Close to regular multipartite tournaments containing a Hamiltonian path

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Abstract

If x is a vertex of a digraph D, then we denote by $d^+(x)$ and $d^-(x)$ the outdegree and the indegree of x, respectively. The global irregularity of a digraph D is defined by $i_g(D) = \max\{d^+(x), d^-(x)\} - \min\{d^+(y), d^-(y)\}$ over all vertices x and y of D (including $x = y$). If $i_g(D) = 0$, then D is regular and if $i_g(D) \leq 1$, then D is called almost regular. The local irregularity is defined as $i_l(D) = \max |d^+(x) - d^-(x)|$ over all vertices x of D. The path covering number of D is the minimum number of directed paths in D that are pairwise vertex disjoint and cover the vertices of D. A semicomplete c-partite digraph is a digraph obtained from a complete c-partite graph by replacing each edge with an arc, or a pair of mutually opposite arcs with the same end vertices. If a semicomplete c-partite digraph D does not contain an oriented cycle of the length two, then D is called a c-partite tournament.

In 2000, Gutin and Yeo [7] proved sufficient conditions for the local irregularity of a semicomplete multipartite digraph to secure a path covering number of at most k. In this paper, we will give a useful supplement to this result by using bounds for the global irregularity that guarantee a path covering number of at most k. As an application, we will present sufficient conditions for close to regular multipartite tournaments containing a Hamiltonian path. Especially,

we will characterize almost regular c-partite tournaments containing a Hamiltonian path.

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1 Terminology and introduction

In this paper all digraphs are finite without loops and multiple arcs. The vertex set and arc set of a digraph D is denoted by $V(D)$ and $E(D)$, respectively. If xy is an arc of a digraph D, then we write $x \to y$ and say x *dominates* y, and if X and Y are two disjoint vertex sets or subdigraphs of D such that every vertex of X dominates every vertex of Y, then we say that X *dominates* Y, denoted by $X \to Y$. Furthermore, $X \leadsto Y$ denotes the fact that there is no arc leading from Y to X. By $d(X, Y)$ we denote the number of arcs from the set X to the set Y, i.e., $d(X,Y) = |\{xy \in E(D) : x \in X, y \in Y\}|$. If D is a digraph, then the *out-neighborhood* $N_D^+(x) = N^+(x)$ of a vertex x is the set of vertices dominated by x and the *in-neighborhood* $N_D^-(x) = N^-(x)$ is the set of vertices dominating x. Therefore, if there is the arc $xy \in E(D)$, then y is an *outer neighbor* of x and x is an *inner neighbor* of y. The numbers $d_D^+(x) = d^+(x) = |N^+(x)|$ and $d_D^-(x) = d^-(x) = |N^-(x)|$ are called the *outdegree* and *indegree* of x, respectively. For a vertex set X of D, we define $D[X]$ as the subdigraph induced by X. If we speak of a *cycle* or *path*, then we mean a directed cycle or directed path, and a cycle of length n is called an *n-cycle.* A cycle or path of a digraph D is *Hamiltonian*, if it includes all the vertices of D. The *path covering number* of a digraph D ($pc(D)$) is the minimum number of paths in D that are pairwise vertex disjoint and cover the vertices of D. A *factor* is a spanning subgraph of a digraph. A factor is a *k-path-cycle*, if it consists of a set of vertex disjoint paths and cycles, where k stands for the number of paths in the set.

There are several measures of how much a digraph differs from being regular. In [14], Yeo defines the *global irregularity* of a digraph D by

$$i_g(D) = \max_{x \in V(D)} \{d^+(x), d^-(x)\} - \min_{y \in V(D)} \{d^+(y), d^-(y)\}$$

and the *local irregularity* as $i_l(D) = \max |d^+(x) - d^-(x)|$ over all vertices x of D. Clearly, $i_l(D) \leq i_g(D)$. If $i_g(D) = 0$, then D is *regular* and if $i_g(D) \leq 1$, then D is called *almost regular.*

A *c-partite* or *multipartite tournament* is an orientation of a complete c-partite graph. A *tournament* is a c-partite tournament with exactly c vertices. A *semicomplete multipartite digraph* is obtained by replacing each edge of a complete multipartite graph by an arc or by a pair of two mutually opposite arcs. If $V_1, V_2, \ldots, V_c$ are the partite sets of a c-partite tournament or semicomplete c-partite digraph D and the vertex x of D belongs to the partite set V_i, then we define $V(x) = V_i$. If D is a c-partite tournament with the partite sets $V_1, V_2, \ldots, V_c$ such that $|V_i| = n_i$ for $i = 1, 2, \ldots, c$, then we speak of the *partition-sequence* $(n_i) = n_1, n_2, \ldots, n_c$.

Hamiltonian cycles in multipartite tournaments are well studied (see e.g. [2, 3, 4, 8, 13, 14]). For example, Yeo [13] presented a result for regular semicomplete multipartite digraphs.

Theorem 1.1 (Yeo [13], 1997) *Every regular semicomplete multipartite digraph D is Hamiltonian.*

On the other hand, it is not paid much attention on the existence of Hamiltonian paths in such digraphs. In 1988, Gutin [5] gave a characterization of semicomplete multipartite digraphs having a Hamiltonian path.

Theorem 1.2 (Gutin [5], 1988) *A semicomplete multipartite digraph D has a Hamiltonian path if and only if it contains a 1-path-cycle factor.*

This result was used to prove another result of Gutin and Yeo [7].

Theorem 1.3 (Gutin, Yeo [7], 2000) *Let D be a semicomplete multipartite digraph with the partite sets $V_1, V_2, \ldots, V_c$ such that $|V_1| \leq |V_2| \leq \ldots \leq |V_c|$. If there exists a positive integer k such that*

$$i_l(D) \leq \min\{|V(D)| - 3|V_c| + 2k + 1, \frac{|V(D)| - |V_{c-1}| - 2|V_c| + 3k + 2}{2}\},$$

then $pc(D) \leq k$.

