k+3} n \right\rceil$, for all $k \in \mathbb{N}_0$,

(b) $\gamma^*_{s,k}(C_n) = \begin{cases} \left\lceil \frac{k+1}{k+3} n \right\rceil, & \text{if } 0 \le k \le n-3 \\ n-2, & \text{if } k \ge n-3. \end{cases}$

Proof: (a) The proof of this result is identical to that of Theorem 3(a).
(b) It can be shown, by exactly the same contradiction argument as in Theorem 3(b), that

$$\gamma^*_{r,k}(C_n) \ge (k+1)\left\lfloor \frac{n}{k+3} \right\rfloor + \left\lceil \frac{k+1}{k+3} c \right\rceil, \quad \text{if } k \le n-3, \qquad (15)$$

with $c \equiv n \pmod{k+3}$. To see that this bound is sharp, partition the cycle C_n into $\lfloor \frac{n}{k+3} \rfloor$ subpaths $P^{(j)}_{k+3} : v_{j(k+3)}, v_{j(k+3)+1}, \ldots, v_{j(k+3)+k+2}$ ($j = 0, \ldots, \lfloor \frac{n}{k+3} \rfloor - 1$) and one subpath $P_c : v_{\lfloor n/(k+3) \rfloor (k+3)}, \ldots, v_{n-1}$ of length $c \equiv n \pmod{k+3}$. Consider the safe guard function $f^{(0)} = (V_0^{(1)}, V_1^{(0)})$, where

$$V(C_n) \cap V_0^{(0)} = \{v_i : i \equiv \lceil k/2 \rceil + 2, k+1 \pmod{k+3}\}, \; j = 0, \ldots, \left\lfloor \frac{n}{k+3} \right\rfloor - 1,$$

and where $V_1^{(0)} = V(C_n) \backslash V_0^{(0)}$. Clearly $f^{(0)}$ is a k-FSDF for C_n, and hence

$$\begin{aligned} \gamma^*_{s,k}(C_n) &\le w(f^{(0)}) & (16) \\ &\le \begin{cases} (k+1)\left\lfloor \frac{n}{k+3} \right\rfloor + c, & \text{if } 0 \le c \le \lceil \frac{k}{2} + 1 \rceil \\ (k+1)\left\lfloor \frac{n}{k+3} \right\rfloor + c - 1, & \text{if } \lceil \frac{k}{2} + 2 \rceil < c \le k+2 \end{cases} \\ &= \left\lceil \frac{k+1}{k+3} n \right\rceil & (17) \end{aligned}$$

for all $k \leq n-3$, exactly as in the proof of Theorem 3(b). The desired result for $k \leq n-3$ therefore follows by a combination of (15) and (17).

Finally note that

$$\left\lceil \frac{k+1}{k+3}n \right\rceil = n-2 \quad \text{if} \quad k = n-3.$$

It follows by Proposition 3 that

$$\gamma^*_{s,k}(C_n) \geq \gamma^*_{s,n-3}(C_n) = n-2 \tag{18}$$

for any $k \geq n-3$. But certainly

$$\gamma^*_{s,k}(C_n) \leq n-2 \tag{19}$$

for all $k \in \mathbb{N}$. A combination of (18) and (19) yields the desired result $\gamma^*_{s,k}(C_n) = n-2$ for all $k \geq n-3$. ■

4.3 Complete bipartite graphs

In this section we consider complete bipartite graphs and find values for $\gamma_{s,k}$ and $\gamma^*_{s,k}$ for this simple graph class.

Theorem 5 *For the complete bipartite graph $K_{p,q}$,*

$$\gamma_{s,k}(K_{p,q}) = \gamma^*_{s,k}(K_{p,q}) = \begin{cases} 4, & k = 1 \text{ and } p \geq 4 \\ 2(k+1), & 1 < k \leq \lfloor \frac{p-2}{2} \rfloor \\ p, & \lfloor \frac{p-2}{2} \rfloor + 1 \leq k < p \\ q, & k \geq p \end{cases}$$

where $p, q \in \mathbb{N}$, with $p \leq q$.

