On complete caps in the projective geometries over $\mathbb{F}_3$. II: New improvements

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Abstract

Hill, Landjev, Jones, Storme and Barát proved in a previous article on caps in PG(5,3) and PG(6,3) that every 53-cap in PG(5,3) is contained in the 56-cap of Hill and that there exist complete 48-caps in PG(5,3). The first result was used to lower the upper bound on $m_2(6,3)$ on the size of caps in PG(6,3) from 164 to 154. Presently, the known upper bound on $m_2(6,3)$ is 148. In this article, using computer searches, we prove that every 49-cap in PG(5,3) is contained in a 56-cap, and that every 48-cap, having a 20-hyperplane with at most 8-solids, is also contained in a 56-cap. Computer searches for caps in PG(6,3) which use the computer results of PG(5,3) then lower the upper bound on $m_2(6,3)$ to $m_2(6,3) \leq 136$. So now we know that $112 \leq m_2(6,3) \leq 136$.

1 Introduction

An n-cap in the projective space PG(N,q) of dimension N over the finite field of order q is a set of n points, no three of which are collinear. A cap is called *complete* when it is not contained in a larger cap of the same projective space. The largest size of caps in PG(N,q) is denoted by $m_2(N,q)$. The size of the second largest complete cap is denoted by $m'_2(N,q)$. Thus any n-cap with $n > m'_2(N,q)$ can be extended to a cap of size $m_2(N,q)$.

Presently, only the following exact values of $m_2(N,q)$ are known. In PG$(2,q)$, q odd, there are at most $(q+1)$-caps [5]. In PG$(2,q)$, q even, there are at most $(q+2)$-caps [5]. In PG$(3,q)$, $q>2$, the maximal size of a cap is q^2+1 [5, 29]. And in PG$(N,2)$, the maximal size of a cap is 2^N [5].

In some cases, a complete characterization is known. Namely, in PG$(2,q)$, q odd, every $(q+1)$-cap is a conic [30, 31]. In PG$(2,q)$, q even, $q \geq 16$, distinct types of $(q+2)$-caps exist; see [24] for a list of the known infinite

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classes of $(q+2)$-caps. In $PG(3,q)$, q odd, every (q^2+1)-cap is an elliptic quadric [2, 27]. In $PG(3,q)$, $q=2^h$, h odd, $h \geq 3$, as well as the elliptic quadric, at least one other type of (q^2+1)-cap exists, called the *Tits ovoid* [35]. In $PG(N,2)$, every 2^N-cap is the complement of a hyperplane [32].

Apart from these results which are valid either for arbitrary q or for arbitrary dimension N, only some other sporadic results are known. Namely, the maximal size of a cap in $PG(4,3)$ is 20 [28], the maximal size of a cap in $PG(5,3)$ is 56 [16], and the maximal size of a cap in $PG(4,4)$ is 41 [9].

Regarding the characterizations, exactly 9 types of 20-caps exist in $PG(4,3)$ [18]. The 56-cap in $PG(5,3)$ is projectively unique [17]. And there are exactly 2 distinct types of 41-caps in $PG(4,4)$ [8].

In the other cases, only upper bounds on the sizes of caps in $PG(N,q)$ are known. We refer to [24] for a list of the known results. We also wish to state the following result of Bierbrauer and Edel [3] which improves the Meshulam upper bound on the size of caps in $AG(N,q)$, q odd [26].

Theorem 1.1 *Let $Q=q^h, q>2$ and $N \geq 4$. Then the size of a cap in $AG(N,Q)$ is upper bounded by*

$$\frac{(Nh+1)Q^N}{(Nh)^2}.$$

Using the known maximal size 45 for a cap in $AG(5,3)$ [11], the upper bound of [3, Theorem 2] gives 114 as upper bound for the size of a cap in $AG(6,3)$ and, then again using [3, Theorem 2] gives 296 for the size of a cap in $AG(7,3)$. The existence of a 149-cap in $PG(6,3)$ would imply the existence of a 298-cap in $AG(7,3)$. This latter cap can be obtained by constructing a cone with vertex P in $PG(7,3)$ and with base a 149-cap in a hyperplane of $PG(7,3)$ skew to P, and then by selecting two points, different from P, on each line of this cone, such that there is a hyperplane skew to all 298 points.

There exists a 112-cap in $PG(6,3)$ obtained by constructing a cone over the 56-cap in $PG(5,3)$, and by taking two points different from the vertex on every line of the cone. So presently,

$$112 \leq m_2(6,3) \leq 148.$$

We will lower the upper bound to 136 by using computer searches and geometrical arguments.

One of the computer search results we rely on is the determination of the value $m_2'(5,3)$ of the size of the second largest complete caps in $PG(5,3)$. We will show in Theorem 3.8 that

$$m_2'(5,3)=48.$$

The following tables show for small values of q and N the known values of $m_2'(N,q)$. Table 1 is [24, Table 2.4]. For the exact references for Table 1, we refer to [24, Table 2.4].

q	7	8	9	11	13	16	17	19	23	25	27	29
$m'(2,q)$	6	6	8	10	12	13	14	14	17	21	22	24

Table 1: $m'(2,q)$ in small planes

N \ q	3	4	5	7
3	8	14	20	32
4	19	40		
5	48			

Table 2: $m_2'(N,q)$

For the values of Table 2, we refer to [13] for $(N,q) = (3,3)$, [22] for $(N,q) = (3,4)$, [1] for $(N,q) = (3,5)$, [12] for $(N,q) = (3,7)$, [34] for $(N,q) = (4,3)$, and [10] for $(N,q) = (4,4)$.

Apart from these results, it is also known that

(1) $m_2'(2,2^{2h}) = 2^{2h} - 2^h + 1$ for $h > 1$ [4, 14, 25],

(2) $m_2'(N,2) = 2^{N-1} + 2^{N-3}$, $N \geq 3$ [7].

An i-solid or an i-hyperplane with respect to a cap K in $PG(N,q)$ is a 3-dimensional space or hyperplane intersecting K in exactly i points. An (in)complete i-solid or i-hyperplane is an i-solid or i-hyperplane intersecting K in an (in)complete i-cap.