In this paper, we will give a useful supplement to Theorem 1.3 by presenting bounds for the global irregularity $i_g(D)$. As an application of this result, we will develop sufficient conditions for close to regular multipartite tournaments containing a Hamiltonian path.

If D is regular, then Theorem 1.1 clearly guarantees the existence of a Hamiltonian path which is already shown by Zhang [15] in 1989. Using a sufficient condition for multipartite tournaments with an arbitrary large irregularity number containing a Hamiltonian path, we will show that every almost regular c-partite tournament D contains a Hamiltonian path with the exception that $c = 2$ and one partite set consists of two vertices more than the other partite set. Furthermore, we will precisely examine the case that $i_g(D) = 2$. If $c \geq 5$, then D contains a Hamiltonian path also in this case. If $c = 4$, then there is only a finite family of graphs that do not contain any Hamiltonian path. Finally, we will show that there are infinitely many 2-partite and 3-partite tournaments with $i_g(D) = 2$ that have no Hamiltonian path at all.

Further new results about Hamiltonian paths can be found in [11, 12]. For more informations about multipartite tournaments, we recommend to read [1, 6, 10].

2 Preliminary results

The following results play an important role in our investigations.

Lemma 2.1 **(Tewes, Volkmann, Yeo [9], 2002)** *Let D be a c-partite tournament with the partite sets $V_1, V_2, \ldots, V_c$ such that $|V_1| \leq |V_2| \leq \cdots \leq |V_c|$. Then $|V_c| \leq |V_1| + 2i_g(D)$.*

Lemma 2.2 **(Yeo [14], 1999)** *Let D be a semicomplete multipartite digraph with the partite sets $V_1, V_2, \ldots, V_c$. Let $X \subset Y \subseteq V(D)$ and let $y_i = |Y \cap V_i|$ for all $i = 1, 2, \ldots, c$. Then*

$$\begin{aligned} & \frac{d(X, Y-X) + d(Y-X, X)}{|X|} + \frac{d(X, Y-X) + d(Y-X, X)}{|Y-X|} \\ \geq \; & |Y| - \max\{y_i | i = 1, 2, \ldots, c\}. \end{aligned}$$

A slight reformulation of a result of Yeo [14] is presented in the following lemma.

Lemma 2.3 **(Yeo [14], 1999)** *If D is a semicomplete c-partite digraph, then the following holds.*

$$i_l(D) \geq \max_{\emptyset \neq X \subseteq V(D)} \left\{ \frac{|d(X, V(D)-X) - d(V(D)-X, X)|}{|X|} \right\}$$

Theorem 2.4 (Gutin, Yeo [7], 2000) *If D is a semicomplete c-partite digraph, then $pc(D) > k$ ($k \geq 1$) if and only if $V(D)$ can be partitioned into subsets Y, Z, R_1, R_2 such that*

$$R_1 \leadsto Y,\ (R_1 \cup Y) \leadsto R_2,\ Y \text{ is an independent set} \qquad (1)$$

and $|Y| > |Z| + k$.

Theorem 2.5 (Volkmann [11]) *If D is an almost regular bipartite tournament with the partite sets X, Y such that $1 \leq |X| \leq |Y|$, then every arc of D is contained in a Hamiltonian path if and only if $|Y| \leq |X| + 1$ and D is not isomorphic to $T_{3,3}$, where $T_{3,3}$ is the bipartite tournament presented in Figure 1.*

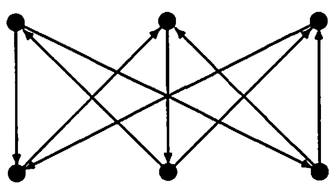

Figure 1: The almost regular bipartite tournament $T_{3,3}$

3 Path covering number and irregularity

The following theorem is a useful supplement to Theorem 1.3, which we will apply in the next section. The proof is similar to the proof of Lemma 4.3 in [14] and Theorem 3.2 in [7].

Theorem 3.1 *Let $V_1, V_2, \ldots, V_c$ be the partite sets of the semicomplete multipartite digraph D such that $|V_1| \leq |V_2| \leq \ldots \leq |V_c|$. Assume that $pc(D) > k$ for an integer $k \geq 1$. According to Theorem 2.4, $V(D)$ can be partitioned into subsets Y, Z, R_1, R_2 satisfying (1) such that $|Z| + k + 1 \leq |Y| \leq |V_c| - t$ with an integer $t \geq 0$. Let V_i be the partite set with the property that $Y \subseteq V_i$. If $Q = V(D) - Z - V_i$, $Q_1 = Q \cap R_1$ and $Q_2 = Q \cap R_2$, then*

$$\begin{aligned} i_l(D) &\geq |V(D)| - 3|V_c| + 2t + 2k + 2 \quad \textit{and} \\ i_g(D) &\geq \frac{|V(D)| - |V_{c-1}| - 2|V_c| + 3k + 3}{2}, \end{aligned}$$

if $Q_1 = \emptyset$ *or* $Q_2 = \emptyset$, *and*

$$i_g(D) \geq i_l(D) \geq \frac{|V(D)| - |V_{c-1}| - 2|V_c| + 3k + 3 + t}{2},$$

if $Q_1 \neq \emptyset$ *and* $Q_2 \neq \emptyset$.

