Extended 7-cycle Systems

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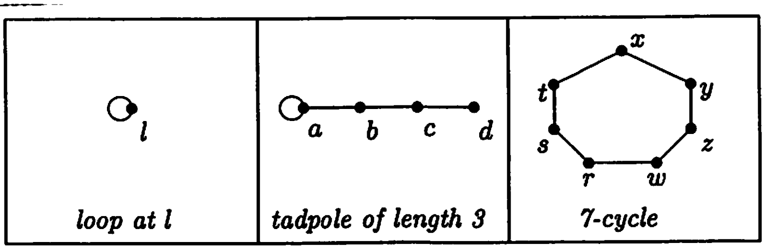
Abstract

An extended 7-cycle system of order n is an ordered pair (V, B), where B is a collection of edge-disjoint 7-cycles, 3-tadpoles and loops which partition the edges of the graph K_n^+ whose vertex set is n-set V. In this paper, we show that an extended 7-cycle system of order n exists for all n except $n = 2, 3$ and 5.

1 Introduction

Let G be a graph and G^+ be a graph which is obtained by attaching a loop to each vertex of G. For terms and notations not defined here, the reader is referred to the book by G. Chartrand and L. Lesniak [1]. As usual, K_n and $K_{m,n}$ denote the complete graph and the complete bipartite graph, respectively. $K_{m(n)}$ [1] denotes the complete partite graph with m parts of size n. Let $\overline{G}$ be the complement of G. If G and H are two graphs then $G \cup H$ will be the graph with vertex set $V(G \cup H) = V(G) \cup V(H)$ and edge set $E(G \cup H) = E(G) \cup E(H)$. If $V(G) \cap V(H) = \emptyset$, then let $G + H$ be the graph with $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H) \cup \{\{g, h\} \mid g \in V(G),\ h \in V(H)\}$. And the graphs $G \cup H$ and $G + H$ will be called the union and join of G and H, respectively. In the following, we depict the subgraphs of K_n^+ which will be used in the decomposition.



loop at l | *tadpole of length 3* | *7-cycle*

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For convenience, we will denote the loop as (l), the tadpole of length 3 (or say 3-tadpole) as $\langle a,b,c,d\rangle$ and the 7-cycle as (x,y,z,w,r,s,t).

A 7-cycle system of a graph G is an ordered pair $(V(G),C)$, where C is a collection of edge-disjoint 7-cycles which partition the edges of G. A 7-cycle system of order n is a 7-cycle system of K_n. In 1989, Hoffman, Lindner and Rodger [2] showed that a 7-cycle system of order n exists precisely when $n \equiv 1,7(mod\ 14)$. Subsequently, 7-cycle systems and their properties have been examined by Manduchi [4]. In 1997, Lindner and Rodger [3] generalized the idea of the cycle system by working on decompositions of the graph K_n^+.

An extended 7-cycle system of a graph G is an ordered pair $(V(G),B)$, where B is a collection of edge-disjoint 7-cycles, 3-tadpoles and loops whose edges partition the edges of G. An extended 7-cycle system of order n is an extended 7-cycle system of the graph K_n^+. We denote by $E7CS(n)$ the class of all extended 7-cycle systems of order n and if (V,B) is an extended 7-cycle system of order n, we write $B \in E7CS(n)$ or $(V,B) \in E7CS(n)$. We say that $E7CS(n)$ exists if there is a system $(V,B) \in E7CS(n)$. Similarly, we can denote the classes of all 7-cycle systems of G, 7-cycle systems of order n and extended 7-cycle systems of G by $7CS(G)$, $7CS(n)$ and $E7CS(G)$, respectively.

2 Small Case

In this section, some extended 7-cycle systems for small order are presented.

Example 2.1 *$B_4 \in E7CS(4)$, where* $B_4 = \{(3),(4),\langle 1,2,4,3\rangle,\langle 2,3,1,4\rangle\}$.

$B_6 \in E7CS(6)$, *where* $B_6 = \{(6), \langle 1, 2, 4, 6\rangle, \langle 2, 3, 5, 1\rangle, \langle 3, 4, 5, 6\rangle, \langle 4, 1, 3, 6\rangle, \langle 5, 2, 6, 1\rangle\}$.