2 Known results on caps in $PG(4,3)$ and $PG(5,3)$

2.1 The 20-caps in $PG(4,3)$

It is known that the maximal size of a cap in $PG(4,3)$ is equal to 20 [28], and that there are exactly 9 types of 20-caps in $PG(4,3)$ [18]. In [18], these caps are called the caps $\Gamma_1,\dots,\Gamma_8$ and Δ. The caps $\Gamma_1,\dots,\Gamma_8$ are all contained in a cone $R\,Q^-(3,3)$, with vertex a point R and base a 3-dimensional elliptic quadric $Q^-(3,3)$, while the cap Δ is contained in a non-singular quadric $Q(4,3)$.

The 20-caps Γ_1 and Δ are the only two 20-caps having at most 8-solids. Moreover, every solid intersects Γ_1 or Δ in 2, 5 or 8 points. The spectrum, i.e. the tuple of the numbers n_i of i-hyperplanes, of both these 20-caps is equal to $(n_2, n_5, n_8) = (10, 36, 75)$.

The caps Γ_2 and Γ_8 are the only 20-caps having 9-solids, but no 10-solids.

2.2 Caps in PG(5,3)

The maximal size of a cap in PG(5,3) is equal to 56. There exists a projectively unique 56-cap in PG(5,3). This 56-cap is contained in a non-singular elliptic quadric $Q^-(5,3)$ of PG(5,3), and only has 11- and 20-hyperplanes. The 11-hyperplanes are the tangent cones to $Q^-(5,3)$ of the points of the 56-cap. This cap also does not have 9- or 10-solids, but does have 8-solids.

The 56-cap has the additional property of intersecting every line of $Q^-(5,3)$ in exactly two points, i.e. the 56-cap is a hemisystem of $Q^-(5,3)$ [6, 33]. In fact, $Q^-(5,3)$ can be described as the union of two disjoint 56-caps. The 56-cap is stabilized by a group of size $56 \cdot 10 \cdot 9 \cdot 8$ acting in three orbits onto PG(5,3). The three orbits consist of the 56-cap on $Q^-(5,3)$, the 56-cap which is its complement on $Q^-(5,3)$, and of $\mathrm{PG}(5,3) \setminus Q^-(5,3)$ [17].

Before [20], few results on other large caps in PG(5,3) were known.

In [20], it was proven that

(1) every 53-, 54-, 55-cap in PG(5,3) is contained in a 56-cap of PG(5,3),
(2) every 52-cap, having at most 8-solids, is contained in a 56-cap of PG(5,3), and
(3) there exists a complete 48-cap in PG(5,3).

Hence, the caps discussed in (1) and (2) are subsets of non-singular elliptic quadrics of PG(5,3).

2.3 Projections of the 56-cap in PG(5,3)

Suppose that δ and σ are disjoint subspaces of $\mathrm{PG}(N,q)$ of dimensions i and j respectively, with $i+j = N-1$. The *projection* $\varphi = \varphi_{\delta,\sigma}$ from δ onto σ is the mapping from $\mathrm{PG}(N,q) \setminus \delta$ onto σ defined by

$$\varphi : \mathrm{PG}(N,q) \setminus \delta \to \sigma : Q \mapsto \langle \delta, Q \rangle \cap \sigma$$

where $\langle \delta, Q \rangle$ is the $(i+1)$-dimensional subspace generated by δ and Q. Note that φ maps $(i+1)$-dimensional subspaces containing δ onto points of σ and maps $(i+2)$-dimensional subspaces containing δ onto lines of σ.

Now suppose K is a subset of $PG(N,q)$. If P is a point of σ, we define $\mu_{\varphi,K}(P)$ to be the cardinality of the set $\{Q \in K \setminus \delta : \varphi(Q) = P\}$. If φ and K are known, we write $\mu(P)$ instead of $\mu_{\varphi,K}(P)$. If a line ℓ in σ consists of the points $P_0, \ldots, P_q$, then we call the $(q+1)$-tuple $(\mu(P_0), \ldots, \mu(P_q))$ the *type* of ℓ (with respect to φ and K). The points $P_0, \ldots, P_q$ will usually be ordered so that $\mu(P_0) \geq \mu(P_1) \geq \cdots \geq \mu(P_q)$. We also define $\mu(\ell)$ to be $\sum_{i=0}^{q} \mu(P_i)$.

In particular, we will project the 56-cap in $PG(5,3)$ from a 2-solid δ onto a line ℓ. Since the hyperplanes of $PG(5,3)$ intersect the 56-cap in either 20 or 11 points, the type of ℓ with respect to δ and this 56-cap is necessarily equal to $(18,18,9,9)$.

3 Computer results in $PG(5,3)$

We first describe the computer searches which have shown that

(1) $m_2'(5,3) = 48$, so every 49-cap in $PG(5,3)$ is contained in a 56-cap of $PG(5,3)$, and

(2) the two 20-caps in $PG(4,3)$ of type Γ_1 and Δ, which have at most 8-solids, extend to a 56-cap of $PG(5,3)$, or maximally to a complete 47-cap of $PG(5,3)$.

First of all, it is known from [19] that every cap of size at least 47 in $PG(5,3)$ contains at least one 19- or 20-hyperplane.

3.1 Caps with 20-hyperplanes

A first sequence of computer searches concerned searches for caps containing hyperplanes with 20 points of the cap. These searches showed

Theorem 3.1 (1) *The* 20-*caps* Δ *and* Γ_1 *in* $PG(4,3)$ *(Subsection* 2.1*), having at most* 8-*solids, extend either to the* 56-*cap of Hill, or maximally to a complete* 47-*cap.*

(2) *The* 20-*caps in* $PG(4,3)$ *having* 9-*solids and/or* 10-*solids are not contained in a complete* 49-, 50-, 51-, *or* 52-*cap of* $PG(5,3)$.

(3) *A complete* 48-*cap in* $PG(5,3)$ *having* 20-*hyperplanes, but also at least one* 19-*hyperplane with at most* 8-*solids in this* 19-*hyperplane, is contained in a quadric.*

(4) *For a* 49-*cap in* $PG(5,3)$ *having at least one* 20-*hyperplane, there is no hyperplane intersecting this* 49-*cap in a complete* 19-*cap.*

3.2 Caps without 20-hyperplanes

We now consider caps of size at least 49, having at most 19-hyperplanes.

The following lemma shows that every cap of size 50 has at least one 20-hyperplane, unless it has at most 8-solids.