Proof. Let $V(D)$ be partitioned into the subsets Y, Z, R_1, R_2 satisfying (1) such that $|Z| + k + 1 \leq |Y| \leq |V_c| - t$ for integers $k \geq 1$ and $t \geq 0$. Define $Y_1 = R_1 \cap V_i$ and $Y_2 = R_2 \cap V_i$. If Q_1 and Q_2 are defined as above, then we observe that $|Z| \leq |Y| - 1 - k \leq |V_c| - 1 - k - t$, $Q_1 \rightsquigarrow Y \rightsquigarrow Q_2$, $(Q_1 \cup Y_1) \rightsquigarrow (Q_2 \cup Y_2)$ and $Y_1 \cup Y_2 \cup Y \subseteq V_i$. If $i = c$, then let $j = c - 1$ and if $i < c$, then let $j = c$. We now consider the following three cases.

Case 1. Let $Q_1 = \emptyset$. Then $Q_2 = Q$ and we obtain

$$\begin{aligned} & d(Y, V(D) - Y) - d(V(D) - Y, Y) \\ \geq\ & |Y||Q_2| - |Y||Z| \geq |Y|(|V(D)| - |V_i| - 2|Z|) \\ \geq\ & |Y|(|V(D)| - |V_c| - 2(|V_c| - 1 - k - t)) \\ =\ & |Y|(|V(D)| - 3|V_c| + 2 + 2k + 2t). \end{aligned}$$

According to Lemma 2.3, this implies that $i_l(D) \geq |V(D)| - 3|V_c| + 2 + 2k + 2t$, and hence, we have one part of the desired result. We will now show the second part.

Let $\delta^* = \min\{d^-(w) | w \in V_i\}$. Since $Y \subseteq V_i$ and thus $d^-(y) \leq |Z|$ for all $y \in Y$ we observe that $\delta^* \leq |Z| \leq |Y| - k - 1 \leq |V_i| - |Y_2| - 1 - k$. Let $\Delta^* = \max\{d^+(w), d^-(w) | w \in V(D) - V_i\}$ and note that $d^+(w) + d^-(w) \geq |V(D)| - |V_j|$ for all $w \in V(D) - V_i$. The fact that $\sum_{x \in Q_2}(d^-(x) - d^+(x)) \geq |Q_2|(|Y| - |Z| - |Y_2|) \geq |Q_2|(1 + k - |Y_2|)$ implies that there is a vertex $q \in Q_2$ such that $d^-(q) > d^+(q) + k - |Y_2|$. This leads to $2d^-(q) - k + |Y_2| > d^+(q) + d^-(q) \geq |V(D)| - |V_j|$, and thus we conclude that $\Delta^* > \frac{|V(D)| - |V_j| + k - |Y_2|}{2}$. This implies

$$\begin{aligned} i_g(D) &\geq \Delta^* - \delta^* > \frac{|V(D)| - |V_j| + k - |Y_2|}{2} - |V_i| + |Y_2| + k + 1 \\ &= \frac{|V(D)| - |V_j| - 2|V_i| + 3k + 2 + |Y_2|}{2} \\ &\geq \frac{|V(D)| - |V_{c-1}| - 2|V_c| + 3k + 2}{2}, \end{aligned}$$

and the second part is proved.

Case 2. Let $Q_2 = \emptyset$. This is analogously to Case 1 (change the orientation of all the arcs in D).

Case 3. Let $Q_1 \neq \emptyset$ and $Q_2 \neq \emptyset$. Since $|V_i| + |V_j| \leq |V_{c-1}| + |V_c|$, we deduce that $|Q| - |V_j| \geq |V(D)| - |V_i| - |Z| - |V_j| \geq |V(D)| - |V_{c-1}| - |V_c| - (|V_c| - 1 - k - t)$. By Lemma 2.2 with $X = Q_1$ and $Y = Q_1 \cup Q_2 = Q$ and because of $Q \cap V_i = \emptyset$, it follows that

$$\begin{aligned} & \frac{d(Q_1,Q_2)+d(Q_2,Q_1)}{|Q_1|} + \frac{d(Q_1,Q_2)+d(Q_2,Q_1)}{|Q_2|} \\ = & \frac{d(Q_1,Q_2)}{|Q_1|} + \frac{d(Q_1,Q_2)}{|Q_2|} \geq |Q| - |V_j| \\ \geq & |V(D)| - |V_{c-1}| - 2|V_c| + 1 + k + t. \end{aligned}$$

Thus,

$$\begin{aligned} \text{(i)} \quad \frac{d(Q_1,Q_2)}{|Q_1|} & \geq \frac{|V(D)| - |V_{c-1}| - 2|V_c| + 1 + k + t}{2} - |Y_2| + |Y_1| \text{ or} \\ \text{(ii)} \quad \frac{d(Q_1,Q_2)}{|Q_2|} & \geq \frac{|V(D)| - |V_{c-1}| - 2|V_c| + 1 + k + t}{2} + |Y_2| - |Y_1|. \end{aligned}$$

Assume that (i) holds as the case when (ii) holds can be treated similarly. Because of $R_1 = Q_1 \cup Y_1$ and $R_2 = Q_2 \cup Y_2$, Lemma 2.3 yields

$$\begin{aligned} i_g(D) \geq i_l(D) & \geq \frac{d(Q_1, V(D) - Q_1) - d(V(D) - Q_1, Q_1)}{|Q_1|} \\ & = \frac{d(Q_1,Q_2)}{|Q_1|} + \frac{d(Q_1, Y \cup Y_2) - d(Y \cup Y_2, Q_1)}{|Q_1|} \\ & + \frac{d(Q_1, Z \cup Y_1) - d(Z \cup Y_1, Q_1)}{|Q_1|} - \frac{d(Q_2,Q_1)}{|Q_1|} \\ & \geq \left(\frac{|V(D)| - |V_{c-1}| - 2|V_c| + 1 + k + t}{2} - |Y_2| + |Y_1|\right) \\ & + (|Y| + |Y_2|) - (|Z| + |Y_1|) \\ & = \frac{|V(D)| - |V_{c-1}| - 2|V_c| + 1 + k + t}{2} + |Y| - |Z| \\ & \geq \frac{|V(D)| - |V_{c-1}| - 2|V_c| + 3 + 3k + t}{2}. \end{aligned}$$

This completes the proof of the theorem. □

Theorem 3.1 with $t = 0$ leads immediately to the following result on the path covering number.

