Bounds for the Independent Domination Number of Graphs and Planar Graphs*

G. MacGillivray
Department of Mathematics and Statistics
University of Victoria
Victoria, British Columbia
Canada V8W 3P4

K. Seyffarth
Department of Mathematics and Statistics
University of Calgary
Calgary, Alberta
Canada T2N 1N4

September 28, 2003

Abstract

We first prove that if G is a connected graph with n vertices and chromatic number $\chi(G) = k \geq 2$, then its independent domination number

$$i(G) \leq \left\lfloor \frac{(k-1)}{k} n \right\rfloor - (k-2).$$

This bound is tight and remains so for planar graphs.

We then prove that the independent domination number of a diameter two planar graph on n vertices is at most $\lceil n/3 \rceil$.

1 Introduction

A subset D of the vertices of a graph G is called a *dominating set* if every vertex of $V(G)\backslash D$ is adjacent to at least one vertex of D, and the *domination number* of G, denoted $\gamma(G)$ is the minimum cardinality of a

*The authors gratefully acknowledge the financial support of the Natural Sciences and Engineering Research Council of Canada.

dominating set. Closely related to this is the notion of the *independent domination number*, denoted $i(G)$, which is the minimum cardinality of a set of vertices that is both a dominating set and an independent set.

In [4], we addressed the problem of obtaining an upper bound for the domination number of a graph using its diameter. For arbitrary graphs, no constant bounds can be given. However, by restricting our attention to *planar* graphs, constant bounds can be obtained for small diameters. In particular, we proved the following.

Theorem 1 [4] *If G is a planar graph of diameter two, then* $\gamma(G) \leq 3$.

Theorem 2 [4] *If G is a planar graph of diameter three, then* $\gamma(G) \leq 10$.

One question that arises naturally is whether or not analogous results hold for the independent domination number. In what follows, we will show that although constant bounds do not exist, it is possible to give a tight bound on the independent domination number of a graph of diameter at least three in terms of the number of vertices and the chromatic number (rather than the diameter). Equality is shown to hold, even for planar graphs, and a bound for the independent domination number due to Gimbel and Vestergaard [3] is derived as a corollary. In the subsequent section, we will show that it is possible to give a tight bound on the independent domination number of a planar graph of diameter two in terms of the number of vertices alone, and to characterise the graphs for which equality holds.

Any terminology not defined here follows that of [1]. Since the independent domination number of a graph is equal to the independent domination number of its underlying simple graph, we assume throughout this paper that the graphs under consideration are simple. By a *k-path* in a graph G we mean a path of length k, and for convenience we will use the term *short path* to mean a path of length at most two.

Let $S \subseteq V(G)$ and let $u \in S$. A *private neighbour* of u (with respect to S) is either a vertex in $V(G)\backslash S$ whose only neighbour in S is u, or vertex u itself if u is not adjacent to any other vertex of S. Let $PN(u) \subseteq N(u) \cup \{u\}$ denote the set of private neighbours of u. We say that u has k *independent private neighbours* if and only if the independent domination number of the subgraph of G induced by $PN(u)$ is at least k.

A plane graph is a planar graph together with an embedding of the graph in the plane. In the proofs involving planar graphs, it is often necessary to begin by assuming that the graph is embedded in the plane. The Jordan Curve Theorem tells us that a cycle C in a plane graph G separates the plane into two regions, the interior of C and the exterior of C. If $u \in V(G)$ lies in the interior of C, and $v \in V(G)$ lies in the exterior of C, then any path in G joining u and v must contain a vertex of C. Throughout this paper we often use this fact implicitly.

Note that the only graphs with diameter one are complete graphs, and $\gamma(K_n) = i(K_n) = 1$ for all $n \geq 1$. Henceforth, we consider only graphs with diameter at least two.

2 Graphs with diameter at least three

Let $n \geq k \geq 2$ be integers, and define the integers q and r by $n = qk+r$, $0 \leq r < k$. Construct a graph $H_{n,k}$ on n vertices as follows: begin with a complete graph on k vertices and a set X of $n-k$ isolated vertices. Add edges joining each of the vertices in X to exactly one vertex of the complete graph in such a way that r vertices of the complete graph are adjacent to q vertices of X, and the remaining $k-r$ vertices of the complete graph are adjacent to $q-1$ vertices of X. It is easy to verify that $H_{n,k}$ has diameter three and chromatic number k. Since an independent dominating set of $H_{n,k}$ contains at most one vertex of the complete subgraph of size k, it follows that

$$i(H_{n,k}) = 1 + (k-1)(q-1) + (r-1) = \left\lfloor \frac{k-1}{k} n \right\rfloor - (k-2).$$

Theorem 3 *If G is a connected graph with n vertices and $\chi(G) = k \geq 2$, then*

$$i(G) \leq \left\lfloor \frac{(k-1)}{k} n \right\rfloor - (k-2).$$

For all integers $n \geq k \geq 2$, equality is achieved when $G = H_{n,k}$.

Proof. Let G be a graph on n vertices with $\chi(G) = k \geq 2$. If $k = 2$, then the result is obvious; thus we may assume $k \geq 3$.

Choose a k-colouring $p : V(G) \to \{1, 2, \ldots, k\}$ such that $\sum_{u \in V(G)} p(u)$ is minimum. Note that such a colouring has the property that any vertex of colour i is adjacent to vertices of colour $1, 2, \ldots i-1$, $2 \leq i \leq k$.

Let C_i denote the set of vertices of colour i, $1 \leq i \leq k$. By our choice of colouring, C_1 is a dominating set, and since C_1 is also independent, if $|C_1| \leq \frac{(k-1)}{k} n - (k-2)$, then $|C_1| \leq \lfloor \frac{(k-1)}{k} n \rfloor - (k-2)$ and we are done. Therefore, we may assume that $|C_1| > \frac{(k-1)}{k} n - (k-2)$.

Since G is connected, $\delta \geq 1$, so every vertex of C_1 is adjacent to some vertex of $\bigcup_{i=2}^{k} C_i$. Thus, for some j, $2 \leq j \leq k$, C_j dominates at least $\frac{1}{k-1}|C_1|$ vertices of C_1. Notice that C_j is an independent set and dominates all vertices of C_i, $j < i \leq k$. Extend C_j to a maximal independent subset,

X, of G; by our previous observation $X \subseteq C_1 \cup C_2 \cup \cdots \cup C_j$. Thus X is an independent dominating set of G, and $|X| = |X \cap C_1| + |X \backslash C_1|$. But,

$$\begin{aligned} |X \cap C_1| &\leq |C_1| - |N(C_j) \cap C_1| \\ &\leq |C_1| - \frac{1}{k-1}|C_1|, \end{aligned}$$

and $|X \backslash C_1| \leq n - |C_1| - (k-2)$ (by the definition of the colouring there is a vertex in each $C_i, i \neq j$ that is adjacent to a vertex in C_j). Therefore,

$$\begin{aligned} |X| &= |X \cap C_1| + |X \backslash C_1| \\ &\leq |C_1| - \frac{1}{k-1}|C_1| + n - |C_1| - (k-2) \\ &= n - \frac{1}{k-1}|C_1| - (k-2) \\ &< n - \frac{1}{k-1}\left(\frac{(k-1)}{k}n - (k-2)\right) - (k-2) \\ &= \frac{(k-1)}{k}n + \frac{(k-2)}{(k-1)} - (k-2). \end{aligned}$$

However, $|X|$ is an integer, and thus

$$|X| \leq \left\lfloor \frac{(k-1)}{k}n + \frac{(k-2)}{(k-1)} \right\rfloor - (k-2) \leq \left\lceil \frac{(k-1)}{k}n \right\rceil - (k-2).$$

If k divides n, then

$$\left\lceil \frac{(k-1)}{k}n \right\rceil - (k-2) = \left\lfloor \frac{(k-1)}{k}n \right\rfloor - (k-2),$$

and the proof is complete.