Lemma 3.2 *A 50-cap K in PG(5,3) having 9- and/or 10-solids has at least one 20-hyperplane.*

Proof: A solid is contained in four hyperplanes. If the solid is a 9-solid and there are no 20-hyperplanes through it, then $|K| \leq 9 + 4 \cdot 10 < 50$, a contradiction. The same argument can be used for a 10-solid. □

Caps of size 49 or larger having at most 19-hyperplanes and at most 8-solids will be studied in Theorem 3.4, so from now on, we concentrate on 49-caps having at most 19-hyperplanes and at least 9-solids.

Consider a 49-cap K having at most 19-hyperplanes. Such a 49-cap has at most 9-solids since if Π were a 10-solid, then there would be a hyperplane through Π containing at least $10 + 39/4 > 19$ points of K.

Regarding the different types of 19-caps in PG(4,3) having at most 9-solids, a computer search showed that the following results are valid. We note that Part (1) of the following theorem also was found by van Eupen and Lisonek [36, Lemma 22].

Theorem 3.3 (1) *There exist exactly two types of 19-caps in PG(4,3) having at most 8-solids. These 19-caps are incomplete.*

(2) *There exist exactly two types of complete 19-caps in PG(4,3) having 9-solids, but no 10-solids.*

(3) *There exist exactly six types of incomplete 19-caps in PG(4,3) having 9-solids, but no 10-solids.*

The computer searches involving large caps in PG(5,3) having at most 19-hyperplanes gave the following results.

Theorem 3.4 (1) *One of the incomplete 19-caps with at most 8-solids is not contained in a complete 48-, 49-, 50-, 51-, or 52-cap of PG(5,3).*

The other incomplete 19-cap, with at most 8-solids, is contained in a complete 48-cap, but not in a complete 49-, 50-, 51-, or 52-cap of PG(5,3).

Moreover, every complete 48-cap, having at most 19-hyperplanes and having a 19-hyperplane with at most 8-solids, is contained in a quadric of PG(5,3).

(2) *The two complete 19-caps in PG(4,3), having 9-solids but no 10-solids, are not contained in 49-caps of PG(5,3) which have at most 19-hyperplanes and at most 9-solids.*

(3) *The six incomplete* 19-*caps in* PG(4, 3), *having* 9-*solids but no* 10-*solids, are not contained in* 48- *and* 49-*caps containing* 9-*solids and incomplete* 19-*hyperplanes, but not having* 10-*solids nor* 20-*hyperplanes nor complete* 19-*hyperplanes.*

Combining the results of Theorem 3.1 (3), (4) with Theorem 3.4 (1), (2), the following corollary is obtained.

Corollary 3.5 (1) *Every complete* 48-*cap having a* 19-*hyperplane with at most* 8-*solids in this hyperplane is contained in a quadric.*

(2) *The complete* 19-*caps of* PG(4, 3) *are not contained in* 49-*caps of* PG(5, 3).

For the computer searches of Theorem 3.4 (3), we remark that we can put a large part of the 48- or 49-cap onto a quadric cone or a non-singular elliptic quadric $Q^-(5,3)$. This simplified the computer searches greatly.

We explain these ideas in the following subsection. A similar idea will be used for the computer searches in PG(6, 3) (Section 6).

3.3 Description of computer searches

We wish to describe the ideas we have used for the computer searches for caps of size 48 and 49 in PG(5, 3) containing

1. incomplete 19-hyperplanes having 9-solids, but
2. not having 10-solids nor 20-hyperplanes nor complete 19-hyperplanes.

We explain the ideas for the corresponding 48-caps K in PG(5, 3).

Consider a 9-solid Π and let $H_0, \ldots, H_3$ be the four hyperplanes through Π. Then we can assume that $|K \cap H_0| = |K \cap H_1| = |K \cap H_2| = 19$ and $|K \cap H_3| = 18$. From the assumptions, we know that the 19-caps $K \cap H_0$ and $K \cap H_1$ are incomplete, so contained in 20-caps, so contained in quadrics Q_0 and Q_1 of H_0 and H_1.

From [18] and Subsection 2.1, we know that these quadrics Q_0 and Q_1 are elliptic quadric cones. The two quadrics Q_0 and Q_1 intersect in Π in an elliptic quadric since a 9-cap in PG(3, 3) lies on a unique elliptic quadric [21, p. 104].

Since Q_0 and Q_1 are 4-dimensional elliptic quadric cones in two hyperplanes H_0 and H_1, intersecting in a 3-dimensional elliptic quadric lying in $H_0 \cap H_1$, they define a pencil of quadrics. This latter pencil of quadrics consists of the four quadrics intersecting in the variety $Q_0 \cup Q_1$ of degree four. One of the four quadrics in this pencil is the union $H_0 \cup H_1$. There are 19 points of K not lying in $H_0 \cup H_1$; so one of the three other quadrics in this pencil contains at least $19 + 10 + 19/3 > 35$ points of K, where we first counted the points of K in H_0 and H_1. So there is a quadric Q in PG(5, 3) containing at least 36 points of the 48-cap K. We show that this latter quadric is either a non-singular elliptic quadric or a quadric $\ell\, Q^-(3,3)$ with vertex a line ℓ and base a 3-dimensional elliptic quadric $Q^-(3,3)$ in a solid skew to ℓ.

Lemma 3.6 *The quadric Q containing at least 36 points of the 48-cap K is either a non-singular elliptic quadric or a quadric $\ell\, Q^-(3,3)$ with vertex line ℓ and with base a 3-dimensional elliptic quadric $Q^-(3,3)$.*

Proof: Assume Q is not a non-singular elliptic quadric.

The quadric Q cannot be a non-singular hyperbolic quadric since Q_0 is a cone over an elliptic quadric.

Also, Q cannot be a quadric cone $P\, Q(4,3)$ with vertex P, for otherwise $P \in H_0 \cap H_1$. If, for instance, $P \notin H_0$, then $Q \cap H_0$ would be a non-singular quadric $Q(4,3)$, and we know that $Q \cap H_0$ is an elliptic quadric cone.

So, $P \in H_0 \cap H_1$, but then $Q \cap (H_0 \cap H_1)$ cannot be an elliptic quadric.

The possibility that Q is a cone $\ell\, Q^+(3,3)$ with vertex a line ℓ and base a hyperbolic quadric cannot occur since such a quadric can be described as the union of 4 solids and this contradicts the fact that H_0 intersects Q in an elliptic quadric cone. The cases that Q is a singular quadric having a vertex of dimension larger than one need not be considered since H_0 intersects Q in an elliptic quadric cone with a point vertex. □

We now show that it is possible to select the 19-cap on Q in H_0 without losing generality.