$B_7 = B_7(1,2,\cdots,7) \in E7CS(7)$, *where* $B_7 = \{(i) \mid i = 1,2,\cdots,7\} \cup \{(1, 2, 3, 4, 5, 6, 7), (1, 3, 5, 7, 2, 4, 6), (1, 4, 7, 3, 6, 2, 5)\}$.

$B_8 = B_8(1,2,\cdots,8) \in E7CS(8)$, *where* $B_8 = \{(8), \langle 1, 3, 5, 7\rangle, \langle 2, 4, 6, 8\rangle, \langle 3, 6, 7, 1\rangle, \langle 4, 8, 7, 1\rangle, \langle 5, 2, 8, 3\rangle, \langle 6, 5, 4, 3\rangle, \langle 7, 2, 6, 1\rangle, (1, 5, 8, 4, 7, 3, 2)\}$.

$B_9 \in E7CS(9)$, *where* $B_9 = \{(2), (3), (8), (9), \langle 1, 8, 3, 4\rangle, \langle 4, 2, 1, 5\rangle, \langle 5, 3, 2, 6\rangle, \langle 6, 1, 3, 7\rangle, (1, 4, 9, 5, 8, 6, 7), (2, 5, 4, 6, 9, 7, 8), (3, 6, 5, 7, 4, 8, 9)\}$.

$B_{10} \in E7CS(10)$, *where* $B_{10} = \{(1), (4), \langle 2, 7, 1, 10\rangle, \langle 3, 6, 4, 9\rangle, \langle 5, 2, 4, 1\rangle, \langle 6, 1, 3, 4\rangle, \langle 7, 4, 5, 3\rangle, \langle 8, 2, 3, 9\rangle, \langle 9, 1, 2, 10\rangle, \langle 10, 4, 8, 3\rangle, (1, 5, 6, 10, 7, 9, 8), (2, 6, 7, 5, 8, 10, 9), (3, 7, 8, 6, 9, 5, 10)\}$.

$B_{11} \in E7CS(11)$, *where* $B_{11} = \{(9), (10), \langle 1, 4, 7, 11\rangle, \langle 2, 6, 9, 3\rangle, \langle 3, 7, 10, 1\rangle, \langle 4, 5, 11, 8\rangle, \langle 5, 1, 8, 2\rangle, \langle 6, 3, 10, 4\rangle, \langle 7, 2, 9, 4\rangle, \langle 8, 3, 11, 4\rangle, \langle 11, 6, 10, 5\rangle, (1, 2, 3, 4, 5, 6, 7), (7, 8, 9, 10, 11, 1, 3), (5, 7, 9, 11, 2, 4, 8), (6, 8, 10, 2, 5, 9, 1)\}$.

$B_{12} \in E7CS(12)$, *where* $B_{12} = \{(5), (7), (11), (12), \langle 1, 6, 11, 5\rangle, \langle 2, 9, 12, 1\rangle, \langle 3, 7, 11, 1\rangle, \langle 4, 8, 3, 11\rangle, \langle 6, 3, 12, 7\rangle, \langle 8, 2, 6, 12\rangle, \langle 9, 4, 10, 3\rangle, \langle 10, 6, 9, 3\rangle, (1, 2, 3, 4, 5, 6, 7), (7, 8, 9, 10, 11, 12, 2), (2, 4, 6, 8, 10, 12, 5), (1, 3, 5, 7, 9, 11, 4), (4, 7, 10, 1, 5, 8, 12), (1, 6, 2, 8, 3, 9, 12)\}$.

$B_{13} \in E7CS(13)$, *where* $B_{13} = \{(2), (3), (4), (5), (6), (7), (9), (11), \langle 1, 8, 3, 10\rangle, \langle 8, 4, 6, 12\rangle, \langle 10, 13, 9, 8\rangle, \langle 12, 2, 7, 13\rangle, \langle 13, 11, 3, 1\rangle, (1, 2, 13, 3, 12, 4, 11), (2, 8, 10, 9, 1, 12, 11), (3, 2, 4, 1, 5, 13, 6), (4, 3, 5, 2, 6, 1, 7), (5, 7, 3, 9, 2, 10, 12), (6, 5, 4, 13, 12, 7, 11), (7, 6, 8, 5, 9, 4, 10), (8, 7, 9, 6, 10, 5), (9, 12, 8, 13, 1, 10, 11)\}$.