We now assume that k does not divide n, and write n as $kq + r$ where q and r are nonnegative integers, $q = \lfloor n/k \rfloor$, and $1 \leq r < k$. In this case, the difference between

$$\left\lceil \frac{(k-1)}{k}n \right\rceil - (k-2) \quad \text{and} \quad \left\lfloor \frac{(k-1)}{k}n \right\rfloor - (k-2)$$

is exactly one. We will show that the bound $\left\lceil \frac{(k-1)}{k}n \right\rceil - (k-2)$ can not be tight, and can therefore be replaced with $\left\lfloor \frac{(k-1)}{k}n \right\rfloor - (k-2)$. Suppose then, that $i(G) = \left\lceil \frac{(k-1)}{k}n \right\rceil - (k-2)$. This implies that equality must hold throughout the argument above.

We focus first on the inequality

$$|X\backslash C_1| \leq n - |C_1| - (k-2);$$

if this inequality is strict, then we are done. Thus we assume that

$$|X\backslash C_1| = n - |C_1| - (k-2),$$

which implies that there are only $k-2$ vertices in $C_2 \cup C_3 \cup \cdots \cup C_{j-1} \cup C_{j+1} \cup \cdots \cup C_{k-1}$ that are adjacent to vertices in C_j. It follows that all vertices in C_j are adjacent to a unique vertex $v_i \in C_i$, $2 \leq i \leq j-1$, and that for $j+1 \leq i \leq k$, $C_i = \{v_i\}$. Furthermore, if any two vertices of

$$S = (C_2\backslash\{v_2\}) \cup (C_3\backslash\{v_3\}) \cup \cdots \cup (C_{j-1}\backslash\{v_{j-1}\})$$

are adjacent then $S \not\subseteq X$ and equality does not hold. Thus S is an independent set.

Recall that the colouring, $p : V(G) \to \{1, 2, \ldots, k\}$, is defined so that $\sum_{u \in V(G)} p(u)$ is minimum. If $j \geq 3$, then it must be the case that $|C_j| = 1$ (the case $j = 2$ will be dealt with at the end of the proof); otherwise, by recolouring all vertices in C_j with colour $j-1$ and v_{j-1} with colour j, we can maintain a proper colouring but reduce the sum of the colours. Thus, we let $C_j = \{v_j\}$.

We now focus on $|X \cap C_1|$. For equality to occur in the bound, we must have $|X \cap C_1| = |C_1\backslash N(C_j)|$; for this to occur, no vertex of $C_1\backslash N(C_j)$ can be adjacent to a vertex of S; i.e., the independent dominating set

$$X = \{v_j\} \cup (C_1\backslash N(C_j) \cup (C_2\backslash\{v_2\}) \cup (C_3\backslash\{v_3\}) \cup \cdots \cup (C_{j-1}\backslash\{v_{j-1}\}).$$

Consider now the subgraph, G', induced by $\{v_2, v_3, \ldots, v_k\}$. If v_s and v_t are non-adjacent in G', then re-colour G as follows.

(i) Assign the vertices in X colour j.

(ii) Assign the vertices in $N(v_j) \cap C_1$ colour 1.

(iii) If $s = j$ or $t = j$, assign v_s and v_t colour j; otherwise, assign v_s and v_t colour s.

(iv) Assign each of the remaining $k-4$ of the v_i's its original colour (colour i).

The result is a $(k-1)$-colouring of G, a contradiction, and thus G' is complete.

Furthermore, there must be a vertex in $N(v_j) \cap C_1$ that is adjacent to each vertex of G'. If this is not the case, then we re-colour the vertices of G as follows.

(i) Assign the vertices in X colour j.

(ii) For $2 \leq i \leq k$ and $i \neq j$, assign vertex v_i colour i.

(iii) Assign each vertex in $N(v_j) \cap C_1$ one of the colours from the set $\{2, 3, \ldots, j-1, j+1, \ldots, k\}$ (at least one of these colours is available for each v_i).

Again, the result is a $(k-1)$-colouring of G, a contradiction. Thus there exists a vertex in $N(v_j) \cap C_1$ adjacent to each vertex of G'.

We can now conclude that G consists of a k-clique and $n-k = (q-1)k+r$ vertices each adjacent to at least one vertex in the clique. Thus, some vertex in the clique is adjacent to at least $\lceil \frac{n-k}{k} \rceil = q$ (since $r > 0$) vertices not in the clique, and so

$$i(G) \leq 1 + n - k - q = (q-1)(k-1) + r = \left\lfloor \frac{(k-1)}{k} n \right\rfloor - (k-2),$$

a contradiction.

What remains is to prove the result when $j = 2$. Notice that in this case $C_i = \{v_i\}$, $i \geq 3$, and that if v_i dominates at least $\left\lceil \frac{1}{k-1}|C_1| \right\rceil$ vertices of C_1, then we can choose $j = i$ and proceed as before. Thus each such v_i dominates at most $\left\lfloor \frac{1}{k-1}|C_1| \right\rfloor$ vertices of C_1.

Since $|X| = \left\lceil \frac{(k-1)}{k} n \right\rceil - (k-2)$, it must be the case that $|X \cap C_1| = |C_1| - \left\lceil \frac{1}{k-1}|C_1| \right\rceil$, and thus

$$|N(C_2) \cap C_1| = \left\lceil \frac{1}{k-1}|C_1| \right\rceil.$$

Each of the remaining

$$\begin{aligned} |C_1 \backslash N(C_2)| &= |C_1| - \left\lceil \frac{1}{k-1}|C_1| \right\rceil \\ &= \left\lfloor \frac{k-2}{k-1}|C_1| \right\rfloor \end{aligned}$$

vertices of C_1 must be dominated by (at least) one of $\{v_3, v_4, \ldots, v_k\}$, so it follows that each v_i, $i \geq 3$, dominates exactly $\left\lfloor \frac{1}{k-1}|C_1| \right\rfloor$ vertices of $C_1 \backslash N(C_2)$.

We will now show that v_i, $i \geq 3$ is adjacent to at least one vertex in $N(C_2) \cap C_1$, implying that v_i dominates at least

$$\left\lfloor \frac{1}{k-1}|C_1| \right\rfloor + 1 = \left\lceil \frac{1}{k-1}|C_1| \right\rceil$$

vertices of C_1, a contradiction.

Claim: There exists a vertex $v_1 \in N(C_2) \cap C_1$ that is adjacent to each v_i, $i \geq 3$.

