A Lower Bound for the Domination Number of Complete Grid Graphs

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Abstract

We use a dynamic programming algorithm to establish a lower bound on the domination number of complete grid graphs $G_{m,n}$. The bound is within 5 of a known upper bound that has been conjectured to be the exact domination number of the complete grid graphs.

keywords: grid graph, domination number

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1 Introduction

A dominating set S for a graph G is a subset of the vertices of G that either contains or is adjacent to every vertex of G. The domination number of G, $\gamma(G)$, is the minimum size of a dominating set, or equivalently the minimum, over all dominating sets S, of $\sum_{v\in S} 1$.

Let P_n denote the path on n vertices; the complete grid graph $G_{m,n}$ is the product $P_m \times P_n$. Fisher [3] has used a dynamic programming algorithm to compute $\gamma(G_{m,n})$ for $m \leq 21$ and all n. In particular, when $16 \leq m \leq 21$, he found that

$$\gamma(G_{m,n}) = \gamma_U \doteq \left\lfloor \frac{(m+2)(n+2)}{5} \right\rfloor - 4.$$

Chang [1] showed that γ_U is an upper bound for $\gamma(G_{m,n})$ when $\min(m,n) \geq 8$, and conjectured that it gives $\gamma(G_{m,n})$ exactly when m and n are large enough. Fisher conjectured that in fact γ_U is the correct value for $\gamma(G_{m,n})$ when $\min(m,n) \geq 16$.

Cockayne, Hare, Hedetniemi, and Wimer [2] showed that $\gamma(G_{n,n}) \geq (n^2+n-3)/5$ when $n \geq 4$. This is substantially smaller than the conjectured value of γ for large n. Other than this result, we know of no other lower bounds for $\gamma(G_{m,n})$. By using a similar dynamic programming algorithm, we establish a lower bound γ_L for all $m, n \geq 22$ that is within 5 of the upper bound, that is, $\gamma_U - \gamma_L \leq 5$.

2 Getting a lower bound

A vertex in the graph $G_{m,n}$ dominates at most five vertices, including itself, so certainly $\gamma(G_{m,n}) \geq nm/5$. If we could keep the sets dominated by individual vertices from overlapping, we could get a dominating set with approximately $nm/5$ vertices, and indeed we can arrange this for 'most' of the graph, as shown in Figure 1. At the edges we are forced to overlap some of the sets dominated by individual vertices, and also to use some vertices with degree less than four.

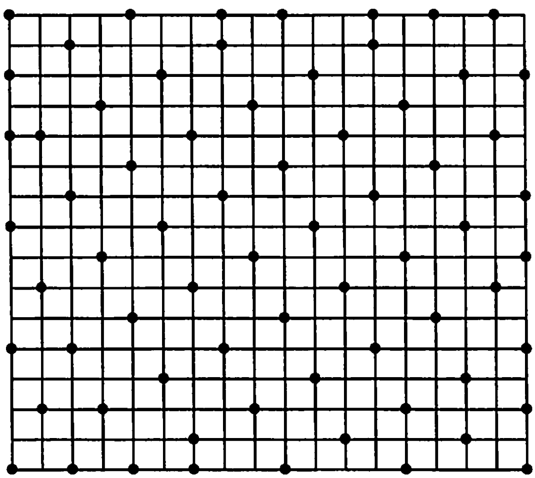

Figure 1: The 16×18 complete grid graph has domination number 68.

Suppose S is a subset of the vertices of G. Let $N[S]$ be the set of vertices that are either in S or adjacent to a member of S, that is, the vertices dominated by S. Define the ***wasted domination*** of S as $w(S) = 5|S| - |N[S]|$, that is, the number of vertices we could dominate with $|S|$ vertices in the best case, less the number actually dominated. When S is a dominating set, $|N[S]| = mn$, and if $w(S) \geq L$ then $|S| \geq (L + mn)/5$. Our goal now is to find a good lower bound L for $w(S)$.

