The Smallest Degree Sum that Yields Potentially C_k-graphical Sequences

Chunhui Lai *
Department of Mathematics, Zhangzhou Teachers College,
Zhangzhou, Fujian 363000, P. R. of CHINA
zjlaichu@public.zzptt.fj.cn

Abstract

In this paper we consider a variation of the classical Turán-type extremal problems. Let S be an n-term graphical sequence, and $\sigma(S)$ be the sum of the terms in S. Let H be a graph. The problem is to determine the smallest even l such that any n-term graphical sequence S having $\sigma(S) \geq l$ has a realization containing H as a subgraph. Denote this value l by $\sigma(H,\ n)$. We show $\sigma(C_{2m+1},\ n) = m(2n-m-1)+2$, for $m \geq 3$, $n \geq 3m$; $\sigma(C_{2m+2},\ n) = m(2n-m-1)+4$, for $m \geq 3,\ n \geq 5m-2$.

Key words: graph; degree sequence; potentially H-graphic sequence

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1 Introduction

If $S = (d_1, d_2, ..., d_n)$ is a sequence of non-negative integers, then it is called graphical if there is a simple graph G of order n, whose degree sequence $(d(v_1), d(v_2), ..., d(v_n))$ is precisely S. If G is such a graph then G is said to realize S or be a realization of S. A graphical sequence S is potentially H graphical if there is a realization of S containing H as a subgraph, while S

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is forcibly H graphical if every realization of S contains H as a subgraph. Let $\sigma(S) = d(v_1) + d(v_2) + ... + d(v_n)$, and $[x]$ denote the largest integer less than or equal to x. If G and G_1 are graphs, then $G \cup G_1$ is the disjoint union of G and G_1. If $G = G_1$, we abbreviate $G \cup G_1$ as $2G$. Let K_k, and C_k denote a complete graph on k vertices, and a cycle on k vertices, respectively.

Given a graph H, what is the maximum number of edges of a graph with n vertices not containing H as a subgraph? This number is denoted $ex(n, H)$, and is known as the Turán number. This problem was proposed for $H = C_4$ by Erdös [2] in 1938 and in general by Turán [11]. In terms of graphic sequences, the number $2ex(n, H) + 2$ is the minimum even integer l such that every n-term graphical sequence S with $\sigma(S) \geq l$ is forcibly H graphical. Here we consider the following variant: determine the minimum even integer l such that every n-term graphical sequence S with $\sigma(S) \geq l$ is potentially H graphical. We denote this minimum l by $\sigma(H, n)$. Erdös, Jacobson and Lehel [3] showed that $\sigma(K_k, n) \geq (k-2)(2n-k+1)+2$ and conjectured that equality holds. They proved that if S does not contain zero terms, this conjecture is true for $k = 3$, $n \geq 6$. The conjecture is confirmed in [4],[7],[8],[9] and [10].

Gould, Jacobson and Lehel [4] also proved that $\sigma(pK_2, n) = (p-1)(2n-2) + 2$ for $p \geq 2$; $\sigma(C_4, n) = 2[\frac{3n-1}{2}]$ for $n \geq 4$. Lai [5, 6] proved that $\sigma(C_5, n) = 4n-4$ for $n \geq 5$, and $\sigma(C_6, n) = 4n-2$ for $n \geq 7$, $\sigma(C_{2m+1}, n) \geq m(2n-m-1)+2$, for $n \geq 2m+1, m \geq 2$, $\sigma(C_{2m+2}, n) \geq m(2n-m-1)+4$, for $n \geq 2m+2, m \geq 2$, $\sigma(K_4 - e, n) = 2[\frac{3n-1}{2}]$ for $n \geq 7$. In this paper we prove that $\sigma(C_{2m+1}, n) = m(2n-m-1)+2$, for $n \geq 3m$, $m \geq 3$; $\sigma(C_{2m+2}, n) = m(2n-m-1)+4$ for $n \geq 5m-2$, $m \geq 3$.

2 Main results

Theorem 1. Let $k \geq 4$. Let S be a potentially C_k-graphical n-term sequence. If there exists $x \notin C_k, w \in C_k$ such that $d(x) \geq [\frac{k}{2}]+1$, $d(w) \geq 3$. Then S has a realization containing a C_{k+1}.

Assume C_k is $w_1w_2\cdots w_kw_1$. Let $w_{k+i} = w_i$. We first give the following three results.

Lemma (a) For any $x \notin C_k$, if there is w_r, w_{r+1} such that $w_rx, w_{r+1}x \in E(G)$, then G contains a C_{k+1} : $w_1w_2\cdots w_rxw_{r+1}\cdots w_kw_1$.