Lemma 3.7 *The 19-cap in $K \cap H_0$ on the quadric Q can be selected without losing generality.*

Proof: We present the proof for $Q = Q^-(5,3)$; the other case is treated analogously.

The stabilizer group of an elliptic quadric $Q^-(3,3)$ has size $|\mathcal{G}(Q^-(3,3))| = 2^5 \cdot 3^2 \cdot 5$. There are 3^4 hyperplanes in H_0 not passing through the vertex of $Q \cap H_0$ and the stabilizer group of a cone $P\, Q^-(3,3)$ acts transitively on the solids not through P. For a fixed solid π not through P, the involutory perspectivity with axis π and center P also stabilizes $P\, Q^-(3,3)$; so the cone $P\, Q^-(3,3)$ is stabilized by a group of size $2^6 \cdot 3^6 \cdot 5$.

The stabilizer group of Q is a group of size $|\mathcal{G}(Q^-(5,3))| = 2^{10} \cdot 3^6 \cdot 5 \cdot 7$ [23, Theorem 22.6.2]. Since Q has 112 points, the stabilizer group of a point $P \in Q$ within $\mathcal{G}(Q)$ has size $2^6 \cdot 3^6 \cdot 5$, and the only transformation fixing the points of the tangent cone $P\, Q^-(3,3)$ point by point is the identity. Hence, the stabilizer group of $P\, Q^-(3,3)$ is a subgroup of $\mathcal{G}(Q^-(5,3))$; so it is possible to select the 19-cap $K \cap H_0$ on $Q^-(5,3)$ without losing generality. □

So, for investigating the extendability of the incomplete 19-caps of PG(4,3) to 48- and 49-caps, we considered the quadrics $Q^-(5,3)$ and $\ell\ Q^-(3,3)$, selected an incomplete 19-cap in one of the hyperplanes intersecting the quadric in a cone $P\ Q^-(3,3)$, and tried to extend the 19-cap to a 48- or 49-cap, under the assumption that at least 36 points of the cap belong to the quadric. This made the computer searches much more efficient.

3.4 The second largest size of a complete cap in PG(5,3)

Theorem 3.8

$$m_2'(5,3) = 48.$$

Proof: This follows from the preceding searches. Every 49-cap in PG(5,3) contains at least one 19- or 20-hyperplane.

Theorem 3.1 shows that there is no complete 49-, 50-, 51-, or 52-cap in PG(5,3) containing a 20-hyperplane. The existence of complete 49-, 50-, 51-, or 52-caps, having at most 8-solids and at most 19-hyperplanes, was eliminated by Theorem 3.4 (1). A cap having 9- or 10-solids, and having at most 19-hyperplanes, has at most size 49 (Lemma 3.2). The existence of such a 49-cap was eliminated in Theorem 3.4 and in the paragraphs preceding Theorem 3.3. The existence of complete 53-, 54-, or 55-caps was already eliminated in [20].

In [20], a complete 48-cap is presented.

Hence, $m_2'(5,3) = 48$. □

4 First consequences for caps in PG(6,3)

Lemma 4.1 *Every 49-cap in PG(5,3) has at least one hyperplane section which is a 20-cap of type Δ.*

Proof: A computer search of all possible 49-caps contained in the 56-cap showed that all these 49-caps contain at least one 20-cap of type Δ. □

Lemma 4.2 *Every 136-cap K in PG(6,3) has at most 8-solids and at least one 4-space intersecting K in a 20-cap of type Δ.*

Proof: Suppose first of all that there is a 9-solid Π_3. This solid Π_3 lies in a 4-space Π_4 intersecting K in at least $9 + 127/13 > 18$ points.

Such a 4-space Π_4 containing 19 or 20 points of K lies in a hyperplane containing at least 49 points of K. But 49-caps in a hyperplane are contained in a 56-cap of this hyperplane (Theorem 3.8) and a 56-cap does not have 9-solids (Subsection 2.2).

The same reasoning excludes the case that there are 10-solids.

Hence, K has at most 8-solids. An elementary counting argument shows that there exist 8-solids. We now prove that there is at least one 4-space intersecting K in 20 points.

Suppose there is no 4-space with 20 points of K. An 8-solid Π_3 lies in at least one 4-space containing at least $8+128/13 > 17$ points of K; so in a 4-space with 18 or 19 points. A 4-space with 19 points lies in a hyperplane with at least 49 points, and a 49-cap has at least one 4-space with 20 points (see Lemma 4.1). A 4-space with 18 points lies in a hyperplane with at least 48 points. In this hyperplane there is a 4-space with 19 points of the cap [19]; so we are reduced to the preceding case.

Now a 4-space with 20 points of K lies in a hyperplane with at least $20 + 116/4 = 49$ points of K. And every 49-cap in PG(5,3) has at least one hyperplane section which is a 20-cap of type Δ (Lemma 4.1). □

The information given by the preceding lemmas will enable us to obtain a lot of information on large caps in PG(6,3). Before presenting the geometrical arguments and computer searches which reduced the upper bound on $m_2(6,3)$ to 136, we deduce some extra information on caps in PG(5,3).

5 Introductory results

Lemma 5.1 *Let K be an n-cap, $n \geq 45$, in* PG(5,3), *which is contained in a quadric Q. Then Q is a non-singular elliptic quadric of* PG(5,3).

Proof: We check the other possibilities of quadrics.

A non-singular hyperbolic quadric $Q^+(5,3)$ has at most $3^3+3^2+3+1 =$ 40-caps, see [15].

For finding an upper bound on the size of a cap contained in a cone $P\,Q(4,3)$, we note that $Q(4,3)$ can be covered by $3^2+2 = 11$ lines. Namely, fix a line ℓ of $Q(4,3)$. Consider the 12 lines of $Q(4,3)$ intersecting ℓ in one point. Replace in one solid through ℓ, which intersects $Q(4,3)$ in a hyperbolic quadric, the four lines of the opposite regulus of ℓ by the three lines different from ℓ in the regulus through ℓ. Then a set of 11 lines of $Q(4,3)$ is obtained covering all the points of $Q(4,3)$. These 11 lines define 11 planes on $P\,Q(4,3)$. These planes contain at most $11 \times 4 = 44$ points of a cap on $P\,Q(4,3)$.