$B_{14} \in E7CS(14)$, *where* $B_{14} = \{(8), (9), (10), (11), (12), (13), (14), \langle 1, 14, 4, 13\rangle, \langle 2, 8, 5, 14\rangle, \langle 3, 11, 2, 10\rangle, \langle 4, 9, 3, 12\rangle, \langle 5, 10, 1, 9\rangle, \langle 6, 7, 1, 11\rangle, \langle 7, 13, 6, 8\rangle, (1, 3, 5, 7, 2, 4, 6), (1, 4, 7, 3, 6, 2, 5), (8, 9, 10, 11, 12, 13, 14), (8, 10, 12, 14, 9, 11, 13), (8, 11, 14, 10, 13, 9, 12), (1, 8, 3, 10, 7, 12, 2), (2, 9, 6, 11, 7, 14, 3), (3, 13, 1, 12, 5, 11, 4), (4, 10, 6, 14, 2, 13, 5), (5, 9, 7, 8, 4, 12, 6)\}$.

$B_{15} \in E7CS(15)$, *where* $B_{15} = \{(i) \mid i = 1, 2, \cdots, 15\} \cup B$ *and* $B \in 7CS(15)$.

$B_{16} \in E7CS(16)$, *where* $B_{16} = \{(6), (9), (12), (13), \langle 1, 16, 15, 9\rangle, \langle 2, 16, 3, 15\rangle, \langle 3, 9, 16, 6\rangle, \langle 4, 15, 11, 12\rangle, \langle 5, 15, 10, 11\rangle, \langle 7, 16, 12, 13\rangle, \langle 8, 6, 13, 16\rangle, \langle 10, 5, 8, 16\rangle, \langle 11, 2, 10, 16\rangle, \langle 14, 5, 16, 11\rangle, \langle 15, 14, 4, 16\rangle, \langle 16, 14, 13, 15\rangle(1, 3, 5, 7, 2, 4, 6), (1, 4, 7, 3, 6, 2, 5), (8, 9, 10, 11, 12, 13, 14), (8, 10, 12, 14, 9, 11, 13), (8, 11, 14, 10, 13, 9, 12), (1, 8, 3, 10, 7, 12, 2), (2, 9, 6, 11, 7, 14, 3), (3, 13, 1, 12, 5, 11, 4), (4, 10, 6, 14, 2, 13, 5), (5, 9, 7, 8, 4, 12, 6), (6, 15, 1, 9, 4, 13, 7), (7, 15, 2, 8, 9, 10, 1), (8, 15, 12, 3, 11, 1, 14)\}$.

$B_{17} \in E7CS(17)$, *where* $B_{17} = \{(3), (5), (6), (7), (11), (12), (13), (14), (15), \langle 1, 12, 15, 9\rangle, \langle 2, 15, 3, 17\rangle, \langle 4, 5, 17, 8\rangle, \langle 8, 15, 16, 10\rangle, \langle 9, 3, 16, 4\rangle, \langle 10, 17, 9, 16\rangle, \langle 16, 2, 17, 1\rangle, \langle 17, 16, 8, 2\rangle, (1, 3, 5, 7, 2, 4, 6), (1, 4, 7, 2,$

5, 3, 6), (8, 9, 10, 11, 12, 13, 14), (8, 10, 12, 14, 9, 11, 13), (8, 11, 14, 10, 13, 9, 12), (1, 8, 4, 12, 7, 13, 2), (2, 12, 5, 10, 1, 13, 3), (3, 8, 6, 10, 2, 14, 4), (4, 9, 7, 11, 3, 14, 5), (5, 9, 1, 11, 4, 13, 6), (6, 9, 2, 11, 5, 8, 7), (7, 10, 3, 12, 6, 14, 1), (4, 10, 15, 5, 16, 11, 17), (5, 13, 15, 6, 16, 12, 17), (6, 11, 15, 7, 16, 14, 17), (7, 14, 15, 1, 16, 13, 17)}.