If this is not the case, then we may re-colour the vertices of G as follows.

(i) Assign the vertices in X colour 1.

(ii) Colour each vertex of $N(C_2) \cap C_1$ with a colour from the set $\{3, 4, \ldots, k\}$ (at least one such colour is always available).

The result is a $(k-1)$-colouring of G, a contradiction. Therefore, there is a $v_1 \in N(C_2) \cap C_1$ adjacent to each v_i, $i \geq 3$, completing the proof of the claim and the theorem. ∎

Notice that the argument in the proof of Theorem 3 shows that

$$i(G) \leq \min\{|C_1|, n - \left\lceil \frac{|C_1|}{k-1} \right\rceil - k + 2\}.$$

Minimizing this as a function of $|C_1|$ gives the bound

$$i(G) + \left\lceil \frac{i(G)}{k-1} \right\rceil \leq n - k + 2.$$

Equality is obtained for the graphs $H_{n,k}$ when k divides n.

We can use Theorem 3 to derive the following result of Gimbel and Vestergaard [3].

Corollary 4 *For any graph G, $i(G) \leq n - 2\sqrt{n} + 2$.*

Proof. Theorem 3 asserts $i(G) \leq n - \left(\frac{1}{k}n + (k-2)\right)$, and hence it suffices to show that $\frac{1}{k}n + (k-2) \geq 2\sqrt{n} - 2$. We have $\frac{1}{k}n + (k-2) \geq 2\sqrt{n} - 2$ if and only if $\frac{1}{k}n + k \geq 2\sqrt{n}$. Since k is positive we can multiply through and obtain that the above statement is equivalent to $n - 2k\sqrt{n} + k^2 \geq 0$. The latter inequality is the same as $(\sqrt{n} - k)^2 \geq 0$, which is true. ∎

For $k = 2, 3, 4$, the graphs $H_{n,k}$ are planar, and thus the bound in Theorem 3 is tight for planar graphs. Combining Theorem 3 with the Four Colour Theorem also gives us the following.

Corollary 5 *If G is a planar graph on n vertices with diameter at least three, then $i(G) < \frac{3}{4}n$.*

3 Planar diameter two graphs

Theorem 6 *If G is a diameter two planar graph on n vertices, then*

$$i(G) \leq \left\lceil \frac{n}{3} \right\rceil.$$

Furthermore, if $i(G) = \lceil n/3 \rceil$, then one of the following holds:

(i) $n = 3$ and G is a path of length two.

(ii) $n = 4$ and G is a cycle of length four.

(iii) $n = 5$ and G arises (by the addition of extra edges) from one of the edge-minimal graphs in Figure 1.

(iv) $n = 6$ and G arises from one of the edge-minimal graphs in Figure 2.

(v) $n \geq 9$, $n \equiv 0 (\text{mod } 3)$, and G is isomorphic to the graph in Figure 3(i), or arises from the edge-minimal graph in Figure 3(ii).

For $n \leq 9$ there are 38 non-isomorphic diameter two planar graphs with independent domination number $\lceil n/3 \rceil$. For each $n \geq 12$, $n \equiv 0(\text{mod } 3)$, there are 13 non-isomorphic examples. We derive natural upper and lower bounds for the degrees of the vertices of such graphs. In Figures 1, 2 and 3, we show only the edge-minimal planar (not plane) graphs. The remaining graphs can be constructed by adding edges while respecting the degree, diameter and domination conditions (the embedding may need to be changed).

Proof of Theorem 6. Many of the arguments that follow depend on G being embedded in the plane in a particular way. The embedding used can always be chosen without loss of generality.

Let G be a diameter two planar graph with n vertices, and suppose that $i(G) \geq \lceil n/3 \rceil$. By Theorem 1, $\gamma(G) \leq 3$, and so we consider three cases.

Case 1. $\gamma(G) = 1$.
In this case $i(G) = 1 \geq \lceil n/3 \rceil$, implying $n \leq 3$. However, if $n = 1$ or 2, then $G \cong K_1$ or $G \cong K_2$, and both these graphs have diameter one, a contradiction. Thus, $n = 3$, and it follows that G is a path of length two.

Case 2. $\gamma(G) = 2$.
Let $\{x, y\}$ be a minimum cardinality dominating set in G. If $xy \notin E(G)$, then $i(G) = 2 \geq \lceil n/3 \rceil$, implying that $n \leq 6$. Recalling that G has diameter two and $\gamma(G) = i(G) = 2$, it is not difficult to see that vertices of G have

degree at least two and at most $n-2$. Therefore, it follows immediately that $n \geq 4$, and thus $i(G) = \lceil n/3 \rceil$.

When $n = 4$, G is 2-regular and is therefore simply a cycle of length four. When $n = 5$, every vertex of G has degree two or three; there are four such non-isomorphic planar graphs, arising from the graphs shown in Figure 1.

Figure 1: $n = 5$

Finally, when $n = 6$, every vertex of G has degree two, three or four. If there are only vertices of degree two, then G is a cycle of length six, which has diameter three, a contradiction. Therefore, G has maximum degree three or four. By examining these two possibilities in detail, we see that there are 23 such non-isomorphic planar graphs, arising from the graphs shown in Figure 2.

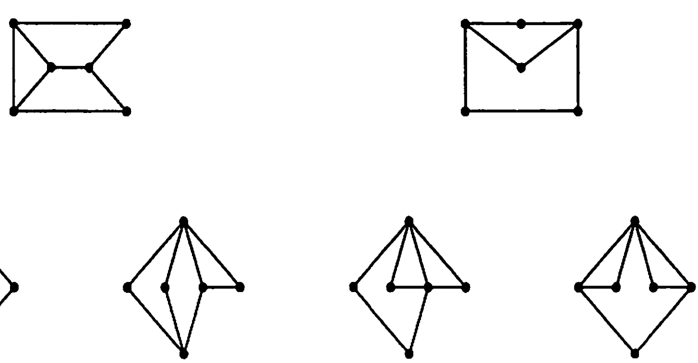


Figure 2: $n = 6$

We now assume that $xy \in E(G)$. If x has only one independent private neighbour (with respect to $\{x, y\}$), say z, then $\{y, z\}$ is an independent dominating set, so $i(G) = 2$. The previous argument applies, and we conclude that $n = 4, 5$ or 6 (and thus $i(G) = \lceil n/3 \rceil$), and furthermore, G is either a 4-cycle or is one of the graphs arising from those in Figures 1 and 2.

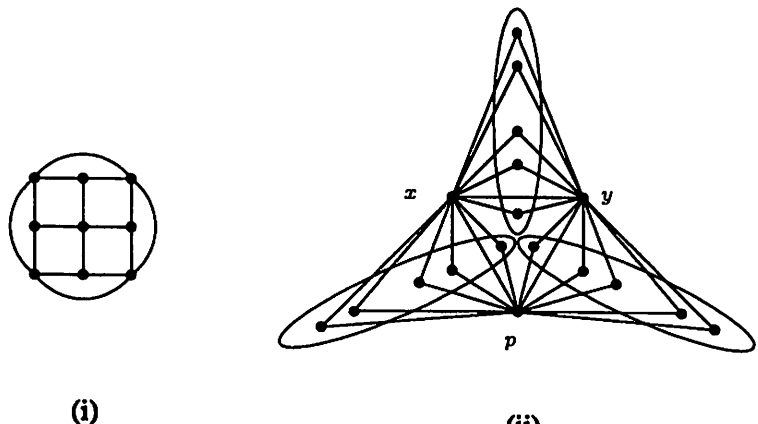


Figure 3: $n \geq 9$

Thus, from now on, we may assume that x and y each have at least two independent private neighbours (with respect to $\{x, y\}$). Let A be a maximal independent set of private neighbours of x, and let $a_1, a_2 \in A$. Similarly, let B be a maximal independent set of private neighbours of y, and let $b_1, b_2 \in B$. Without loss of generality, we may assume that G is embedded as a plane graph as shown in Figure 4(i).