For the quadric cone $\ell\,Q^+(3,3)$, this quadric can be described as the union of four solids; so has at most $4 \times 10 = 40$ points of a cap. The quadric

$\ell\ Q^-(3,3)$ is in fact a union of 10 planes. So at most $4 \times 10 = 40$ points of a cap can lie on $\ell\ Q^-(3,3)$. And a cone $\pi\ Q(2,3)$ is in fact a union of four solids; so the same result as for $\ell\ Q^+(3,3)$ is obtained.

Finally, a quadric consisting of two hyperplanes contains at most $2 \times 20 = 40$ points of a cap. □

Lemma 5.2 *A cap K of size at least 45 in* PG(5, 3), *which lies on a quadric, cannot lie on a pencil of quadrics.*

Proof: Suppose such a cap exists. The preceding lemma shows that all the quadrics of this pencil are non-singular elliptic quadrics. Suppose that the quadrics of this pencil intersect in x points. Then, since $|Q^-(5,3)| = 112$ and $|\,\mathrm{PG}(5,3)| = 364$,

$$364 = 4 \times (112 - x) + x$$

and so $x = 28$ which is false. □

Lemma 5.3 *Suppose K is a cap of size at least 17 in* PG(4, 3). *Suppose also that a solid shares at most 8 points with this cap, and that this cap is contained in a quadric of* PG(4, 3). *Then it is contained in exactly one quadric of* PG(4, 3).

Proof: Suppose these 17 points lie on a pencil of quadrics which intersect in $17 + x$ points.

There are $|\,\mathrm{PG}(4,3)| - 17 - x = 104 - x$ points left. So one quadric has at least $104/4 - x/4$ other points. So one quadric has at least $26 + 17 + 3x/4$ points in total, and so it is a quadric with at least 43 points.

We check the two possibilities $R\ Q^+(3,3)$ and $\pi\ Q^+(1,3)$, where R is a point and $Q^+(3,3)$ is a 3-dimensional hyperbolic quadric in a solid not containing R, and where π is a plane and where $Q^+(1,3)$ is a hyperbolic quadric on a line skew to π.

In the former case, $R\ Q^+(3,3)$ can be considered to be 4 planes defined by R and a regulus of $Q^+(3,3)$. These 4 planes can have at most 16 points of a cap, a contradiction.

The quadric $\pi\ Q^+(1,3)$ is the union of two solids. For K, we know that solids contain at most 8 points of this 17-cap. So at most 16 points of K lie in these 2 solids.

So the 17 points do not lie on a pencil of quadrics. □

6 Improvements to the upper bound on $m_2(6,3)$

6.1 First results

We want to make a computer search for 138-caps K in PG(6, 3). We know that a solid contains at most 8 points of a 138-cap. Also there is a PG(4, 3) = Π_4 intersecting this 138-cap in a 20-cap of type Δ (Lemma 4.2).

Now $(138 - 20)/4 > 29$, so Π_4 lies in a hyperplane H_0 containing at least 50 points of the 138-cap; so $H_0 \cap K$ is contained in a 56-cap lying on a quadric Q_0.

Also $(138 - 56)/3 = 82/3 > 27$, hence Π_4 lies in a hyperplane H_1 containing at least 48 points of this 138-cap; and by Theorem 3.1, $H_1 \cap K$ lies on a 56-cap lying on a quadric Q_1.

In this section, we rely on the following computer result.

Theorem 6.1 *A 20-cap of type Δ lying in a hyperplane section Π of a non-singular elliptic quadric $Q^-(5,3)$ is contained in exactly two 56-caps of $Q^-(5,3)$.*

For the computer search of the preceding theorem, we relied on the property that a 56-cap of $Q^-(5,3)$ intersects every line of $Q^-(5,3)$ in two points (Subsection 2.2). We will give more details on this computer search in Subsection 6.5 since it also led us to the unique starting configuration for the main computer searches.

6.2 Second results

Lemma 6.2 *Consider two elliptic quadrics Q_0 and Q_1 in distinct hyperplanes H_0 and H_1 of PG(6, 3) sharing a common $Q(4,3)$ in $H_0 \cap H_1$. The quadrics Q_0, Q_1 define a pencil of quadrics in PG(6, 3) consisting of the hyperplane pair $H_0 \cup H_1$, one $Q(6,3)$ and two cones with point vertices and bases non-singular 5-dimensional elliptic quadrics $Q^-(5,3)$.*

Proof: The only quadrics different from $H_0 \cup H_1$ that can occur are non-singular quadrics $Q(6,3)$ and cones with base $Q^-(5,3)$.

Now $|\text{PG}(6,3) \setminus (H_0 \cup H_1)| = 486$.

Suppose there are α quadrics of type $Q(6,3)$ in the pencil, and β cones with base $Q^-(5,3)$.

Then $\alpha + \beta = 3$ and $180\alpha + 153\beta = 486$ which gives us the solution $\alpha = 1$ and $\beta = 2$. □

From the preceding subsection, we know that H_0 intersects K in a subset of a 56-cap lying on the elliptic quadric Q_0, and that H_1 intersects K in a subset of a 56-cap lying on the elliptic quadric Q_1, where $K \cap H_0 \cap H_1$ is a 20-cap of type Δ.

We now show that, using projective equivalence, there is a unique configuration for Q_0 and Q_1, and that it is also possible to fix uniquely the 56-caps within Q_0 and Q_1 containing the intersections $H_0 \cap K$ and $H_1 \cap K$. We will do this by studying in detail the stabilizer group of the unique quadric $Q = Q(6,3)$ containing Q_0 and Q_1, and also by using the stabilizer group of the 56-cap, and of the quadric $Q(4,3) = Q \cap H_0 \cap H_1$.

6.3 The possible starting configurations for Q_0 and Q_1

Lemma 6.3 *Under $\mathcal{G}(Q(6,3))$, there exist exactly two orbits of 4-spaces of* PG(6,3) *which intersect a given quadric $Q(6,3)$ in non-singular quadrics $Q(4,3)$.*

Proof: The 4-spaces of PG(6,3) intersecting a given quadric $Q(6,3)$ in non-singular quadrics $Q(4,3)$ are the polar spaces of the external lines and the bisecant lines to $Q(6,3)$; so there are two orbits. □

Theorem 6.4 *Suppose $Q(4,3)$ in the 4-space Π_4 on $Q(6,3)$ is the polar space of an external line to $Q(6,3)$. Then two of the 4 hyperplanes through Π_4 intersect $Q(6,3)$ in a non-singular quadric $Q^-(5,3)$, and two in a non-singular quadric $Q^+(5,3)$.*

Proof: The 4 hyperplanes through Π_4 can only intersect $Q(6,3)$ in elliptic quadrics $Q^-(5,3)$ or hyperbolic quadrics $Q^+(5,3)$ since the polar line of Π_4 is skew to $Q(6,3)$.