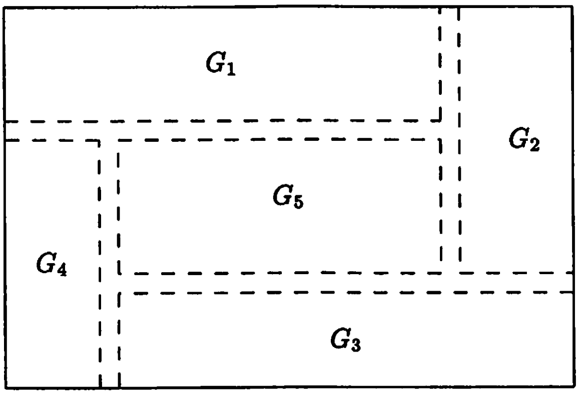

Figure 2: Partitioned complete grid graph.

Suppose a complete grid graph G is partitioned into five subgraphs as shown in Figure 2 and that S is a dominating set for G. Let $S_k = S \cap V(G_k)$. Then

$$w(S) \geq \sum_{k=1}^{5} w(S_k) \geq \sum_{k=1}^{4} w(S_k).$$

Note that in computing $w(S_k)$ we consider S_k to be a subset of $V(G)$, not of $V(G_k)$ (this affects the computation of $N[S_k]$). Since we expect that $w(S_5) = 0$, we aren't giving up anything in the second inequality, but the first might well be strict, so it may be too much to hope that this technique will close the gap between the upper and lower bounds.

Note that G_k is a complete grid graph and that S_k is a set that dominates all the vertices of G_k except possibly the vertices in one row and one column on the boundary. Let us say that a set that dominates a complete grid graph G, except possibly the vertices in the bottom row and the rightmost column, *almost dominates* G. Suppose $H = G_{i,j}$; what we would like to know is the value of

$$\min_{A} w(A),$$

taking the minimum over sets A that almost dominate H and computing $w(A)$ as if A were a subset of a larger grid, in which H occupies the northwest corner. If we can compute this minimum for fixed i and any j, we can choose G_1 through G_4 with width i and get lower bounds on $w(S_k)$ for any dominating set S of the original $G_{m,n}$. It seems possible that the minimum value of $w(A)$ over almost dominating sets A could be strictly smaller than the minimum value of $w(S_k)$, again suggesting that we may not be able to close the gap between the upper and lower bounds. Ideally, we want to choose i so that the resulting lower bound is close to the upper bound, i is small enough to give us a bound on $\gamma(G_{m,n})$ for $m, n \geq 22$, and computing the minimum is computationally feasible. Using $i = 10$ satisfies all of these criteria, giving a small constant difference between the upper and lower bounds.

3 The algorithm

We now describe an algorithm based closely on one in Hare, Hedetniemi, and Hare [5] and Fisher [3]. Imagine a complete grid graph $G_{m,n}$ with a designated subset S of the vertices, as in Figure 1. Number the vertices in $G_{m,n}$ in the obvious way: $v_{i,j}$, $1 \leq i \leq m$, $1 \leq j \leq n$. We describe a column, say column number i, in such a diagram by a state vector $\mathbf{s}$, in which $\mathbf{s}_j$ is 0 if vertex $v_{i,j}$ on the path is in S, 1 if vertex $v_{i,j}$ is adjacent to a member of S in column i or column $i-1$, and 2 otherwise. For example, the second column from the right in Figure 1 has state vector $(2,1,2,1,2,1,0,1,1,2,1,0,1,1,1,0)$; note that no state vector can ever have a 0 adjacent to a 2. Let $|\mathbf{s}|$ denote the number of zeros in $\mathbf{s}$.

An $\mathbf{s}$-*almost-domination* of $G_{m,n}$ is a subset S of the vertices that dominates the first $n-1$ columns, except possibly vertices in the first (i.e., bottom) row, and for which the state vector of the final column is $\mathbf{s}$. Suppose S is a subset of the vertices of $G_{i,j}$ and denote by $w_{i,j}(S)$ the value of $w(S)$ computed in $G_{i+1,j+1}$, in which $G_{i,j}$ occupies the northwest corner. Let

$$w_{i,j}(\mathbf{s}) = \min_{S} w_{i,j}(S),$$

taking the minimum over all s-almost-dominations of $G_{i,j}$. If there is no s-almost-domination of $G_{i,j}$, let $w_{i,j}(\mathbf{s}) = \infty$.