Proof: It was shown in [3] that $\gamma^*_{s,1}(K_{p,q}) = 4$ if $p \geq 4$. In order to establish the other three cases, let $\mathcal{P}$ and $\mathcal{Q}$ denote the partite sets of $K_{p,q}$, with $|\mathcal{P}| = p$ and $|\mathcal{Q}| = q$. First, consider the case $1 < k \leq \lfloor \frac{p-2}{2} \rfloor$. Note that $f^{(0)} = (V_0^{(0)}, V_1^{(0)})$ is a k-FSDF if $|V_1^{(0)} \cap \mathcal{P}| = k+1$, $|V_1^{(0)} \cap \mathcal{Q}| = k+1$ and $V_0^{(0)} = V(K_{p,q}) \backslash V_1^{(0)}$ (as shown in Figure 4.1(a)). Hence

$$\gamma_{s,k}(K_{p,q}) \leq \gamma^*_{s,k}(K_{p,q}) \leq 2(k+1) \quad \text{if} \quad 1 < k \leq \left\lfloor \frac{p-2}{2} \right\rfloor \tag{20}$$

by Theorem 2. Now assume that $\gamma_{s,k}(K_{p,q}) < 2(k+1)$, then either $|V_1^{(0)} \cap \mathcal{P}| \leq k$ or $|V_1^{(0)} \cap \mathcal{Q}| \leq k$. Assume, without loss of generality, that $|V_1^{(0)} \cap \mathcal{Q}| \leq k$. Then no move sequence of the form $f^{(i+1)} = \text{move}(f^{(i)}, u_i \to v_i)$

renders a safe guard function $f^{(k)}$ for any sequence of vertices $u_i \in \mathcal{P}$ if $v_i \in \mathcal{Q}$ $(i = 0, \ldots, k-1)$, since

$$2\left\lfloor \frac{p-2}{2} \right\rfloor + 1 = \left\{ \begin{array}{ll} p-1 & \text{if } p \text{ is even} \\ p-2 & \text{if } p \text{ is odd} \end{array} \right\} < p.$$

This contradiction shows that

$$\gamma^*_{s,k}(K_{p,q}) \geq \gamma_{s,k}(K_{p,q}) \geq 2(k+1) \text{ if } 1 < k \leq \left\lfloor \frac{p-2}{2} \right\rfloor. \tag{21}$$

The second case of the theorem therefore follows by a combination of (20) and (21).

Now, consider the case $\lfloor \frac{p-2}{2} \rfloor + 1 \leq k < p$ and let $f^{(0)} = (V_0^{(0)}, V_1^{(0)})$ be a k-FSDF with $|V_1^{(0)}| = \ell < p-1$. Suppose $|\mathcal{Q} \cap V_1^{(0)}| = c$ and $|\mathcal{P} \cap V_1^{(0)}| = \ell - c$ for some $0 \leq c \leq \ell$. Denote the elements of the non-empty set $\mathcal{P} \cap V_0^{(0)}$ by $\{v_0, \ldots, v_{p-\ell+c-1}\}$, and consider the problem vertex sequence v_i $(i = 0, \ldots, c-1)$. Clearly any move sequence of the form move$(f^{(i)}, u_i \rightarrow v_i)$ with $u_i \in \mathcal{Q} \cap V_1^{(i)}$ $(i = 0, \ldots, c-1)$ will render an unsafe guard function $f^{(c)}$ in $K_{p,q}$. This contradiction shows that

$$\gamma^*_{s,k}(K_{p,q}) \geq \gamma_{s,k}(K_{p,q}) \geq p \text{ if } \left\lfloor \frac{p-2}{2} \right\rfloor + 1 \leq k < p. \tag{22}$$

To see that this bound is sharp, consider the k–FSDF $f^{(0)} = (V_0^{(0)}, V_1^{(0)})$ depicted in Figure 4.1(b), where dark vertices denote elements of $V_1^{(0)}$, by which it follows that

$$\gamma_{s,k}(K_{p,q}) \leq \gamma^*_{s,k}(K_{p,q}) \leq p \text{ if } \left\lfloor \frac{p-2}{2} \right\rfloor + 1 \leq k < p. \tag{23}$$

The third case of the theorem therefore follows by a combination of (22) and (23).