Suppose α hyperplanes intersect $Q(6,3)$ in elliptic quadrics and β hyperplanes intersect $Q(6,3)$ in hyperbolic quadrics. Then standard counting gives us the result $\alpha = \beta = 2$. □

A similar argument proves the following result.

Theorem 6.5 *Suppose $Q(4,3)$ in the 4-space Π_4 on $Q(6,3)$ is the polar space of a bisecant to $Q(6,3)$. Then two of the 4 hyperplanes through Π_4 intersect $Q(6,3)$ in tangent hyperplanes, one in a non-singular elliptic quadric $Q^-(5,3)$ and one in a non-singular hyperbolic quadric $Q^+(5,3)$.*

Corollary 6.6 *Using the notations of Subsection 6.1, the 4-space intersecting K in a 20-cap of type Δ is the polar space of an external line to $Q(6,3)$.*

Proof: Since we know that the 20-cap on $Q(4,3)$ in Π_4 lies on two elliptic quadrics in two hyperplanes through Π_4, necessarily Π_4 is the polar space of an external line to $Q(6,3)$.

6.4 The stabilizer group of a quadric $Q(4,3)$ on $Q(6,3)$ corresponding to a polar external line

We know that Π_4 intersects K in a 20-cap of type Δ lying on two elliptic quadrics Q_0 and Q_1, which lie on a non-singular quadric $Q(6,3)$, and that Π_4 is the polar space of an external line to $Q(6,3)$. We now show that we can select this 20-cap in Π_4 without losing generality.

Let $Q: X_0^2 + X_1X_2 + X_3X_4 + X_5^2 + X_6^2 = 0$ be the quadric $Q(6,3)$.

Then $L: X_0 = X_1 = X_2 = X_3 = X_4 = 0$ is an external line to $Q(6,3)$. Its polar 4-space is $X_5 = X_6 = 0$. Hence $Q(4,3)$:

$$\begin{cases} X_5 = X_6 = 0 \\ X_0^2 + X_1X_2 + X_3X_4 = 0 \end{cases}$$

The transformations fixing $Q(6,3)$, fixing $X_5 = X_6 = 0$ and its polar space $X_0 = X_1 = X_2 = X_3 = X_4 = 0$, are of type

$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} \mapsto \begin{pmatrix} & & & & & 0 & 0 \\ & & & & & 0 & 0 \\ & & A & & & 0 & 0 \\ & & & & & 0 & 0 \\ & & & & & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & B & \\ 0 & 0 & 0 & 0 & 0 & & \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}$$

where A stabilizes the part $X_0^2 + X_1X_2 + X_3X_4$ and where B stabilizes the part $X_5^2 + X_6^2$.

If $(x_0, x_1, x_2, x_3, x_4)^t \mapsto A(x_0, x_1, x_2, x_3, x_4)^t$ maps $X_0^2 + X_1X_2 + X_3X_4$ onto $X_0^2 + X_1X_2 + X_3X_4$, then select $B = I_2$; if $(x_0, x_1, x_2, x_3, x_4)^t \mapsto A(x_0, x_1, x_2, x_3, x_4)^t$ maps $X_0^2 + X_1X_2 + X_3X_4$ onto $-(X_0^2 + X_1X_2 + X_3X_4)$, then select $B = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}^{-1}$. This maps $X_5^2 + X_6^2$ onto

$$(X_5 + X_6)^2 + (X_5 - X_6)^2 = -(X_5^2 + X_6^2).$$

So the stabilizer group of $Q(4,3)$ is still available as a subgroup of $\mathcal{G}(Q(6,3))$ when $Q(4,3)$ is the polar space of an external line to $Q(6,3)$. This implies that it is possible to select the 20-cap of K in $H_0 \cap H_1$ uniquely without losing generality.

We can also select H_0, H_1 without losing generality.

The hyperplanes $X_5 = 0$ and $X_6 = 0$ intersect $Q(6,3)$ respectively in the elliptic quadrics $X_0^2 + X_6^2 + X_1X_2 + X_3X_4 = 0$ and $X_0^2 + X_5^2 + X_1X_2 + X_3X_4 = 0$.

The involution $(X_0, \ldots, X_4, X_5, X_6) \mapsto (X_0, \ldots, X_4, X_6, X_5)$ fixing $Q(6,3)$, fixes $H_0 \cap H_1$ point by point, and interchanges $X_5 = 0$ and $X_6 = 0$; so let $H_0 : X_6 = 0$ and $H_1 : X_5 = 0$.

6.5 The starting configuration

The hyperplanes H_0 and H_1 intersect K in subsets of 56-caps lying on Q_0 and Q_1. We now show that it is possible to select these two 56-caps without losing generality.

Consider the elliptic quadric $Q^-(5,3) : X_0^2+X_1^2+X_2^2+X_3^2+X_4^2+X_5^2 = 0$. This elliptic quadric consists of the projective points whose coordinates have weight 3 or 6. Let D be the 2-(6,3,2)-design with blocks

$$\{1,2,3\}, \{1,2,4\}, \{2,4,5\}, \{3,4,5\}, \{1,3,5\},$$

$$\{1,4,6\}, \{1,5,6\}, \{2,3,6\}, \{2,5,6\}, \{3,4,6\}.$$

Consider the 20-cap on this quadric $Q^-(5,3)$ lying in $X_5 = 0$ where the points have coordinates whose supports correspond to the first five blocks of D.

Then this 20-cap is a 20-cap of type Δ since $X_5 = 0$ intersects $Q^-(5,3)$ in a non-singular quadric. This 20-cap of type Δ extends in two ways to a 56-cap contained in $Q^-(5,3)'$(Theorem 6.1). The first 56-cap consists of the points having coordinates of weight 3 corresponding to the blocks of D and the points having coordinates of weight 6 having an even number of 1's. The other 56-cap is obtained by taking the same points of weight three and the complementary points of weight 6.