Since G has diameter two, there must be a short path from a_1 to b_2. If this short path is an edge, then since a_1 and a_2 are independent, as are b_1 and b_2, it follows from the planarity of G that there is no short path from b_1 to a_2, a contradiction. Therefore, there must be a vertex, $p \neq a_1, b_1, a_2, b_2$ such that a_1pb_2 is a 2-path. In particular, this implies that $n \geq 7$. This same vertex p must be on a short path from a_2 to b_1, and thus G contains the subgraph shown in Figure 4(ii). It now follows that vertex p is adjacent to all vertices in $A \cup B$.

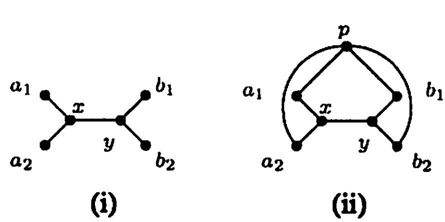


Figure 4: Case 2

Observe that $|A| \geq \lfloor (n-1)/3 \rfloor$; otherwise, a maximal independent set containing y has cardinality at most

$$1 + \lfloor (n-1)/3 \rfloor - 1 = \lfloor (n-1)/3 \rfloor,$$

and $\lfloor (n-1)/3 \rfloor < \lceil n/3 \rceil$. This maximal independent set is an independent dominating set, a contradiction. Similarly, $|B| \geq \lfloor (n-1)/3 \rfloor$.

Since $\{x, y\}$ is a dominating set, $px \in E(G)$ or $py \in E(G)$; without loss of generality, assume that $px \in E(G)$. This, along with the planarity of G, implies that p is adjacent to *all* private neighbours of x. But then either $\{p, y\}$ is an independent dominating set of cardinality two, implying that $i(G) = 2$, or $py \in E(G)$. In the first case, since $i(G) \geq \lceil n/3 \rceil$, it follows that $n \leq 6$, a contradiction, since $n \geq 7$.

In the second case, $py \in E(G)$; this, along with the planarity of G, implies that p is adjacent to all private neighbours of y. Thus, it follows that the degree of vertex p satisfies

$$d(p) \geq 2\lfloor (n-1)/3 \rfloor + 2,$$

and a maximal independent set in G containing vertex p has cardinality at most

$$1 + (n-1) - d(p) \leq n - 2\lfloor (n-1)/3 \rfloor - 2 \leq \lceil n/3 \rceil,$$

with equality only if $n \equiv 0 \pmod 3$ and $d(p) = 2\lfloor (n-1)/3 \rfloor + 2$. This implies that x and y each have exactly $\lfloor (n-1)/3 \rfloor$ private neighbours. Furthermore, by our previous remark, the private neighbours of x are independent, and the private neighbours of y are independent. The remaining vertices of the graph, $\lfloor (n-1)/3 \rfloor$ in all, must be shared neighbours of x and y. In addition, these vertices must form an independent set. Otherwise, there would exist a maximal independent set, S, consisting of p and shared neighbours of x and y such that $|S| < \lceil n/3 \rceil$.

Therefore, G consists of the triangle $xypx$, and each pair from $\{x, y, p\}$ has $\lfloor (n-1)/3 \rfloor = (n/3) - 1$ shared neighbours. The only such graphs arise from the edge-minimal graph shown in Figure 3(ii).

Case 3. $\gamma(G) = 3$.
In this case, choose a minimum dominating set $D = \{x, y, z\}$ of G such that $G[D]$ has a minimum number of edges.

Claim. *If $uv \in E(G[D])$, then u and v each has at least two independent private neighbours.*

Since D is a minimal dominating set, the vertices u and v each have a private neighbour. Further, since $uv \in E$, neither vertex can be its own private neighbour. Suppose u has only one private neighbour, a. Then $D' = (D - \{u\}) \cup \{a\}$ is a dominating set, and $G[D']$ has fewer edges than $G[D]$, a contradiction. This proves the claim.

The proof of Case 3 proceeds by considering two cases, depending on whether $|E(G[D])| > 0$.

Case 3.1. $G[D]$ has at least one edge.
Without loss of generality, $xy \in E(G[D])$. By the claim, the vertices x and y each have at least two independent private neighbours. Therefore, we may assume without loss of generality that G contains the subgraph shown in Figure 5(i), where a_1 and a_2 are independent private neighbours of x, and b_1 and b_2 are independent private neighbours of y.

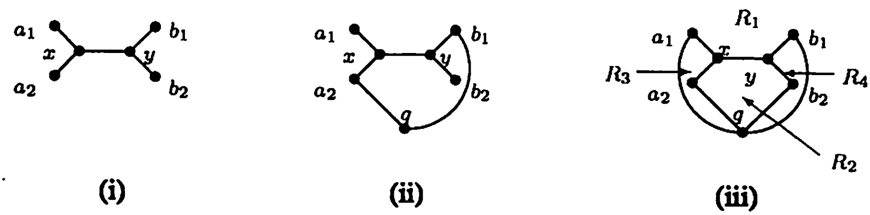


Figure 5: Case 3.1

Since G has diameter two, there is a short path from a_2 to b_1; if this short path is the edge a_2b_1, then because of the planarity of G, the independence of a_1 and a_2, and the independence of b_1 and b_2, there is no short path from a_1 to b_2, a contradiction. Therefore, a_2 and b_1 have some common neighbour, q. Note that $q \neq a_2, b_1, x, y, z$ (since a_1, a_2 are independent private neighbours of x, and b_1, b_2 are independent private neighbours of y). Thus, without loss of generality, G contains the subgraph shown in Figure 5(ii).

Now, there exists a short path from a_1 to b_2; using analogous arguments, this short path must contain q, and thus G contains the subgraph shown in Figure 5(iii).

We now claim that q dominates all private neighbours of x. To see this, let t be a private neighbour of x. If t lies in R_1 (see Figure 5(iii)), then a short path from t to b_2 contains q; if t lies in R_2, then a short path from t to b_1 contains q.

Finally, if t lies in R_3, then if t is not adjacent to q, it must be the case that a short path from t to b_1 contains a_1 and a short path from t to b_2 contains a_2, implying that G contains the subgraph shown in Figure 6. Since t, a_1, a_2 are private neighbours of x, and b_1, b_2 are private neighbours of y, the vertex z is not adjacent to t, a_1, a_2, b_1, or b_2. Thus, in order for z to be distance at most two from x, it must be that z lies in R_5, R_6, R_7, or R_8 (see Figure 6). But it is now impossible for there to be a short path from z to q, a contradiction. Therefore, no such t exists, and we conclude that

q dominates all private neighbours of x. An analogous argument implies that q dominates all private neighbours of y.