$B_{18} \in E7CS(18)$, *where* B_{18} = {(2), (5), (6), (7), (10), (11), (16), (17), (18), ⟨1, 18, 3, 16⟩, ⟨3, 9, 16, 17⟩, ⟨4, 15, 18, 6⟩, ⟨8, 16, 18, 7⟩, ⟨9, 17, 8, 2⟩, ⟨12, 13, 18, 11⟩, ⟨13, 14, 18, 5⟩, ⟨14, 8, 9, 10⟩, ⟨15, 17, 10, 18⟩, (1, 3, 5, 7, 2, 4, 6), (1, 4, 7, 3, 6, 2, 5), (8, 10, 12, 14, 9, 11, 13), (8, 11, 14, 10, 13, 9 , 12), (1, 8, 4, 12, 7, 13, 2), (2, 12, 5, 10, 1, 13, 3), (3, 8, 6, 10, 2, 14, 4), (4, 9, 7, 11, 3, 14, 5), (5, 9, 1, 11, 4, 13, 6), (6, 9, 2, 11, 5, 8, 7), (7, 10, 3, 12, 6, 14, 1), (4, 10, 15, 5, 16, 11, 17), (5, 13, 15, 6, 16, 12, 17), (6, 11, 15, 7, 16, 14, 17), (7, 14, 15, 1, 16, 13, 17), (2, 17, 1, 12, 15, 9, 18), (4, 16, 2, 15, 3, 17, 18), (8, 15, 16, 10, 11, 12, 18)}.

$B_{19} \in E7CS(19)$, *where* B_{19} = {(1), (2), (3), (4), (5), (7), (10), (12), (13), (16), (17), ⟨6, 19, 18, 15⟩, ⟨8, 14, 9, 11⟩, ⟨9, 16, 17, 19⟩, ⟨11, 19, 10, 9⟩, ⟨14, 12, 13, 18⟩, ⟨15, 4, 19, 14⟩, ⟨18, 1, 19, 8⟩, ⟨19, 3, 18, 6⟩, (1, 3, 5, 7, 2, 4, 6), (1, 4, 7, 3, 6, 2, 5), (8, 11, 14, 10, 13, 9, 12), (1, 8, 4, 12, 7, 13, 2), (2, 12, 5, 10, 1, 13, 3), (3, 8, 6, 10, 2, 14, 4), (4, 9, 7, 11, 3, 14, 5), (5, 9, 1, 11, 4, 13, 6), (6, 9, 2, 11, 5, 8, 7), (7, 10, 3, 12, 6, 14, 1), (4, 10, 15, 5, 16, 11, 17), (5, 13, 15, 6, 16, 12, 17), (6, 11, 15, 7, 16, 14, 17), (7, 14, 15, 1, 16, 13, 17), (2, 17, 1, 12, 15, 9, 18), (4, 16, 2, 15, 3, 17, 18), (8, 15, 16, 10, 11, 12, 18), (3, 9, 17, 8, 2, 19, 16), (10, 8, 16, 18, 7, 19, 12), (8, 13, 14, 18, 5, 19, 9), (17, 10, 18, 11, 13, 19, 15)}.

$B_{20} \in E7CS(20)$, *where* B_{20} = {(10), (12), (13), (15), (16), (17), (19), (20), ⟨1, 19, 10, 14⟩, ⟨2, 20, 6, 18⟩, ⟨3, 20, 1, 18⟩, ⟨4, 20, 11, 19⟩, ⟨5, 20, 9, 10⟩, ⟨6, 19, 20, 16⟩, ⟨7, 20, 13, 12⟩, ⟨8, 20, 10, 13⟩, ⟨9, 13, 18, 15⟩, ⟨11, 9, 12, 14⟩, ⟨14, 9, 16, 17⟩, ⟨18, 19, 17, 20⟩, (1, 3, 5, 7, 2, 4, 6), (1, 4, 7, 3, 6, 2, 5), (1, 8, 4, 12, 7, 13, 2), (2, 12, 5, 10, 1, 13, 3), (3, 8, 6, 10, 2, 14, 4), (4, 9, 7, 11, 3, 14, 5), (5, 9, 1, 11, 4, 13, 6), (6, 9, 2, 11, 5, 8, 7), (7, 10, 3, 12, 6, 14, 1), (4, 10, 15, 5, 16, 11, 17), (5, 13, 15, 6, 16, 12, 17), (6, 11, 15, 7, 16, 14, 17), (7, 14, 15, 1, 16, 13, 17), (2, 17, 1, 12, 15, 9, 18), (4, 16, 2, 15, 3, 17, 18), (8, 15, 16, 10, 11, 12, 18), (3, 9, 17, 8, 2, 19, 16), (10, 8, 16, 18, 7, 19, 12), (8, 13, 14, 18, 5, 19, 9), (17, 10, 18, 11, 13, 19, 15), (15, 4, 19, 14, 8, 12, 20), (18, 3, 19, 8, 11, 14, 20)}.