Corollary 3.2 *Let $V_1, V_2, \ldots, V_c$ be the partite sets of a semicomplete multipartite digraph D such that $|V_1| \leq |V_2| \leq \ldots \leq |V_c|$. If there exists a positive integer k such that $i_g(D) \leq \frac{|V(D)| - |V_{c-1}| - 2|V_c| + 3k + 2}{2}$, then $pc(D) \leq k$.*

4 Hamiltonian paths

Another consequence of Theorem 3.1 is presented in the following result.

Theorem 4.1 *Let $V_1, V_2, \ldots, V_c$ be the partite sets of the semicomplete c-partite digraph D such that $1 \leq r = |V_1| \leq |V_2| \leq \ldots \leq |V_c| \leq r+p$ for an integer $p \geq 0$. If $c \geq \max\{2, 3 + \frac{2i_g(D)-5+p}{r}\}$, then D contains a Hamiltonian path.*

Proof. Clearly, D contains a Hamiltonian path if and only if $pc(D) = 1$. Hence, according to Corollary 3.2 with $k = 1$, it is sufficient to show that $i_g(D) \leq \frac{|V(D)|-|V_{c-1}|-2|V_c|+5}{2}$. Because of $c \geq 3 + \frac{2i_g(D)-5+p}{r}$, we conclude that $i_g(D) \leq \frac{(c-3)r+5-p}{2}$, and together with $|V_1|, |V_2|, \ldots, |V_{c-2}| \geq r$, $|V_c| \leq r+p$ and $c \geq 2$ this implies

$$\begin{aligned} \frac{|V(D)|-|V_{c-1}|-2|V_c|+5}{2} &= \frac{|V_1|+|V_2|+\ldots+|V_{c-2}|-|V_c|+5}{2} \\ &\geq \frac{(c-3)r-p+5}{2} \geq i_g(D), \end{aligned}$$

the desired result. □

If D is a multipartite tournament, then, according to Theorem 2.1, we can choose $p = 2i_g(D)$ in the previous theorem.

Corollary 4.2 *Let $V_1, V_2, \ldots, V_c$ be the partite sets of a c-partite tournament D such that $1 \leq r = |V_1| \leq |V_2| \leq \ldots \leq |V_c|$. If $c \geq \max\{2, \frac{4i_g(D)-5}{r} + 3\}$, then D contains a Hamiltonian path.*

Theorem 4.3 *Let D be an almost regular c-partite tournament with the partite sets $V_1, V_2, \ldots, V_c$ such that $|V_1| \leq |V_2| \leq \ldots \leq |V_c|$. Then D contains a Hamiltonian path if and only if $c \geq 3$ or $c = 2$ and $|V_2| \leq |V_1|+1$.*

Proof. Firstly, let $c = 2$. Suppose that $|V_2| = |V_1|+2$. Then it is obvious that D does not contain any Hamiltonian path, because the vertices of this path would alternately be part of the partite sets V_1 and V_2. Hence let $|V_2| \leq |V_1| + 1$. In this case Theorem 2.5 shows that D contains a Hamiltonian path, since $T_{3,3}$ is Hamiltonian.

Secondly, let $c \geq 3 \geq \max\{2, 3 - \frac{1}{r}\}$ for all $r \in \mathbb{N}$. In this case, Corollary 4.2 yields the desired result. □

The case that $i_g(D) = 2$ is more complicated as the following considerations demonstrate.

Theorem 4.4 *Let $V_1, V_2, \ldots, V_c$ be the partite sets of a c-partite tournament D such that $1 \le r = |V_1| \le |V_2| \le \ldots \le |V_c|$. If $i_g(D) = 2$, $c \ge 4$ and D doesn't have one of the partition-sequences* $1,1,2,4$; $1,2,3,5$; $1,1,3,4$ *and* $2,2,4,6$, *then D contains a Hamiltonian path.*

Proof. Since $\max\{\frac{8-5}{r} + 3 | r \in \mathbb{N}\} = 6 > 3$, Corollary 4.2 yields the desired result, if $c \ge 6$.

Hence, let $c = 4$ or $c = 5$ and assume that D does not contain any Hamiltonian path. Let the sets $Y, Z, R_1, R_2, Q, Q_1, Q_2, Y_1, Y_2$ be defined as in the proof of Theorem 3.1. Since the case that $Q_2 = \emptyset$ follows analogously as the case that $Q_1 = \emptyset$, in the following we will always distinguish the two cases that $Q_1 = \emptyset$ or $Q_1, Q_2 \ne \emptyset$.

Case 1. Let $c = 5$. If $r \ge 2$, then because of $c > \max\{\frac{8-5}{r} + 3 | r \ge 2\} = \frac{9}{2} > 3$ and Corollary 4.2 D contains a Hamiltonian path and the proof is finished. If $|V_5| \le 4$, then $\frac{|V_1|+|V_2|+|V_3|-|V_5|+5}{2} \ge 2 = i_g(D)$, a contradiction to Corollary 3.2. If $|V_5| = 5$ and $|V_1| + |V_2| + |V_3| \ge 4$, then we analogously arrive at a contradiction. Altogether, we see that there remain to consider the following five partition-sequences.