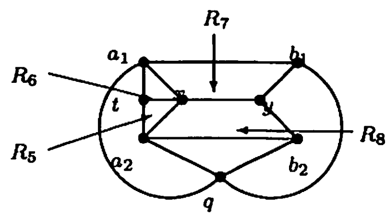


Figure 6: Case 3.1

Recall that G contains the subgraph shown in Figure 6(iii), and that q dominates all private neighbours of both x and y. Now, vertex z must lie in R_1, R_2, R_3 or R_4. Because of the symmetry of this subgraph, we may assume, without loss of generality, that z lies in R_2 or z lies in R_3.

First suppose that z lies in R_2. Since D is a minimal dominating set, the vertex z has a private neighbour, possibly itself. By the planarity of G, all private neighbours of z (except possibly q) also lie in R_2. A short path from a_1 to any private neighbour of z in R_2 must contain q, and thus q dominates all private neighbours of z. If x and z have a shared neighbour (also in R_2), then a short path from this vertex to b_1 must contain q, and therefore q also dominates shared neighbours of x and z. Therefore, $\{q, y\}$ is a dominating set, contradicting the fact that $\gamma(G) = 3$.

Thus, z must lie in R_3 and hence all private neighbours of z (except possibly q) also lie in R_3. Further, z can not be adjacent to y, and y and z can nor have common neighbours other than (possibly) x. Since a_1 and a_2 are private neighbours of x, a short path from z to a private neighbour of y contains q. Thus, q dominates z and all private neighbours of y. Similarly, a short path from y to a private neighbour of z in R_3 contains q. Thus, q dominates y and all private neighbours of z. But then $\{x, q\}$ is a dominating set, contradicting the fact that $\gamma = 3$. It follows that $G[D]$ has no edges.

Case 3.2: $G[D]$ is an independent set.

It follows that $i(G) = 3$; but, since we have assumed that $i(G) \geq \lceil n/3 \rceil$, we have $3 = i(G) \geq \lceil n/3 \rceil$, and therefore $n \leq 9$.

Observe that whatever the value of n, the maximum degree in the graph, Δ, must satisfy $\Delta \leq n - 4$. If $\Delta = n - 2$ or $\Delta = n - 1$, then a maximal independent set in G containing a vertex of degree Δ has cardinality less than three; this maximal independent set is a dominating set, and thus $\gamma(G) < 3$, a contradiction. If $\Delta = n - 3$, then let x be a vertex of degree Δ, and let a and b be the two vertices of G that are distance two from

x. If $ab \in E(G)$, then $\{x, a\}$ is a dominating set, a contradiction; on the other hand, if $ab \notin E(G)$, then a and b must have a common neighbour, $u \in N(x)$, and then $\{x, u\}$ is a dominating set, again a contradiction.

The minimum degree of the graph, δ, must satisfy $\delta \geq 3$; since the neighbours of any vertex in a diameter two graph form a dominating set, if $\delta = 1$ or $\delta = 2$ then $\gamma(G) < 3$. Therefore, $3 \leq \Delta \leq n - 4$. These conditions immediately rule out the possibility that $n \leq 6$, and thus we are left with three cases: $n = 7, 8$ or 9.

(a) Suppose that $n = 7$; by our previous observations, $\Delta \leq 3$. But, since $\delta \geq 3$, this implies that G is a 3-regular graph on seven vertices, which is impossible.

(b) Suppose that $n = 8$; by the conditions given on Δ, we have $\Delta = 4$ or $\Delta = 3$.

Suppose that $\Delta = 4$, and let x be a vertex with degree four; let $N(x) = \{u, v, w, y\}$, and let $S = \{a, b, c\}$ be the remaining vertices of G that are distance two from x. If $G[S]$ has a vertex of degree two, say a, then $\{x, a\}$ is a dominating set, a contradiction. However, if $G[S]$ is an independent set, then, since $\delta \geq 3$, each vertex of S has at least three neighbours in $N(x)$. It follows from the pigeonhole principle that some vertex u of $N(x)$ is adjacent to all three vertices of S, and thus $\{x, u\}$ is a dominating set, a contradiction. The only remaining possibility for $G[S]$ is that it has exactly one edge. Without loss of generality, assume that $ab \in E(G)$, and that $ac, bc \notin E(G)$. Then, since c is adjacent to at least three vertices of $N(x)$, G contains the subgraph shown in Figure 7(i), and the edge ab lies in R_1, R_2 or R_3.

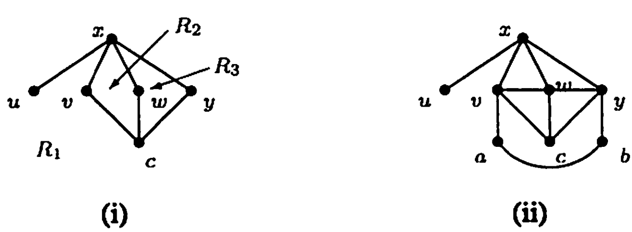
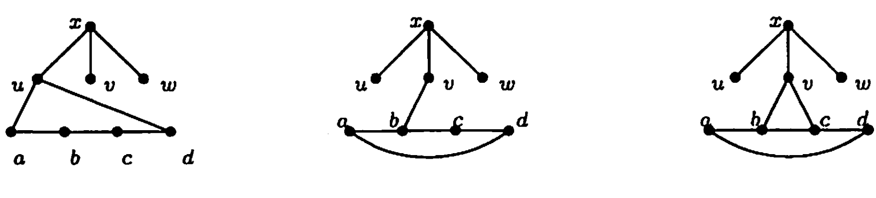


Figure 7: Case 3.2(b)

If ab lies in R_2, then a short path from u to a contains v, as does a short path from u to b, implying that $d(v) \geq 5$, a contradiction. If ab lies in R_3 we arrive at a similar contradiction, and thus the only possibility is that ab lies in R_1. It now follows that a short path from a to w contains v or y, as does a short path from b to w. Because of the restriction that $\Delta = 4$, we may assume, without loss of generality, that G contains the subgraph shown in Figure 7(ii). However, because v and y have degree $\Delta = 4$, and because c in not adjacent to a or b, there is no possible short path from u to c, a contradiction. Therefore, $\Delta \neq 4$.

Next, suppose that $\Delta = 3$; then G is a 3-regular graph with eight vertices. Let x be a vertex of G, let $N(x) = \{u, v, w\}$, and let $S = \{a, b, c, d\}$. Each vertex of $N(x)$ has at most two neighbours in S (contributing at most six to the degrees of vertices in S), and the sum of the degrees of the vertices of S is twelve, implying that $G[S]$ has at least three edges. However, no vertex of $G[S]$ has degree three (otherwise, $\gamma(G) = 2$), and it therefore follows that $G[S]$ is a path on three edges, a cycle on four edges, or consists of a cycle on three edges and in isolated vertex.