3 Main Theorem

We know that the spectrum of $7CS(n)$ is $n \equiv 1, 7 (mod\ 14)$. From this system, we can construct an extended 7-cycle system of order n as follows.

Theorem 3.1 *$E7CS(n)$ exists for $n \equiv 1, 7(mod\ 14)$.*

Proof: Let $B = \{(i) \mid i = 1, 2, \cdots, n\} \cup C$, where $C \in 7CS(n)$, then $B \in E7CS(n)$. ∎

In 1989, Hoffman, Lindner and Rodger [2] showed that there is a group divisible k-cycle system with m groups of size k, for all odd m and k. Thus we have the following theorem.

Theorem 3.2 *[2] $7CS(K_{m(7)})$ exists for all non-negative odd integer m.*

Let $G_1 = K_7^+ + K_1$, where the vertex sets of K_7 and K_1 are $\{1, 2 \cdots, 7\}$ and $\{a\}$, respectively. Let $S_1 = B_8(1, 2, \cdots, 7, a) \setminus \{(a)\}$. Then $E7CS(G_1)$ exists by $S_1 \in E7CS(G_1)$.

Theorem 3.3 *$E7CS(n)$ exists for $n \equiv 8(mod\ 14)$.*

Proof: For $n \equiv 8(mod\ 14)$, we write $n = 14k + 8$. Let the vertex set of K_n^+ be $S = \{i_j \mid i = 1, 2, \cdots, 2k + 1;\ j = 1, 2, \cdots, 7\} \cup \{a\}$. The graph K_n^+ can be regarded as a union of subgraphs $K_{(2k+1)(7)}$, $2k + 1$ copies of K_7^+ and the join of K_1^+ and $\overline{K}_{14k+7}$ and we will denote it by $K_n^+ = K_{(2k+1)(7)} \cup (2k + 1)K_7^+ \cup (K_1^+ + \overline{K}_{14k+7})$. The complete partite graph $K_{(2k+1)(7)}$ with partition sets $\{i_j \mid j = 1, 2, \cdots, 7\}$, $i = 1, 2, \cdots, 2k + 1$, can be decomposed into 7-cycles using Theorem 3.2. For $i = 1, 2, \cdots, 2k + 1$, taking the join of the graph K_7^+ on the vertices $i_1, i_2, \cdots, i_7$ and K_1 on the vertices a form a graph G_1 which can be decomposed into 3-tadpoles and 7-cycles as above. Finally, taking a loop at the vertex a, we obtain a decomposition of K_n^+ consisting of loops, 3-tadpoles and 7-cycles. So, $E7CS(n)$ exists for $n \equiv 8(mod\ 14)$. ∎

Now, we take two disjoint 7-cycles $c_1 = (1, 2, \cdots, 7)$ and $c_2 = (8, 9, \cdots, 14)$ and a graph $\overline{K}_n$, where $V(\overline{K}_n) = \{x_1, x_2, \cdots, x_n\}$. A consideration of the following three graphs shows that the corresponding extended 7-cycle systems exist.

1). Let $G_2 = (c_1 \cup c_2)^+ + \overline{K}_2$ and $S_2 = \{\langle 1, x_1, 8, 9\rangle, \langle 2, x_1, 9, 10\rangle, \langle 3, x_1, 10, 11\rangle, \langle 4, x_1, 11, 12\rangle, \langle 5, x_1, 12, 13\rangle, \langle 6, x_1, 13, 14\rangle, \langle 7, x_1, 14, 8\rangle, \langle 8, x_2,$

1, 2⟩, ⟨9, x_2, 2, 3⟩, ⟨10, x_2, 3, 4⟩, ⟨11, x_2, 4, 5⟩, ⟨12, x_2, 5, 6⟩, ⟨13, x_2, 6, 7⟩, ⟨14, x_2, 7, 1⟩}. Then $S_2 \in E7CS(G_2)$.