Subcase 1.1. Let $(n_i) = 1,1,1,1,5$. In this case, we see that $d^+(x) = d^-(x) = 4$, if $x \in V_1 \cup V_2 \cup V_3 \cup V_4$ and $d^+(x) = d^-(x) = 2$, if $x \in V_5$, that means $i_l(D) = 0$. Since $\frac{|V(D)|-|V_4|-2|V_5|+6}{2} = 2 > i_l(D)$, it remains to consider the case that $Q_1 = \emptyset$ in Theorem 3.1. If $Y = V_5$, then it follows that $|Z| \le 3$ and thus $|Q_2| \ge 1$. This yields $d^-(x) \ge 5$ for all $x \in Q_2$, a contradiction to $d^-(x) \le 4$ for all $x \in V(D)$. If $|Y| = 4$, then it follows that $|Z| \le 2$ and thus $|Q_2| \ge 2$. This implies that there is an arc $pq \in E(D[Q_2])$. Since $d^-(q) \ge 5$, we arrive at a contradiction. If $|Y| \le |V_5| - 2$, then $|V(D)| - 3|V_5| + 8 = 2 > i_l(D)$ contradicts Theorem 3.1 with $t = 2$.

Subcase 1.2. Let $(n_i) = 1,1,1,2,5$. Since $i_g(D) = 2$, this is impossible.

Subcase 1.3. Let $(n_i) = 1,1,1,3,5$. This yields $d^+(x) = d^-(x) = 5$, if $x \in V_1 \cup V_2 \cup V_3$, $d^+(x) = d^-(x) = 4$ or $\{d^+(x), d^-(x)\} = \{3,5\}$, if $x \in V_4$ and $d^+(x) = d^-(x) = 3$, if $x \in V_5$.

Firstly, we assume that $Q_1 = \emptyset$. If $Y = V_5$, then we conclude that $|Z| \le 3$ and thus $|Q_2| \ge 3$. Since $d^+(x) = d^-(x) = 3$ for all $x \in V_5$, it follows that $|Z| = |Q_2| = 3$ and $Z \to Y$. It is obvious that there are either in Q_2 or in Z two vertices of different partite sets. Hence, there is an arc $p \to q$ that is either in $E(D[Q_2])$ or in $E(D[Z])$. If $pq \in E(D[Q_2])$, then $d^-(q) \geq 6$, and if $pq \in E(D[Z])$, then $d^+(p) \geq 6$, in both cases a contradiction. If $|Y| = 4$, then we see that $|Z| \leq 2$ and thus $|Q_2| \geq 4$, a contradiction to $d^+(x) = 3$ for all $x \in V_5$. Hence, let $|Y| \leq |V_5| - 2$. But now, $|V(D)| - 3|V_5| + 8 = 4 > i_l(D)$ contradicts Theorem 3.1.

Consequently, it remains to consider the case that $Q_1 \neq \emptyset$ and $Q_2 \neq \emptyset$. Note that $\frac{|V(D)|-|V_4|-2|V_5|+6}{2} = 2$. According to the proof of Theorem 3.1 the inequalities in the last inequality-chain of Case 3 have to be equalities, that means especially $t = 0$ and $|Y| = |Z| + 2$. This implies $Y = V_5$ and $|Z| = |Y| - 2 = 3$. Hence, we conclude that $|Q| = 3$ and, without loss of generality, let $|Q_1| = 1$ and $|Q_2| = 2$. If there is an arc leading from Q_1 to Q_2 and $q_1 \in Q_1$, then it follows that $d^+(q_1) \geq |Y| + 1 = 6$, a contradiction. Since $Q_1 \leadsto Q_2$, we obtain $Q = V_4$ and thus $Z = V_1 \cup V_2 \cup V_3$. Let $D' = D[V_1 \cup V_2 \cup V_3]$. Because of $Q_2 \to Z \to Q_1$, we have on the one hand that

$$\begin{aligned} 15 = \sum_{x \in V(D')} d^+(x) &= d(V(D'), Q_1) + \sum_{x \in V(D')} d^+_{D'}(x) + d(V(D'), Y) \\ &= 3 + 3 + d(V(D'), Y), \end{aligned}$$

that means $d(V(D'), Y) = 9$. Since $Q_1 \to Y \to Q_2$, we observe on the other hand that

$$15 = \sum_{y \in Y} d^-(y) = d(Q_1, Y) + d(V(D'), Y) = 5 + d(V(D'), Y),$$

that means $d(V(D'), Y) = 10$, a contradiction.

Subcase 1.4. Let $(n_i) = 1, 1, 1, 4, 5$. Since $i_g(D) = 2$, this is impossible.

Subcase 1.5. Let $(n_i) = 1, 1, 1, 5, 5$. This yields that $d^+(x) = d^-(x) = 6$, if $x \in V_1 \cup V_2 \cup V_3$ and $d^+(x) = d^-(x) = 4$, if $x \in V_4 \cup V_5$, that means $i_l(D) = 0$. Since $|V(D)| - 3|V_5| + 3 = 1 > i_l(D)$ and $\frac{|V(D)|-|V_4|-2|V_5|+5}{2} = \frac{3}{2} > i_l(D)$, this contradicts Theorem 1.3.

Case 2. Let $c = 4$. If $r \geq 3$, then $4 \geq \max\{2, \frac{3}{r} + 3\}$ and Corollary 4.2 yields that D contains a Hamiltonian path, a contradiction. If $r = 2$ and $|V_4| \leq 5$ or $r = 2$, $|V_4| = 6$ and $|V_1| + |V_2| \geq 5$, then $\frac{|V(D)|-|V_3|-2|V_4|+5}{2} \geq 2 = i_g(D)$ leads to a contradiction to Corollary 3.2. If $r = 1$ and $|V_4| \leq 3$ or $r = 1$, $|V_4| = 4$ and $|V_1| + |V_2| \geq 3$ or $r = 1$, $|V_4| = 5$ and $|V_1| + |V_2| \geq 4$, then $\frac{|V_1|+|V_2|-|V_4|+5}{2} \geq 2 = i_g(D)$, a contradiction to Corollary 3.2.