First suppose that $G[S]$ is a path; without loss of generality, $G[S]$ is the path $abcd$. Since G has diameter two, a and d must have a common neighbour, and this neighbour must be in $N(x)$. Let this common neighbour be u; we may assume that G contains the subgraph shown in Figure 8(i). Now b and c must each be adjacent to a vertex of $N(x)$. If b and c are both adjacent to v, then w must be adjacent to a and d (in order for G to be 3-regular), but this is impossible because of the planarity of G. Similarly, if b and c are both adjacent to w, then v must be adjacent to a and d, which is again impossible because of the planarity of G. Therefore, b and c must each be adjacent to exactly one of v and w; by planarity, $bv \in E(G)$ and $cw \in E(G)$. But now there can be no short path from a to v. Therefore $G[S]$ is not a path.



(i) (ii) (iii)

Figure 8: Case 3.2(b)

Next, suppose $G[S]$ is the cycle $abcda$. Since G is 3-regular, each vertex in S has exactly one neighbour in $N(x)$, and since each vertex of $N(x)$ has at most two neighbours in S, it follows that some vertex of $N(x)$ has exactly two neighbours in S. Without loss of generality, assume that v is adjacent to exactly two vertices of S, one of them being b, and that G contains the subgraph shown in Figure 8(ii). If v is adjacent to d, then by the planarity of G, either a or c must also be adjacent to v (in order to be adjacent to some vertex of $N(x)$), giving v degree greater than three, a contradiction.

Thus, without loss of generality, we may assume that v is adjacent to c, and that G contains the subgraph shown in Figure 8(iii). Notice that if either u or w is adjacent to both a and d, then $\{u, v\}$ or $\{w, v\}$ is a dominating set, a contradiction. Therefore, u is adjacent to exactly one of a and d; similarly, w is adjacent to exactly one of a and d. Because of the planarity of G, it must be that $ua \in E(G)$ and $wd \in E(G)$. Since G is 3-regular, $uw \in E(G)$; but G has no short path from u to c, a contradiction. Thus, $G[S]$ is neither a path nor a cycle, and hence must consist of a cycle on three edges and an isolated vertex.

Finally, suppose $G[S]$ consists of the isolated vertex a and the 3-cycle $bcdb$. Then, the subgraph $G[N(x) \cup \{a\}]$ is isomorphic to $K_{2,3}$. Since each vertex in $\{u, v, w\}$ is adjacent to exactly one vertex in $\{b, c, d\}$, we may assume by symmetry that the remaining edges of G are bu, cv and dw. But then the graph $G - \{bd\}$ is a subdivision of $K_{3,3}$, contradicting of the planarity of G.

(c) We are now left with the case $n = 9$. Recall G is planar, has diameter two, and that $\gamma(G) = i(G) = 3$. Also, by our earlier remarks, $3 \leq \Delta \leq 5$, and $\delta \geq 3$. First notice that $\Delta = 3$ is impossible, since there is no 3-regular graph on nine vertices. We are therefore left with the two cases: $\Delta = 5$ and $\Delta = 4$.

First suppose that $\Delta = 5$, and let x be a vertex of degree five; let $N(x) = \{u, v, w, y, z\}$ be the neighbours of x, and let $S = \{a, b, c\}$ be the three remaining vertices of G that are distance two from x. Suppose that S is an independent set; since G has diameter two, each pair of vertices in S has a common neighbour in $N(x)$. If a, b, and c have a common neighbour, $u \in N(x)$, then $\{x, u\}$ is a dominating set, a contradiction. Therefore, a and b have a common neighbour, $u \in N(x)$, b and c have a common neighbour, $v \in N(x)$, and a and c have a common neighbour $w \in N(x)$, $u \neq v \neq w$. Without loss of generality, we may assume that G contains the subgraph shown in Figure 9. Furthermore, the remaining two vertices of $N(x)$ must lie in R_1, R_2 or R_3. However, to ensure that a, b and c each have degree at least three, there would need to be a vertex of $N(x)$ in each of R_1, R_2, and R_3, which is impossible. Therefore, S is not an independent set. But, if $G[S]$ contains a vertex of degree two, say vertex a, then $\{x, a\}$ is a dominating set, a contradiction. It follows that $G[S]$ has maximum degree one; since S is not an independent set, this implies that $G[S]$ contains a single edge.

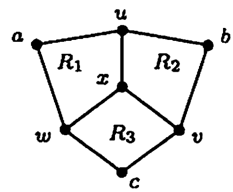


Figure 9: Case 3.2(c)

Without loss of generality, we may assume that $ab \in E(G)$ and that $bc, ac \notin E(G)$. Since $ac \notin E(G)$, a and c have a common neighbour, say u, in $N(x)$; if b is also adjacent to u, then $\{x, u\}$ is a dominating set, a contradiction. Thus, $bu \notin E(G)$, but because $bc \notin E(G)$, b and c have a common neighbour, say v, in $N(x)$ ($v \neq u$). Also, a is not adjacent to v, since this would result in $\{x, v\}$ being a dominating set. Thus, we may assume, without loss of generality, that G contains the subgraph shown in Figure 10(i).

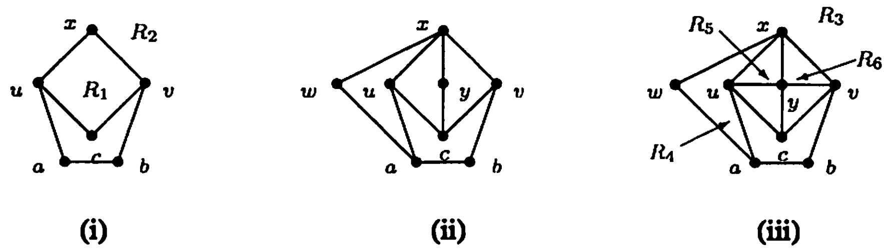


Figure 10: Case 3.2(c)

Notice that the remaining (three) vertices of $N(x)$ must all lie in R_1 or R_2. Furthermore, to ensure that a has degree at least three, there must be at least one vertex of $N(x)\backslash\{u, v\}$ in R_2; also, to ensure that c has degree at least three, there must be at least one vertex of $N(x)\backslash\{u, v\}$ in R_1. We may therefore assume that y lies in R_1 and is adjacent to c, that w lies in R_2 and is adjacent to a, and that G contains the subgraph shown in Figure 10(ii). From this figure, we see that the only short path from y to b contains v, and that the only short path from y to a contains u, implying that $yu, yv \in E(G)$. Thus, G contains the subgraph shown in Figure 10(iii).

The remaining vertex of $N(x)$, z, lies in R_3, R_4, R_5, or R_6. If z lies in R_5, there is no short path from z to b. If z lies in R_6, then because b has degree at least three, $wb \in E(G)$, and thus, by planarity, there is no short path from z to a. If z lies in R_4, then a short path from z to b contains w, and a short path from z to c contains u, so $zu, zw, wb \in E(G)$. But then G has a dominating set $\{u, b\}$ of cardinality two, a contradiction. Finally, if z lies in R_3, then a short path from z to c contains v, so $vz \in E(G)$, and so $\{v, a\}$ is a dominating set of cardinality two, a contradiction. Therefore, $\Delta = 5$ is impossible, and the only possibility $\Delta = 4$.