Consider now the quadric $Q : X_0^2 + X_1^2 + X_2^2 + X_3^2 + X_4^2 + X_5^2 + X_6^2 = 0$ in PG(6,3). The intersections of $H_0 : X_6 = 0$ and $H_1 : X_5 = 0$ are elliptic quadrics and $H_0 \cap H_1$ intersects Q in a non-singular 4-dimensional quadric. The polar space of $H_0 \cap H_1$ with respect to Q is an external line with respect to Q.

From the results of the preceding subsection, we know that we can select the 20-cap of type Δ in $Q \cap H_0 \cap H_1$ without losing generality. Select this 20-cap to be equal to the one described by D. Then this 20-cap extends in two ways to a 56-cap on $Q \cap H_0$ and in two ways to a 56-cap on $Q \cap H_1$.

Consider first of all the involution $(X_0, \ldots, X_5, X_6) \mapsto (X_0, \ldots, X_5, -X_6)$. This involution fixes H_0 point by point and interchanges the two 56-caps in H_1. Hence, we can select the 56-cap in H_1 without losing generality. Similarly, the involution $(X_0, \ldots, X_4, X_5, X_6) \mapsto (X_0, \ldots, X_4, -X_5, X_6)$ fixes H_1 point by point, but interchanges the two 56-caps in H_0. Hence, also in H_0, we can select the 56-cap without losing generality.

<u>The conclusion is:</u> Regarding the intersections of H_0 and H_1 with the 138-cap K, we can make sure that without loss of generality:

1. Π_4 is a 4-dimensional space intersecting K in a 20-cap of type Δ.

2. If H_0, H_1, H_2, H_3 are the hyperplanes through Π_4, where $|K \cap H_0| \geq |K \cap H_1| \geq |K \cap H_2| \geq |K \cap H_3|$, then $|H_0 \cap K| \geq 50$ and $|H_1 \cap K| \geq 48$.

3. There is a non-singular parabolic quadric Q in PG(6, 3) containing the intersections $K \cap H_0$ and $K \cap H_1$.
4. We may select on Q the 56-cap C_0 in H_0 containing $K \cap H_0$ without losing generality.
5. We may select on Q the 56-cap C_1 in H_1 containing $K \cap H_1$ without losing generality.

6.6 Computer results

This configuration was the starting configuration for a computer search for large caps in PG(6, 3).

This search led to the following results.

In Table 3, $x = |K \cap H_0|$ and $y = |K \cap H_1|$.

The number n means that there is no n-cap in PG(6, 3) having x points of C_0 in H_0, and y points of C_1 in H_1, where $H_0 \cap H_1$ shares 20 points with this n-cap; 20 points lying on a non-singular quadric $Q(4, 3)$. Here, we assume $x \geq y$.

x	y	no n-cap for $n =$
56	≥ 52	113
55	$55 \geq y \geq 53$	113
54	54	113
56	$51 \geq y \geq 48$	132
55	$52 \geq y \geq 48$	132
54	$53 \geq y \geq 49$	132
53	$53 \geq y \geq 50$	132
52	$52 \geq y \geq 51$	132
54	48	135
53	$49 \geq y \geq 48$	135
52	$50 \geq y \geq 48$	135
51	$51 \geq y \geq 49$	135
50	50	135

Table 3

These results enable us to eliminate the existence of 138-caps in PG(6, 3).

Theorem 6.7 *There does not exist a* 138*-cap in* PG(6, 3).

Proof: Suppose there is a 138-cap in PG(6, 3). From the preceding results, we know it is possible to find a 4-space Π_4 intersecting this 138-cap in a 20-cap of type Δ. This 4-space lies in two hyperplanes H_0 and H_1

intersecting K in caps of size at least 50 and 48, lying on elliptic quadrics Q_0 and Q_1.

Using the notations x and y of Table 3, we get the following possibilities

x	y
56	≥ 48
55	≥ 48
54	≥ 48
53	≥ 49
52	≥ 49
51	≥ 49
50	$= 50$

From Table 3, no such 138-cap exists. □

7 No 137-caps in PG(6,3)

Using some extra computer searches and geometrical arguments, it is also possible to prove the non-existence of 137-caps in PG(6,3). Let K be a 137-cap in PG(6,3). We first state the following computer search result.

Lemma 7.1 *A 137-cap K in PG(6,3) does not contain a 20-cap of type Δ lying in a 56-hyperplane and in three 47-hyperplanes.*

The results of Table 3 and the preceding lemma show that a 4-space Π_4 intersecting K in a 20-cap of type Δ lies in one 50-hyperplane and in three 49-hyperplanes. Let H_0 be the 50-hyperplane passing through Π_4 and let H_1, H_2, H_3 be the three 49-hyperplanes passing through Π_4. Denote by Q_i the elliptic quadrics in H_i, $i = 0, 1, 2, 3$, containing the intersections $H_i \cap K$.

Consider Q_1 and Q_2. These two 5-dimensional elliptic quadrics intersect in the 4-dimensional parabolic quadric containing the 20-cap $K \cap \Pi_4$ of type Δ. Hence, analogously as in Subsection 3.3, they define a pencil of four 6-dimensional quadrics intersecting pairwise in the variety $Q_1 \cup Q_2$ of degree four. One of these four 6-dimensional quadrics is $H_1 \cup H_2$. We show that one of the three other quadrics of this pencil contains at least 113 points of K. This will give us the desired contradiction.

This latter contradiction is obtained by geometrical arguments and the following additional computer search results.

Lemma 7.2 *Consider an arbitrary 50-cap K_{50} in PG(5,3), then K_{50} contains a 20-cap Δ_0 of type Δ which is intersected in eight points of K_{50} by $s \geq 8$ other 20-caps Δ_i, $i = 1, \ldots, s$, of type Δ, contained in K_{50}.*

We now construct inductively a subset G of $K_{50} \setminus \Delta_0$. We describe the construction of the set G in pseudo-code.

$G := \Delta_1 \setminus \Delta_0$

REPEAT

$g := |G|$

FOR $i = 2$ **TO** s **DO**

IF $(\Delta_i \setminus \Delta_0) \cap G \neq \emptyset$ **THEN**

$G := G \cup (\Delta_i \setminus \Delta_0)$

END IF

END FOR

UNTIL $|G| = g$

Then the computer searches showed the following result.

Lemma 7.3 *Consider an arbitrary* 50*-cap* K_{50} *in* PG(5,3)*, then* K_{50} *contains a* 20*-cap* Δ_0 *of type* Δ *which is intersected in eight points of* K_{50} *by* $s \geq 8$ *other* 20*-caps* Δ_i*,* $i = 1, \ldots, s$*, of type* Δ*, contained in* K_{50}*, and for which* $|G| \geq 28$.