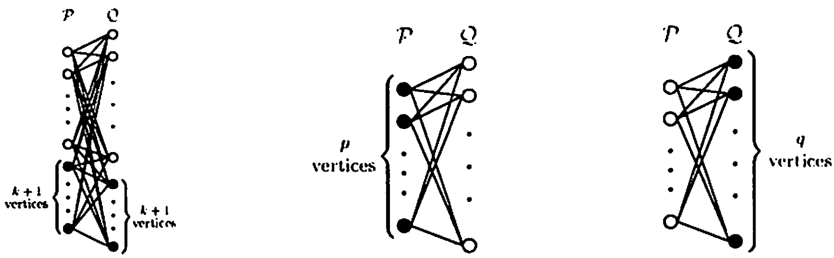
Finally, consider the case where $k \geq p$ and suppose $f^{(0)} = (V_0^{(0)}, V_1^{(0)})$ is a p–FSDF with $|V_1^{(0)}| = \ell < q$. Suppose $|\mathcal{P} \cap V_1^{(0)}| = d$ and $|\mathcal{Q} \cap V_1^{(0)}| = \ell - d$ for some $0 \leq d \leq \ell$. Denote the elements of the non-empty set $\mathcal{Q} \cap V_0^{(0)}$ by $\{v_0, \ldots, v_{q-\ell+d-1}\}$, and consider the problem vertex sequence v_i $(i = 0, \ldots, d-1)$. Clearly any move sequence of the form move$(f^{(i)}, u_i \rightarrow v_i)$ with $u_i \in \mathcal{Q} \cap V_1^{(0)}$ $(i = 0, \ldots, d-1)$ will render an unsafe guard function $f^{(d)}$ in $K_{p,q}$. This contradiction shows that $\gamma_{s,p}(K_{p,q}) \geq q$. Hence

$$\gamma_{s,k}(K_{p,q}) \geq q \text{ if } k \geq p. \tag{24}$$

But certainly

$$\gamma_{s,k}(K_{p,q}) \leq q. \tag{25}$$

Hence the fourth case of the theorem follows by a combination of (24) and (25). ■



(a) $1 < k \leq \lfloor \frac{p-2}{2} \rfloor$ (b) $\lfloor \frac{p-2}{2} \rfloor + 1 \leq k < p$ (c) $k \geq p$

Figure 4.1: Foolproof k-secure dominating functions $f^{(0)} = (V_0^{(0)}, V_1^{(0)})$ for the complete bipartite graph $K_{p,q}$ $(p \leq q)$. Dark vertices denote elements of $V_1^{(0)}$.

5 Conclusion

In this paper the previously studied notions of secure domination and of weak Roman domination were generalised in the sense that safe configurations in a simple graph were not merely sought after *one* move, but rather after $k \geq 1$ moves. Some general properties of these generalised domination numbers were established in §3, after which the parameter values were found for certain simple graph structures in §5. There is ample scope for the determination of parameter values for specific graph structures, such as the values of $\gamma^*_{r,k}(P_n)$ and $\gamma^*_{r,k}(C_n)$, the various parameter values for complete multipartite graphs, etc.

Further work may involve a number of interesting generalisations: (i) In our work the problem vertex sequence was always known completely in advance by the strategist. However, the situation where these problem vertices are made known (and are dealt with) one at a time might be a more realistic scenario in terms of games of strategy, and this generalisation deserves to be investigated. (ii) It might also be worth while allowing for a number of consecutive moves *before* requiring the graph to be protected, instead of requiring safe configurations after *each* move (similar to the *watchman's walk problem* described in [4]).

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References

[1] E.J. Cockayne, P.A. Dreyer, S.M. Hedetniemi and S.T. Hedetniemi, *Roman domination in graphs*, to appear in *Discrete Math.*

[2] E.J. Cockayne, O. Favaron and C.M. Mynhardt, *Secure domination, weak Roman domination and forbidden subgraphs*, to appear in *Bull. Inst. Combin. Appl.*

[3] E.J. Cockayne, P.J.P. Grobler, W.R. Gründlingh, J. Munganga and J.H. van Vuuren, *Protection of a graph*, to appear in *Utilitas Math.*

[4] B.L. Hartnell, D.F. Rall and C.A. Whitehead, *The watchman's walk problem: An introduction*, Cong. Numer., **130** (1998), 149–155.

[5] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.

[6] M.A. Henning and S.M. Hedetniemi, *Defending the Roman Empire – A new strategy*, to appear in *Discrete Math.*

[7] I. Stewart, *Defend the Roman Empire!*, Scientific American, December 1999, 136–138.