Suppose that $\Delta = 4$, and let x be a vertex of degree four; let $N(x) = \{u, v, y, z\}$ be the neighbours of x (in clockwise order around x in a plane embedding of G), and let $S = \{a, b, c, d\}$ be the vertices distance two from x. We claim that the graph shown in Figure 3(i) is the unique planar graph (up to isomorphism) that satisfies all of the necessary conditions: planar, diameter two, nine vertices, $\Delta = 4$, $\delta = 3$, $\gamma(G) = i(G) = 3$. Notice that the graph in Figure 3(i) is 3-connected, and thus has essentially only one embedding as a plane graph. Furthermore, the addition of any edge to this graph results in a violation of planarity, the degree condition ($\Delta = 4$), or of the domination number condition ($\gamma(G) = 3$).

To show that the graph in Figure 3(i) is unique, the first step is to show that $G[S]$ is a cycle. First notice that $G[S]$ has maximum degree two. Otherwise, a degree three vertex in $G[S]$, along with x, shows that $\gamma(G) \leq 2$, a contradiction.

Secondly, $G[S]$ has no isolated vertices. To see this, assume that $G[S]$ has an isolated vertex; without loss of generality, we may assume that a is an isolated vertex in $G[S]$. Since $\delta \geq 3$, a must be adjacent to at least three vertices in $N(x)$; we may assume that a is adjacent to u, v and y, and that G contains the subgraph shown in Figure 11.

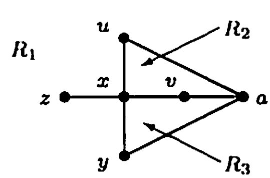


Figure 11: Case 3.2(c)

Suppose that exactly one of b, c, d lies in R_2 or R_3; without loss of generality, b lies in R_2. Then, since b is not adjacent to x or a, $d(b) \leq 2$, a contradiction. However, if at least two of b, c, d lie in R_2 or R_3, then we also obtain a contradiction. To see this, suppose without loss of generality that b and c lie in R_2. Then a short path from z to b or c contains u, implying that $d(u) \geq 5$, and contradicting the fact that $\Delta = 4$. Therefore, b, c and d all lie in R_1. However, a short path from v to b, c or d now uses u or y, and it is impossible to have these short paths and also have $d(u) \leq 4$ and $d(y) \leq 4$. Thus, $G[S]$ has no isolated vertices.

Since $G[S]$ has maximum degree two and minimum degree one, there are three possibilities: $G[S]$ is a matching with two edges, a path of length three, or a cycle of length four. We begin by eliminating the first two possibilities.

Suppose that $G[S]$ is a matching with two edges; without loss of generality, ab and cd are the edges of $G[S]$. A short path from b to c contains a vertex of $N(x)$; similarly, a short path from a to d contains a vertex of $N(x)$. Because $\Delta(G) \leq 4$, the common neighbour of b and c is not the same as the common neighbour of a and d. We may therefore assume, without loss of generality, that b and c are both adjacent to v, and that a and d are both adjacent to y or to z. These two possibilities are depicted in Figure 12.

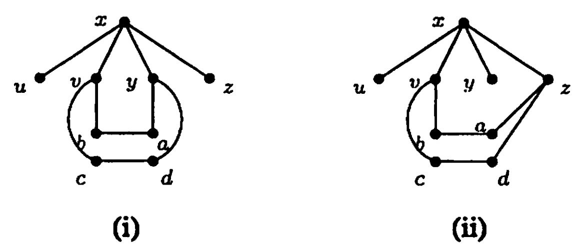


Figure 12: Case 3.2(c)

First consider the situation in Figure 12(i). A short path from b to d contains either v or y; if this path contains v, then $dv \in E(G)$, and thus $d(v) = 4$. If a short path from b to d contains y, then $by \in E(G)$ and $d(y) = 4$. In either case, there is no possible short path from a to c, a contradiction. The same argument applies to the situation in Figure 12(ii), and thus $G[S]$ is not a matching with two edges.

Now suppose that $G[S]$ is a path of length three, say $abcd$. Since a and d are not adjacent in S, they must have a common neighbour in $N(x)$; without loss of generality, a and d are both adjacent to u, and G contains the subgraph shown in Figure 13(i).

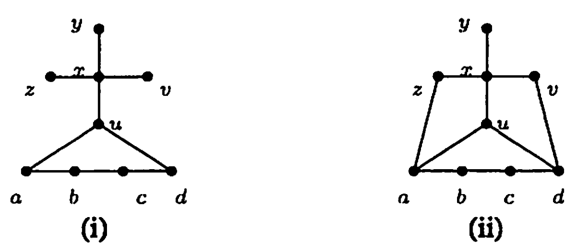


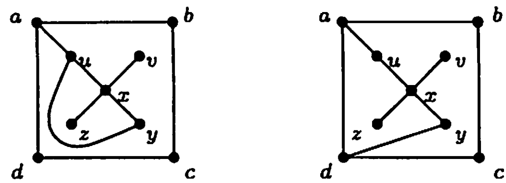
Figure 13: Case 3.2(c)

Since $\delta \geq 3$, a is adjacent to at least one more vertex in $N(x)$, and we claim that a must be adjacent to z. To see this, first suppose that a is adjacent to y but not to z. Then, for z to have degree at least three, $zy, zu \in E(G)$. Hence $d(u) = 4$ and $d(y) \geq 3$, and it is impossible to have short paths from z to both b and c. Therefore if $ay \in E(G)$ then $az \in E(G)$. Secondly, suppose that a is adjacent to v. Then, since d has degree at least three, $dv \in E(G)$; also since b and c each have degree at least three, $bu, cv \in E(G)$ or $bv, cu \in E(G)$. In either case, u and v have degree four, and there is no possible way to have short paths from z and y to b and c. Therefore, $az \in E(G)$, and by symmetry, we may assume that $dv \in E(G)$ (see Figure 13(ii)); also, $av \notin E(G)$, and symmetrically, $dz \notin E(G)$.

Since G has diameter two, b is adjacent to at least one vertex in $N(x)$. If b is adjacent to v, then either $cu \in E(G)$ or $cv \in E(G)$. In either case, there are no short paths from d to both y and z, a contradiction. Thus, b is adjacent to y, z or u, and, by symmetry, c is adjacent to y, z or u. If b is adjacent to y, then a short path from z to c contains b or y. In the first case, $bz \in E(G)$ and $d(b) = 4$; in the second case, $zy, yc \in E(G)$ and $d(y) = 4$. In both cases, the only way to have a short path from a to v is if $uv \in E(G)$. However, this leads to $\{u, b\}$ being a dominating set in the first case, and $\{u, y\}$ being a dominating set in the second case, a contradiction. Therefore, b is adjacent to z or u, and by symmetry, c is adjacent to v or u. Not both b and c can be adjacent to u, and thus because of the symmetry we may assume that $bu \notin E(G)$, and thus $bz \in E(G)$.

Consider a short path from a to y; since $by \notin E(G)$, such a path contains z, so $yz \in E(G)$ and $d(z) = 4$. Now consider a short path from a to v; since $d(z) = 4$ and since $bv \notin E(G)$, this short path contains u, and thus $uv \in E(G)$. This implies that $cu \notin E(G)$, and so $cv \in E(G)$. But then there is no short path from d to z, a contradiction.