Let $\mathcal{P}(\mathbf{s})$ be the set of state vectors $\mathbf{t}$ such that $\mathbf{t}$ is the state vector of the next to last column in an s-almost-domination of a complete grid graph. Then

$$w_{m,n}(\mathbf{s}) = \min_{\mathbf{t} \in \mathcal{P}(\mathbf{s})} \left(5|s| - \mathrm{nd}(\mathbf{t}, \mathbf{s}) + w_{m,n-1}(\mathbf{t})\right),$$

where $\mathrm{nd}(\mathbf{t}, \mathbf{s})$, the number of newly dominated vertices, may be computed as follows.

1. $\mathrm{nd} = 0$
2. If $n > 1$, for each $j = 1, \ldots, m$ for which $\mathbf{s}_j = 0$ and $\mathbf{t}_j = 2$, add 1 to nd. This counts the newly dominated vertices $v_{i-1,j}$.

3. For each $j = 1, \ldots, m$ for which $\mathbf{s}_j \leq 1$ and either $n = 1$ or $\mathbf{t}_j \geq 1$, add 1 to nd. This counts the newly dominated vertices $v_{i,j}$.
4. For each $j = 1, \ldots, m$ for which $\mathbf{s}_j = 0$, add 1 to nd. This counts the newly dominated vertices $v_{i,j+1}$.

Now the algorithm to compute $\min_S w_{i,j}(S)$, $j = 1, \ldots, N$, is:

1. **Initialization.** Set $w_{m,0}(s) = 0$ if $\mathbf{s} = \mathbf{1}$, and ∞ otherwise.
2. **Iteration.** Suppose that $n \leq N$ and that $w_{m,n-1}(\mathbf{s})$ has been computed for all $\mathbf{s}$. Then set
$$w_{m,n}(\mathbf{s}) = \min_{\mathbf{t} \in \mathcal{P}(\mathbf{s})} (5|s| - \mathrm{nd}(\mathbf{t}, \mathbf{s}) + w_{m,n-1}(\mathbf{t})) .$$
3. The final values of interest are $\min_S w_{i,j}(S) = \min_{\mathbf{s}} w_{i,j}(\mathbf{s})$, $j = 1, \ldots, N$.

Of course, this does not quite solve our problem: knowing $\min w_{i,j}(S)$ only for $j \leq N$ is not enough to give us a lower bound on $G_{m,n}$ for all m and n. Livingston and Stout [6] and Fisher [4] independently thought of looking for periodicity in the values of $\gamma(G_{m,n})$ for fixed m. Since they succeeded, we might hope that for fixed i, there are N, p, and q so that for $j \geq N$ and all $\mathbf{s}$
$$w_{i,j}(\mathbf{s}) = w_{i,j-p}(\mathbf{s}) + q.$$
In this case, after a finite amount of computation, we could determine $\min_{\mathbf{s}} w_{i,j}(\mathbf{s})$ for all j.

It is easy to modify the program so to check for this periodicity. When we do this, we find that for $j \geq 22$, $\min_S w_{10,j}(S) = j - 1$, taking the minimum over all S that almost dominate $G_{10,j}$. Thus, for $m, n \geq 32$, if S is a dominating set in $G_{m,n}$, $w(S) \geq 2(m - 10 - 1) + 2(n - 10 - 1)$ and $|S| \geq \gamma_L = \lceil (mn + 2m + 2n - 44)/5 \rceil$. The difference $\gamma_U - \gamma_L$ depends only on the values of m and n modulo 5; computing all possible differences, we find that all the differences are 4 or 5. Thus, when m and n are at least 32, the lower bound is within 5 of the upper bound. It remains to check the difference when at least one of m and n is between 22 and 31; for this, it suffices to check all m between 22 and 31, and for each such m, all n between 22 and 36. When we do this, we find that all differences are 3, 4, or 5.

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