2). Let $G_4 = (c_1 \cup c_2)^+ + \overline{K}_4$ and S_4 = {(1), (2), (3), (6), (9), (10), (13), ⟨4, x_4, 1, 7⟩, ⟨5, 6, x_2, 4⟩, ⟨7, 6, x_4, 5⟩, ⟨8, 14, x_4, 12⟩, ⟨11, 12, x_3, 14⟩, ⟨12, 13, x_4, 8⟩, ⟨14, 13, x_1, 11⟩, (1, x_1, 8, x_2, 9, x_3, 2), (2, x_1, 10, x_2, 11, x_3, 3), (3, x_1, 12, x_2, 13, x_3, 4), (4, x_1, 14, x_2, 7, x_3, 5), (8, x_3, 6, x_1, 7, x_4, 9), (9, x_1, 5, x_2, 2, x_4, 10), (10, x_3, 1, x_2, 3, x_4, 11)}. Then $S_4 \in E7CS(G_4)$.

3). Let $G_6 = (c_1 \cup c_2) + \overline{K}_6$ and S_6 = {(1, x_1, 3, x_2, 4, x_3, 2), (2, x_4, 4, x_6, 5, x_3, 3), (3, x_4, 5, x_2, 6, x_5, 4), (4, x_1, 6, x_3, 7, x_5, 5), (5, x_1, 7, x_2, 8, x_6, 6), (6, x_4, 8, x_1, 9, x_6, 7), (7, x_4, 9, x_2, 10, x_6, 2), (8, x_3, 10, x_1, 12, x_5, 9), (9, x_3, 11, x_2, 12, x_4, 10), (10, x_5, 3, x_6, 13, x_1, 11), (11, x_4, 13, x_2, 14, x_3, 12), (12, x_6, 14, x_4, 1, x_5, 13), (13, x_3, 1, x_2, 2, x_5, 14), (14, x_1, 2, x_6, 11, x_5, 8)}. Then $S_6 \in 7CS(G_6)$.

Theorem 3.4 *E7CS(n) exists for* $n \equiv 9, 11, 13 (mod\ 14)$.

Proof: For $n \equiv 9, 11, 13 (mod\ 14)$, we write $n = (14k + 7) + r$, where $r = 2, 4, 6$, respectively. Let the vertex set of K_n^+ be $S = \{i_j \mid i = 1, 2, \cdots, 2k+1;\ j = 1, 2, \cdots, 7\} \cup V$, where $V = \{x_i \mid i = 1, 2, \cdots, r\}$. Then $K_n^+ = K_{(2k+1)(7)} \cup (2k+1)K_7^+ \cup (K_r^+ + \overline{K}_{14k+7})$. The complete partite graph $K_{(2k+1)(7)}$ with partition sets $\{i_j \mid j = 1, 2, \cdots, 7\}$, $i = 1, 2, \cdots, 2k+1$, can be decomposed into 7-cycles by Theorem 3.2. And the graph $2kK_7^+$ can be decomposed by $B_7(i_1, i_2, \cdots, i_7)$, for $i = 1, 2, \cdots, 2k$, where $B_7(i_1, i_2, \cdots, i_7)$ contains three 7-cycles (say $c_{i_1}, c_{i_2}, c_{i_3}$). For the remaining decompositions, there are two cases to consider according to r.

Case 1. $r = 2, 4$. For each $i = 1, 2, \cdots, k$, we take the 14 loops $\{(l_j) \mid l = 2i-1, 2i \text{ and } j = 1, 2, \cdots, 7\}$ and two 7-cycles $c_{(2i-1)_1}, c_{(2i)_1}$ from the decomposition of $2kK_7^+$. Then, combining with $K_{r,14}$ with partition sets V and $\{l_j \mid l = 2i-1, 2i \text{ and } j = 1, 2, \cdots, 7\}$, we have G_r which can be decomposed into 3-tadpoles and 7-cycles.

Case 2. $r = 6$. For each $i = 1, 2, \cdots, k$, we take two 7-cycles $c_{(2i-1)_1}$ and $c_{(2i)_1}$ from the decomposition of $2kK_7^+$. Then, combining with $K_{6,14}$ with partition sets V and $\{l_j \mid l = 2i-1, 2i \text{ and } j = 1, 2, \cdots, 7\}$, we have G_6 which can be decomposed into 7-cycles.