Summarizing our results, we see that, according to the assertion of this theorem, there remain to treat 14 partition-sequences.

Subcase 2.1. Since $i_g(D) = 2$, the partition-sequences $1,1,1,5$; $1,1,3,5$; $1,1,5,5$; $1,2,2,5$; $1,2,4,5$; $2,2,3,6$ and $2,2,5,6$ are impossible.

Subcase 2.2. Let $(n_i) = 1,1,2,5$. In this case, we obtain that $d^+(x) = d^-(x) = 4$, if $x \in V_1 \cup V_2$, $\{d^+(x), d^-(x)\} = \{3,4\}$, if $x \in V_3$, and $d^+(x) = d^-(x) = 2$, if $x \in V_5$, that means $i_l(D) = 1$. Since $\frac{|V_1|+|V_2|-|V_4|+6}{2} = \frac{3}{2} > i_l(D)$, it remains to consider the case that $Q_1 = \emptyset$ in Theorem 3.1. If $Y = V_4$, then we conclude that $|Z| \leq 3$ and thus $|Q_2| \geq 1$, a contradiction to $d^-(x) \leq 4$ for all $x \in V(D)$. If $|Y| = 4$, then it follows that $|Z| \leq 2$ and $|Q_2| \geq 2$. This implies that there is an arc $p \to q$ that is either in $E(D[Q_2])$ or in $E(D[Z])$. Since $d^-(x) = 2$ for all $x \in Y$, we conclude that $|Z| = 2$ and $Z \to Y$. If $pq \in E(D[Q_2])$, then $d^-(q) \geq 5$ and if $pq \in E(D[Z])$, then $d^+(p) \geq 5$, in both cases a contradiction. Hence, let $|Y| \leq 3 = |V_4| - 2$. Then because of $|V(D)| - 3|V_4| + 8 = 2 > i_l(D)$ we arrive at a contradiction to Theorem 3.1.

Subcase 2.3. Let $(n_i) = 1,1,4,5$. This yields that $d^+(x) = d^-(x) = 5$, if $x \in V_1 \cup V_2$, $\{d^+(x), d^-(x)\} = \{3,4\}$, if $x \in V_3$, and $d^+(x) = d^-(x) = 3$, if $x \in V_5$, that means $i_l(D) = 1$. Since $\frac{|V_1|+|V_2|-|V_4|+6}{2} = \frac{3}{2} > i_l(D)$, it remains to consider the case that $Q_1 = \emptyset$ in Theorem 3.1. If $Y = V_4$, then it follows that $|Z| \leq 3$ and thus $|Q_2| \geq 3$. This implies that there is a vertex $q_2 \in Q_2 \cap V_3$, since $Y \to Q_2$ a contradiction to $d^-(x) \leq 4$ for all $x \in V_3$. Consequently, let $|Y| \leq 4$. In this case, the fact that $|V(D)| - 3|V_4| + 6 = 2 > i_l(D)$ contradicts Theorem 3.1.

Subcase 2.4. Let $(n_i) = 1,2,5,5$. This yields that $d^+(x) = d^-(x) = 6$, if $x \in V_1$, $\{d^+(x), d^-(x)\} = \{5,6\}$, if $x \in V_2$, and $d^+(x) = d^-(x) = 4$, if $x \in V_3 \cup V_4$, that means $i_l(D) = 1$. Since $\frac{|V_1|+|V_2|-|V_4|+5}{2} = \frac{3}{2} > i_l(D)$ and $|V(D)| - 3|V_4| + 3 = 1 = i_l(D)$, we have a contradiction to Theorem 1.3.

Subcase 2.5. Let $(n_i) = 1,1,1,4$. In this case, we observe that $d^+(x) = d^-(x) = 3$, if $x \in V_1 \cup V_2 \cup V_3$, and $\{d^+(x), d^-(x)\} = \{1,2\}$, if $x \in V_4$, that means $i_l(D) = 1$. Because of $\frac{|V_1|+|V_2|-|V_4|+6}{2} = 2 > i_l(D)$, it remains to consider the case that $Q_1 = \emptyset$ in Theorem 3.1. If $Y = V_4$, then we conclude that $|Z| \leq 2$ and thus $|Q_2| \geq 1$, a contradiction to $d^-(x) \leq 3$ for all $x \in V(D)$. If $|Y| = 3$, then $|Z| \leq 1$ and $|Q_2| \geq 2$. Hence, there exists an arc $p \to q$ with $p, q \in Q_2$. Since $d^-(q) \geq 4$, we arrive at a contradiction. Consequently, let $|Y| \leq 2$. In this case, $|V(D)| - 3|V_4| + 8 = 3 > i_l(D)$ contradicts Theorem 3.1.

Subcase 2.6. Let $(n_i) = 1,1,4,4$. This yields $\{d^+(x), d^-(x)\} = \{4,5\}$, if $x \in V_1 \cup V_2$ and $d^+(x) = d^-(x) = 3$, if $x \in V_3 \cup V_4$, that means $i_l(D) = 1$. Since $\frac{|V_1|+|V_2|-|V_4|+5}{2} = \frac{3}{2} > i_l(D)$ and $|V(D)| - 3|V_4| + 3 = 1 = i_l(D)$, we arrive at a contradiction to Theorem 1.3.