Since all possibilities lead to a contradiction, $G[S]$ is not a path of length three, and therefore we conclude that $G[S]$ is a cycle. Without loss of generality, we may assume that $G[S]$ is the cycle $abcda$. Suppose that some vertex of S has exactly one neighbour in $N(x)$; we may assume that $u \in N(x)$ is the unique neighbour of $a \in S$. Since $N(a) = \{d, u, b\}$, a short path from a to y contains one of these vertices. Suppose a short path from a to y contains u; then $uy \in E(G)$, and by the symmetry of the graph, we may assume that G contains the subgraph shown in Figure 14(i).



(i) (ii)

Figure 14: Case 3.2(c)

It follows that a short path from z to any vertex of S contains u or y, so $\{u, y\}$ dominates S. Therefore $N(u) \cup N(y) \supseteq S \cup \{u, x, y, z\}$, and since both u and y are adjacent to x, $d(u) + d(y) \geq 9$, a contradiction (since $\Delta = 4$). Therefore, a short path from a to y contains b or d; by the symmetry of the graph, we may assume that a short path from a to y contains d, and thus G contains the subgraph shown in Figure 14(ii). From Figure 14(ii), we see that a short path from z to b contains either y or u. If such a short path contains y, then $zy, yb \in E(G)$ and $d(y) = 4$. This implies that the subgraph induced by the vertices in $V(G)\backslash N[y]$ is a cycle of length four; i.e., $G[\{a, u, v, c\}]$ is a cycle of length four. But this is impossible since a is not adjacent to v or c. Therefore a short path from z to b must contain u, so $zu, ub \in E(G)$.

Observe now that $d(u) = 4$, and thus the subgraph induced by the vertices of $V(G)\backslash N[u] = \{c, d, y, v\}$ is a 4-cycle. Since yd, dc are edges of G, this 4-cycle must be $vydcv$, implying that $cv, vy \in E(G)$. A short path from c to z must contain one of a, u, x, y, d; however, this path can not contain u or x since $d(u) = d(x) = 4$, and can not contain a since $ca \notin E(G)$. If such a path contains y, then $zy, cy \in E(G)$ and $d(y) \geq 5$, a contradiction. Therefore, the short path from c to z contains d, implying that $zd \in E(G)$. Since $d(z) = 4$, there must be a 4-cycle on the vertices in $V(G)\backslash N[z] = \{b, u, v, x\}$, and because $d(u) = d(x) = 4$ and $d(b) = d(v) = 3$, this 4-cycle must be $bvxub$. Thus $bv \in E(G)$ and G contains the subgraph shown in Figure 15(i). However, this can be re-drawn, as in Figure 15(ii), and we see that this graph is isomorphic to the graph in Figure 3(i). Therefore, we have shown that if some vertex in S has degree three (and thus has a unique neighbour in $N(x)$), then G is isomorphic to the graph in Figure 3(i). The only other possibility is that every vertex of S has degree exactly four, and thus is adjacent to two vertices of $N(x)$ (since $G[S]$ is a cycle).

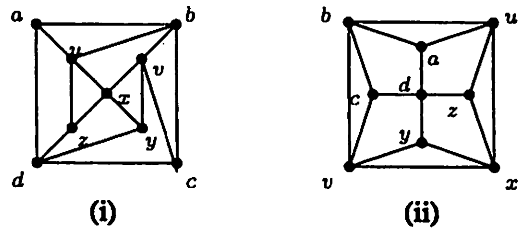


Figure 15: Case 3.2(c)

Suppose that every vertex of S is adjacent to exactly two vertices in $N(x)$. Recall that $N(x) = \{u, v, y, z\}$ are in clockwise order around x in a plane embedding of G. Because of symmetry, we may assume without loss of generality that $a \in S$ is adjacent to v and z (nonconsecutive vertices of $N(x)$), or to u and v (consecutive vertices of $N(x)$). First suppose that a is adjacent to v and z; we may assume that G contains the subgraph shown in Figure 16(i). In this case, short paths from u to b, c and d must contain v or z. It is impossible to have these short paths and also $d(v), d(z) \leq 4$. Therefore, a must be adjacent to consecutive vertices of $N(x)$, and, in fact, every vertex of S must be adjacent to two consecutive vertices of $N(x)$. The facts that a is adjacent to u and v, and $abcda$ is the cycle of $N[S]$ in clockwise order, uniquely determine G as shown in Figure 16(ii).

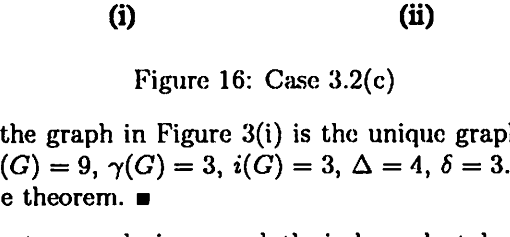


Figure 16: Case 3.2(c)

Therefore, the graph in Figure 3(i) is the unique graph, up to isomorphism, with $V(G) = 9$, $\gamma(G) = 3$, $i(G) = 3$, $\Delta = 4$, $\delta = 3$. This completes the proof of the theorem. ■

For diameter two graphs in general, the independent domination number can be arbitrarily large, as illustrated in the following example. For positive integers m and k, let $G_{m,k}$ be the graph constructed as follows. Take a set of mk independent vertices partitioned into m sets of k vertices; denote these sets $I_1, I_2, \ldots, I_m$. For each p, q with $1 \leq p < q \leq m$, add a new vertex x_p^q and join this vertex to all vertices in I_p and to all vertices in I_q. Finally, construct a complete subgraph on the $\binom{m}{2}$ new vertices x_p^q, $1 \leq p < q \leq m$.

It is not difficult to verify that $G_{m,k}$ is a diameter two graph on $mk + \binom{m}{2}$ vertices. Since at most one vertex of the complete subgraph can be in an independent dominating set of $G_{m,k}$, we see that $i(G_{m,k}) = 1 + (m-2)k$. If we now consider that ratio of $i(G_{m,k})$ to $|V(G_{m,k})|$, we see that

$$\frac{i(G_{m,k})}{|V(G_{m,k})|} = \frac{2(km - 2k + 1)}{m^2 - m + 2mk} = \frac{k(2m-4)+1}{k(2m) + m^2 - m},$$

and thus

$$\lim_{k\to\infty} \frac{i(G_{m,k})}{|V(G_{m,k})|} = \frac{m-2}{m};$$

i.e., for k sufficiently large, $i(G_{m,k}) \sim \left(\frac{m-2}{m}\right) |V(G_{m,k})|$.

References

[1] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications (Elsevier, Amsterdam, 1976).

[2] P. Erdős, S. Fajtlowicz and A.J. Hoffman, Maximum degree in graphs of diameter 2, *Networks* **10** (1980) 87–90.

[3] J. Gimbel and P.D. Vestergaard, Inequalities for total matchings of graphs, *Ars Combin.* **39** (1995), 109–119.

[4] G. MacGillivray and K. Seyffarth, Domination numbers of planar graphs, *Journal of Graph Theory* **22** (1996) 213–229.