Finally, by putting B_{7+r} on the vertex set $\{(2k+1)_j \mid j = 1, 2, \cdots, 7\} \cup V$, we obtain a decomposition of K_n^+ consisting of loops, 3-tadpoles and 7-cycles. So, $E7CS(n)$ exists for $n \equiv 9, 11, 13 (mod\ 14)$. ∎

Now, we take one 7-cycle $c_1 = (1, 2, \cdots, 7)$ and a graph $\overline{K}_n$, where $V(\overline{K}_n) = \{x_1, x_2, \cdots, x_n\}$. A consideration of the following three graphs shows that the corresponding extended 7-cycle systems exist.

1). Let $G_3 = (c_1)^+ + \overline{K}_3$ and $S_3 = \{(1, x_1, 5, x_2, 6, x_3, 2), \langle 1, 7, x_3, 4\rangle, \langle 2, x_1, 7, x_2\rangle, \langle 3, 2, x_2, 1\rangle, \langle 4, x_2, 3, x_3\rangle, \langle 5, 4, 3, x_1\rangle, \langle 6, 5, x_3, 1\rangle, \langle 7, 6, x_1, 4\rangle\}$. Then $S_3 \in E7CS(G_3)$.

2). Let $G_5 = (c_1)^+ + \overline{K}_5$ and $S_5 = \{\langle 1, x_2, 2, x_3\rangle, \langle 2, x_4, 3, x_5\rangle, \langle 3, 4, 5, x_1\rangle, \langle 4, x_5, 1, x_4\rangle, \langle 5, 6, 7, x_3\rangle, \langle 6, x_3, 4, x_1\rangle, \langle 7, x_2, 6, x_1\rangle, (1, x_1, 3, x_3, 5, x_5, 2), (1, x_2, 4, x_4, 6, x_5, 7), (2, x_1, 7, x_4, 5, x_2, 3)\}$. Then $S_5 \in E7CS(G_5)$.

3). Let $G_7 = (c_1)^+ + \overline{K}_7$ and $S_7 = \{\langle 1, x_7, 4, x_6\rangle, \langle 2, x_1, 5, x_7\rangle, \langle 3, x_4, 2, x_3\rangle, \langle 4, x_2, 3, x_5\rangle, \langle 5, x_3, 1, x_2\rangle, \langle 6, 7, 1, x_4\rangle, \langle 7, x_6, 6, x_1\rangle, (1, x_1, 3, x_3, 7, x_5, 2), (2, x_2, 6, x_4, 7, x_7, 3), (3, x_6, 1, x_5, 5, x_4, 4), (4, x_3, 6, x_7, 2, x_6, 5), (5, x_2, 7, x_1, 4, x_5, 6)\}$. Then $S_7 \in E7CS(G_7)$.

Theorem 3.5 *E7CS(n) exists for* $n \equiv 10, 12, 14 (mod\ 14)$.

Proof: **Proof:** For $n \equiv 10, 12, 14(mod\ 14)$, we write $n = (14k + 7) + r$, where $r = 3, 5, 7$, respectively. Let the vertex set of K_n^+ be $S = \{i_j \mid i = 1, 2, \cdots, 2k+1;\ j = 1, 2, \cdots, 7\} \cup V$, where $V = \{x_i \mid i = 1, 2, \cdots, r\}$. Then $K_n^+ = K_{(2k+1)(7)} \cup (2k + 1)K_7^+ \cup (K_r^+ + \overline{K}_{14k+7})$. The complete partite graph $K_{(2k+1)(7)}$ with partition sets $\{i_j \mid j = 1, 2, \cdots, 7\}$, $i = 1, 2, \cdots, 2k + 1$, can be decomposed into 7-cycles using Theorem 3.2. And the graph $2kK_7^+$ can be decomposed by $B_7(i_1, i_2, \cdots, i_7)$, for $i = 1, 2, \cdots, 2k$, where $B_7(i_1, i_2, \cdots, i_7)$ contains three 7-cycles (say $c_{i_1}, c_{i_2}, c_{i_3}$). For each $i = 1, 2, \cdots, 2k$, we take the 7 loops $\{(i_j) \mid j = 1, 2, \cdots, 7\}$ and one 7-cycle c_{i_1} from the decomposition of $2kK_7^+$. Then, combining with $K_{r,7}$ with partition sets V and $\{i_j \mid j = 1, 2, \cdots, 7\}$, we have G_r which can be decomposed into 3-tadpoles and 7-cycles. Finally, by putting B_{7+r} on the vertex set $\{(2k + 1)_j \mid j = 1, 2, \cdots, 7\} \cup V$, we have obtained a decomposition of K_n^+ consisting of loops, 3-tadpoles and 7-cycles. So, $E7CS(n)$ exists for $n \equiv 10, 12, 14(mod\ 14)$. ∎

Theorem 3.6 *E7CS(n) exists for* $n \equiv 2, 3, 4, 5, 6(mod\ 14)$.