Subcase 2.7. Let $(n_i) = 2,2,2,6$. In this case, we observe that $d^+(x) = d^-(x) = 5$, if $x \in V_1 \cup V_2 \cup V_3$, and $d^+(x) = d^-(x) = 3$, if $x \in V_4$, that means $i_l(D) = 0$. Because of $\frac{|V_1|+|V_2|-|V_4|+6}{2} = 2 > i_l(D)$, it remains to consider the case that $Q_1 = \emptyset$ in Theorem 3.1. If $Y = V_4$, then we conclude that $|Z| \leq 4$ and thus $|Q_2| \geq 2$. Since $Y \to Q_2$, we have a contradiction to $d^-(x) \leq 5$ for all $x \in V(D)$. If $|Y| = 5$, then it follows $|Z| \leq 3$ and $|Q_2| \geq 3$. This yields the existence of an arc $p \to q$ with $p, q \in Q_2$. Hence, $d^-(q) \geq 6$, a contradiction. Consequently, let $|Y| \leq 4$. But now $|V(D)| - 3|V_4| + 8 = 2 > i_l(D)$ contradicts Theorem 3.1.

Subcase 2.8. Let $(n_i) = 2,2,6,6$. This implies that $d^+(x) = d^-(x) = 7$ for all $x \in V_1 \cup V_2$ and $d^+(x) = d^-(x) = 5$ for all $x \in V_3 \cup V_4$, that means $i_l(D) = 0$. Since $\frac{|V_1|+|V_2|-|V_4|+5}{2} = \frac{3}{2} > i_l(D)$ and $|V(D)| - 3|V_4| + 3 = 1 > i_l(D)$, this is a contradiction to Theorem 1.3. □

For the partition-sequences 1, 1, 2, 4; 2, 2, 4, 6; 1, 2, 3, 5 and 1, 1, 3, 4 there are multipartite tournaments that do not have any Hamiltonian path as the following four examples demonstrate.

Example 4.5 *Let D be a 4-partite tournament with the partite sets $V_1 = \{u\}$, $V_2 = \{v\}$, $V_3 = \{x_1, x_2\}$ and $V_4 = \{y_1, y_2, y_3, y_4\}$ such that $(V_1 \cup V_2) \to x_1 \to V_4 \to x_2 \to (V_1 \cup V_2)$ and $u \to v \to y_4 \to u \to y_1 \to v \to y_3 \to u \to y_2 \to v$ (see Figure 2). Then $i_g(D) = 2$ and D has the partition-sequence 1, 1, 2, 4 but no Hamiltonian path.*

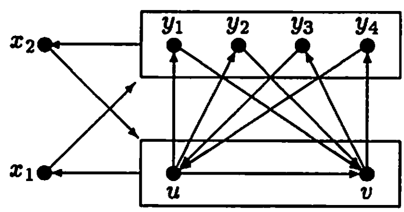

Figure 2: A 4-partite tournament D with $i_g(D) = 2$ and the partition-sequence 1, 1, 2, 4 that does not contain a Hamiltonian path

Example 4.6 *Let D be a 4-partite tournament with the partite sets $V_1 = \{u_1, u_2\}$, $V_2 = \{v_1, v_2\}$, $V_3 = \{x_1, x_2, x_3, x_4\}$ and $V_4 = \{y_1, y_2, y_3, y_4, y_5, y_6\}$ such that $V_4 \to \{x_3, x_4\} \to (V_1 \cup V_2) \to \{x_1, x_2\} \to V_4$, $u_1 \to v_1 \to u_2 \to v_2 \to u_1$ and $\{y_1, y_2, y_3\} \to \{u_2, v_2\} \to \{y_4, y_5, y_6\} \to \{u_1, v_1\} \to \{y_1, y_2, y_3\}$ (see Figure 3). Then D is a 4-partite tournament with $i_g(D) =$*

2 and the partition-sequence $2, 2, 4, 6$ *that does not contain any Hamiltonian path.*

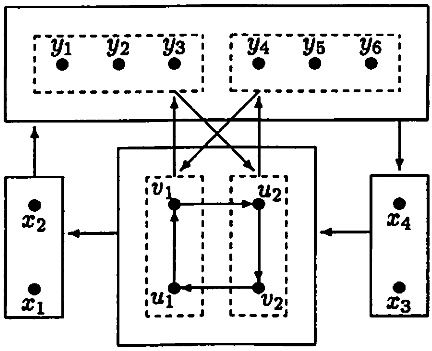

Figure 3: A 4-partite tournament D with $i_g(D) = 2$ and the partition-sequence $2, 2, 4, 6$ that does not contain a Hamiltonian path

Example 4.7 *Let D be a 4-partite tournament with the partite sets $V_1 = \{u\}$, $V_2 = \{v_1, v_2\}$, $V_3 = \{x_1, x_2, x_3\}$ and $V_4 = \{y_1, y_2, y_3, y_4, y_5\}$ such that $V_4 \to \{x_1, x_2\} \to (V_1 \cup V_2) \to x_3 \to V_4$, $V_2 \to V_1$, $\{v_2, u\} \to \{y_1, y_2\} \to v_1$, $\{v_1, u\} \to \{y_3, y_4\} \to v_2$ and $V_2 \to y_5 \to V_1$ (see Figure 4). Then D is a 4-partite tournament with $i_g(D) = 2$ and the partition-sequence $1, 2, 3, 5$ that does not contain a Hamiltonian path.*

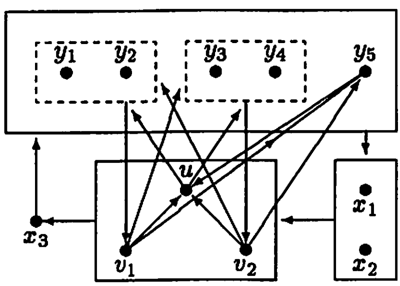

Figure 4: A 4-partite tournament D with $i_g(D) = 2$ and the partition-sequence $1, 2, 3, 5$ that does not contain a Hamiltonian path