We now have the required computer results to exclude the existence of 137-caps in PG(6,3).

Lemma 7.4 *If* K *is a* 137*-cap in* PG(6,3)*, then there is a quadric* Q *of* PG(6,3) *which contains at least* 113 *points of* K.

Proof: We start with the information on the 4-space Π_4 and the quadrics $Q_0, \ldots, Q_3$ in the hyperplanes $H_0, \ldots, H_3$, described at the beginning of Section 7. Select the 20-cap of type Δ in $H_0 \cap K$ in such a way that for this 20-cap, the corresponding set G of the 50-cap $K \cap H_0$ satisfies $|G| \geq 28$ (Lemma 7.3). Similarly, using the notations of Lemma 7.3, denote $\Pi_4 \cap K$ by Δ_0 and denote by $\Delta_i, i = 1, \ldots, s$, $s \geq 8$, the 20-caps of type Δ in $K \cap H_0$ intersecting Δ_0 in 8 points.

We now use the points of the set G in the same order as they were added to the set G in the description above of the construction of G.

First of all, consider the 20-cap Δ_1 lying in the 4-space $\Pi_4^{(1)}$ of H_0. Let $R \in \Delta_1 \setminus \Delta_0$. Consider the unique 6-dimensional quadric Q containing R and the quadrics Q_1 and Q_2.

Then also this 20-cap Δ_1 lies in four elliptic quadrics $Q'_0, \ldots, Q'_3$, lying in the four hyperplanes through $\Pi_4^{(1)}$, and sharing respectively 50, 49, 49, 49 points with K. Let $Q'_0 = Q_0$.

Consider the solid $\Pi_3^{(1)} = \Pi_4 \cap \Pi_4^{(1)}$ sharing 8 points with K. The four hyperplanes $H'_0, \ldots, H'_3$ through $\Pi_4^{(1)}$ intersect H_1 in the four 4-spaces of H_1 through $\Pi_3^{(1)}$. These four 4-spaces all intersect the 56-cap containing $H_1 \cap K$ in 20 points since $|\Pi_3^{(1)} \cap K| = 8$. At most one of these four 4-spaces intersects $H_1 \cap K$ in less than 17 points.

The same arguments are valid for the cap $H_2 \cap K$.

This implies that at least one of the quadrics Q'_1, Q'_2, Q'_3, for instance Q'_3, shares at least 17-caps with $H_1 \cap K$ and $H_2 \cap K$. These latter 17-caps are contained in a unique 4-dimensional quadric (Lemma 5.3). Hence, Q'_3 shares the point R, and two 4-dimensional quadrics $Q'_3 \cap Q_1$ and $Q'_3 \cap Q_2$ with Q. Then Bézout's theorem implies that $Q'_3 \subset Q$. This also implies that $\Delta_1 \subset Q$.

We now prove that the other points of G belong to Q. We prove that they belong to Q in the same order as they were added to the set G.

Suppose that at a certain point in the **REPEAT UNTIL** loop of the pseudo-code above, the set of points $\Delta_i \setminus \Delta_0$, $i \geq 2$, was added to G. This was done because of the fact that the intersection $\Delta_i \cap G$ was not empty. Assume by induction that it is already proven that all points of G, which belonged to G before $\Delta_i \setminus \Delta_0$ was added to G, belong to the quadric Q.

Then we repeat the arguments above for Δ_1, but now for Δ_i. The cap Δ_i lies in at least one 5-dimensional elliptic quadric Q''_3, different from Q_0, sharing at least 17 points with $H_1 \cap K$ and $H_2 \cap K$. This latter elliptic quadric Q''_3 then already shares two 4-dimensional quadrics $Q''_3 \cap Q_1$ and $Q''_3 \cap Q_2$ with Q. This quadric Q''_3 also contains the 20-cap Δ_i, containing at least one point R' of the set G. Note that $R' \notin \Delta_0$ and that, by assumption, $R' \in Q$. Hence, following Bézout's theorem, $Q''_3 \subset Q$.

So, the 20-cap Δ_i is contained in Q.

Since $|G| \geq 28$ and since Δ_0 is a 20-cap, Q necessarily contains at least 48 points of the 50-cap $K \cap H_0$. This latter cap of size at least 48 only lies on the quadric Q_0 (Lemma 5.2). Hence, $Q \cap H_0 = Q_0$.

This already implies that the quadric Q contains at least $50 + 49 + 49 - 2 \times 20 = 108$ points of K.

In fact, Q contains at least 113 points of K. Consider again the quadric Q'_3 which is contained in Q. It shares at least 49 points with K, of which at least 13 lie in H_3. Eight of those points are the points of $\Delta_0 \cap \Delta_1$; so adding at least 5 to $|K| \geq 108$ gives $|K| \geq 113$. □

Theorem 7.5 *There does not exist a 137-cap K in* PG(6, 3).

Proof: The preceding lemma shows that if there is a 137-cap K in PG(6, 3), then there is a quadric Q containing at least 113 points of K. We know that there is at least one 5-dimensional elliptic quadric $Q \cap H_0$ on Q.

Hence $Q = Q(6,3)$ or a cone with a point vertex and as base a non-singular 5-dimensional elliptic quadric $Q^-(5,3)$.

Case 1: $Q = Q(6,3)$. The quadric $Q(6,3)$ has a spread, i.e. a partition into 28 planes, see e.g. [23, p. 348]. Each plane has at most a 4-cap. So $Q(6,3)$ has at most a $28 \times 4 = 112$-cap.

Case 2: $Q =$ cone with base $Q^-(5,3)$. The quadric $Q^-(5,3)$ has a spread, i.e. a partition into 28 lines. For instance, consider a spread of $Q(6,3)$ as stated in the preceding case. Consider a hyperplane intersecting $Q(6,3)$ in a non-singular elliptic quadric; then this hyperplane intersects each of the 28 planes of the spread of $Q(6,3)$ in a line, yielding a spread of lines of $Q^-(5,3)$.

Hence it is possible to cover Q with 28 planes, so Q has at most a $28 \times 4 = 112$-cap, a contradiction. □

There exists a 112-cap in PG(6, 3) (Section 1). Hence, the following corollary is valid.

Corollary 7.6

$$112 \leq m_2(6,3) \leq 136.$$

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