Proof: For $n \equiv 2, 3, 4, 5, 6(mod\ 14)$, we write $n = (14k + 7) + r$, where $r = 9, 10, 11, 12, 13$, respectively. Let the vertex set of K_n^+ be $S = \{i_j \mid i = 1, 2, \cdots, 2k+1;\ j = 1, 2, \cdots, 7\} \cup V$, where $V = \{x_i \mid i = 1, 2, \cdots, r\}$. Then $K_n^+ = K_{(2k+1)(7)} \cup (2k + 1)K_7^+ \cup (K_r^+ + K_{14k+7}^c)$. The complete partite graph $K_{(2k+1)(7)}$ with partition sets $\{i_j \mid j = 1, 2, \cdots, 7\}$, $i = 1, 2, \cdots, 2k + 1$, can be decomposed into 7-cycles using Theorem 3.2. And the graph $2kK_7^+$ can be decomposed by $B_7(i_1, i_2, \cdots, i_7)$, for $i = 1, 2, \cdots, 2k$, where $B_7(i_1, i_2, \cdots, i_7)$ contains three 7-cycles (say $c_{i_1}, c_{i_2}, c_{i_3}$). First, for each

$i = 1, 2, \cdots, k$, we take two cycles $c_{(2i-1)_1}$, $c_{(2i)_2}$ and $K_{6,14}$ with partition sets $\{x_i \mid i = 1, 2, \cdots, 6\}$ and $\{l_j \mid l = 2i-1, 2i$ and $j = 1, 2, \cdots, 7\}$ combining a graph G_6 which can be decomposed into 7-cycles. For the remaining decompositions, there are three cases to consider according to r.

Case 1. $r = 9, 11, 13$. For each $i = 1, 2, \cdots, 2k$, we take the 7 loops $\{(i_j) \mid j = 1, 2, \cdots, 7\}$ and 7-cycle c_{i_2} from the decomposition of $2kK_7^+$. Then, combining with and $K_{r-6,7}$ with partition sets $\{x_i \mid i = 7, 8, \cdots, r\}$ and $\{i_j \mid j = 1, 2, \cdots, 7\}$, we have G_{r-6} which can be decomposed into 3-tadpoles and 7-cycles.

Case 2. $r = 10$. For each $i = 1, 2, \cdots, k$, we take the 14 loops $\{(l_j) \mid l = 2i-1, 2i$ and $j = 1, 2, \cdots, 7\}$ and two cycles $c_{(2i-1)_2}, c_{(2i)_2}$ from the decomposition of $2kK_7^+$. Then, combining with and $K_{4,14}$ with partition sets $\{x_i \mid i = 7, 8, 9, 10\}$ and $\{l_j \mid l = 2i-1, 2i$ and $j = 1, 2, \cdots, 7\}$, we have G_4 which can be decomposed into loops, 3-tadpoles and 7-cycles.

Case 3. $r = 12$. For each $i = 1, 2, \cdots, k$, we take two cycles $c_{(2i-1)_2}$ and $c_{(2i)_2}$ from the decomposition of $2kK_7^+$. Then, combining with and $K_{6,14}$ with partition sets $\{x_i \mid i = 7, 8, \cdots, 12\}$ and $\{l_j \mid l = 2i-1, 2i$ and $j = 1, 2, \cdots, 7\}$, we have G_6 which can be decomposed into 7-cycles.

Finally, by putting B_{7+r} on the vertex set $\{(2k+1)_j \mid j = 1, 2, \cdots, 7\} \cup V$, we obtain a decomposition of K_n^+ consisting of loops, 3-tadpoles and 7-cycles. So, $E7CS(n)$ exists for $n \equiv 2, 3, 4, 5, 6 (mod\ 14)$. ∎

Using Example 2.1 and Theorem 3.1, 3.3, 3.4 and 3.5, we obtain the following result:

Main Theorem An extended 7-cycle system of order n exists for all n except $n = 2, 3$ and 5.

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