Example 4.8 *Let D be a 4-partite tournament with the partite sets $V_1 = \{u\}$, $V_2 = \{v\}$, $V_3 = \{x_1, x_2, x_3\}$ and $V_4 = \{y_1, y_2, y_3, y_4\}$ such that $V_4 \to \{x_1, x_2\} \to (V_1 \cup V_2) \to x_3 \to V_4$, $u \to v \to \{y_3, y_4\} \to u$, $u \to \{y_1, y_2\}$ and $y_1 \to v \to y_2$ (see Figure 5). Then D is a 4-partite tournament with $i_g(D) = 2$ and the partition-sequence $1, 1, 3, 4$ that does not contain any Hamiltonian path.*

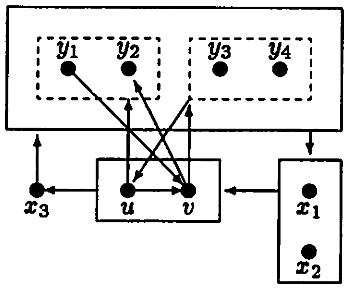

Figure 5: A 4-partite tournament D with $i_g(D) = 2$ and the partition-sequence $1, 1, 3, 4$ that does not contain a Hamiltonian path

In the case that $c = 2$ or $c = 3$ and $i_g(D) \geq 2$, there are infinitely many digraphs D that do not contain any Hamiltonian path as we can see in the following example.

Example 4.9 *Let D be a bipartite tournament with $i_g(D) \geq 2$. If V_1, V_2 are the partite sets of D such that $|V_1| + 2 \leq |V_2| \leq |V_1| + 2i_g(D)$, then clearly, D does not contain a Hamiltonian path.*

Now, let D be a 3-partite tournament with the partite sets V_1, V_2, V_3. If $|V_1| = |V_2| = r$ and $|V_3| = r + i_g(D)$ with $i_g(D) \geq 2$ such that $V_1 \to V_2 \to V_3 \to V_1$, then D does not contain a Hamiltonian path.

All the presented examples show that Theorem 4.4 is best possible. Combining Corollary 4.2 with Example 4.9, we observe the following.

Corollary 4.10 *Let $i \geq 2$ be an arbitrary integer. Then all, except a finite number, of c-partite tournaments with $i_g \leq i$ and $c \geq 4$ have a Hamiltonian path. Furthermore the bound $c \geq 4$ is best possible.*

Proof. If $r \geq 4i - 5$, then Corollary 4.2 yields that a c-partite tournament D with $c \geq 4$ and at least r vertices in each partite set contains a Hamiltonian path. Because of Lemma 2.1 there are only finitely many c-partite tournaments with $i_g \leq i$ and at most $4i - 6$ vertices in one partite set. Thus, the first part of this corollary is proved.

Example 4.9 demonstrates that there are infinitely many 3-partite tournaments with $i_g \geq 2$ that do not contain any Hamiltonian path and the proof of this corollary is complete. □

An interesting extension to our results would be the solution of the following open problem.

Problem 4.11 *For all i find the smallest value, $g(i)$, with the property that all c-partite tournaments with $i_g \leq i$ and $c \geq g(i)$ have a Hamiltonian path.*

According to the Theorems 1.1, 4.3 and 4.4, it is already shown that $g(0) = 1$, $g(1) = 3$ and $g(2) = 5$.

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References

[1] J. Bang-Jensen, G. Gutin, *Digraphs: Theory, Algorithms and Applications*, Springer, London, 2000.

[2] J. Bang-Jensen, G. Gutin, J. Huang, A sufficient condition for a semicomplete multipartite digraph to be Hamiltonian, *Discrete Math.* **161**, (1996), 1-12.

[3] J. Bang-Jensen, G. Gutin, A. Yeo, On k-strong and k-cyclic digraphs, *Discrete Math.* **162**, (1996), 1-11.

[4] Y. Guo, M. Tewes, L. Volkmann, A. Yeo, Sufficient conditions for semicomplete multipartite digraphs to be Hamiltonian, *Discrete Math.* **212**, (2000), 91-100.

[5] G. Gutin, Characterization of complete n-partite digraphs that have a Hamiltonian path, *Kibernetica* **1**, (1988), 107-108 (in Russian).

[6] G. Gutin, Cycles and paths in semicomplete multipartite digraphs, theorems and algorithms: a survey, *J. Graph Theory* **19**, (1995), 481-505.

[7] G. Gutin, A. Yeo, Note on the path covering number of a semicomplete multipartite digraph, *J. Comb. Math. Comb. Comput.* **32**, (2000), 231-237.

[8] B. Jackson, Long paths and cycles in oriented graphs, *J. Graph Theory* **5**, (1981), 145-157.

[9] M. Tewes, L. Volkmann, A. Yeo, Almost all almost regular c-partite tournaments with $c \geq 5$ are vertex pancyclic, *Discrete Math.* **242** (2002), 201-228.

[10] L. Volkmann, Cycles in multipartite tournaments: results and problems, *Discrete Math.* **245**, (2002), 19-53.

[11] L. Volkmann, Hamiltonian paths containing a given arc, in almost regular bipartite tournaments, submitted.

[12] L. Volkmann, A. Yeo, Hamiltonian paths, containing a given path or collection of arcs, in close to regular multipartite tournaments, *Discrete Math.*, to appear.

[13] A. Yeo, One-diregular subgraphs in semicomplete multipartite digraphs, *J. Graph Theory* **24**, (1997), 175-185.

[14] A. Yeo, How close to regular must a semicomplete multipartite digraph be to secure Hamiltonicity? *Graphs Combin.* **15**, (1999), 481-493.

[15] C.-Q. Zhang, Hamilton paths in multipartite oriented graphs, *Ann. Discrete Math.* **41**, (1989), 499-514.