Lemma(b) For any $x, y \notin C_k, xy \in E(G)$, if there is w_r such that $w_rx \in E(G)$, $w_ry \notin E(G)$, then S has a realization containing a C_{k+1}. (We see the edge $w_{r+1}x$ is not in G or a C_{k+1} would exist, but then the edge interchange which removes the edges $w_r w_{r+1}$ and xy and inserts the edges $w_{r+1}x$ and $w_r y$ produces a realization containing a $C_{k+1}: w_1 w_2 \cdots w_r x w_{r+1} \cdots w_k w_1$)

Lemma(c) For any $x, y \notin C_k$, $xy \in E(G)$, if there is w_r, w_{r+2} such that $w_r x$, $w_{r+2}x \in E(G)$, then S has a realization containing a C_{k+1}. (If $w_{r+2}y \notin E(G)$, then by Lemma(b), S has a realization containing a C_{k+1}. Otherwise, $w_{r+2}y \in E(G)$ and so G contains a $C_{k+1}: w_1 w_2 \cdots w_r x y$ $w_{r+2} w_{r+3} \cdots w_k w_1$)

Proof of theorem 1. Assume every realization of S does not contain a C_{k+1}. By Lemma(a), x is adjacent to at most $[\frac{k}{2}]$ vertices of C_k. Since $d(x) \geq [\frac{k}{2}] + 1$ there exists $x_1 \notin C_k$ such that $xx_1 \in E(G)$. Thus, by Lemma(c), x is adjacent to at most $[\frac{k}{3}]$ vertices of C_k. Note that $[\frac{k}{3}] \leq [\frac{k}{2}] - 1$ since $k \geq 4$. Hence there is $x_2 \notin C_k$, $x_2 \neq x_1$, such that $xx_2 \in E(G)$.

Case 1. Suppose that there is $w_i \in C_k$ such that $w_i x \in E(G)$. By Lemma(b), $w_i x_1, w_i x_2 \in E(G)$. By Lemma(a), $w_{i+1}x, w_{i+1}x_1, w_{i+1}x_2 \notin E(G)$. By Lemma(c) $w_{i+2}x, w_{i+2}x_1, w_{i+2}x_2 \notin E(G)$. Then the edge interchange which removes the edges $w_{i+1}w_{i+2}$ and xx_2 and inserts the edges $w_{i+2}x$ and $w_{i+1}x_2$ produces a realization containing a $C_{k+1}: w_1 w_2 \cdots w_i x_1 x w_{i+2} w_{i+3} \cdots w_k w_1$. This is a contradiction.

Case 2. Suppose for any $w_i \in C_k, w_i x \notin E(G)$. Since $d(x) \geq [\frac{k}{2}] + 1 \geq 2 + 1 = 3$, hence there is $x_3 \notin C_k$, $x_3 \neq x_1$, $x_3 \neq x_2$ such that $xx_3 \in E(G)$. By Lemma(b), $w_i x_1, w_i x_2, w_i x_3 \notin E(G)$. Since there is $w \in C_k$ such that $d(w) \geq 3$, then there is x_4 such that $wx_4 \notin E(C_k)$, $wx_4 \in E(G)$. By Lemma(b), x_4 is not one of x_1,x_2,x_3. If $x_3x_4 \in E(G)$, then by Lemma(b) $wx_3 \in E(G)$ and thus, by Lemma(b) as well, so is $wx \in E(G)$. This is a contradiction. Thus $x_3x_4 \notin E(G)$. Then the edge interchange which removes the edges wx_4 and xx_3 and inserts the edges wx and x_3x_4 produces a realization containing the edge wx. By Case 1, S has a realization containing a C_{k+1}. This is a contradiction.

Theorem 2. Let $m \geq 3$. Let S be an n-term graphical sequence. Suppose S satisfies the following two conditions: (i) there is a realization G of S containing a C_{2m+1}, such that for all $x, y \notin C_{2m+1}$, $d(x) = d(y) = m$ and $xy \notin E(G)$, (ii) there is no realization of S containing a C_{2m+2}. Then $\sigma(S) \leq m(2n - m - 1) + 2$.

Proof. Let C_{2m+1} be $w_1 w_2 \cdots w_{2m+1} w_1$, and $w_{2m+1+i} = w_i$. Since every realization of S does not contain a C_{2m+2}, by Lemma(a), for any $v \notin C_{2m+1}$, there is not w_r, w_{r+1} such that $w_r v, w_{r+1} v \in E(G)$. Since for any $x, y \notin C_{2m+1}$, $xy \notin E(G)$, $d(x) = d(y) = m$, then x, y are all adjacent to m vertices of C_{2m+1}. Assume without loss of generality

$w_1x, w_4x, w_6x, \cdots, w_{2m}x \in E(G)$.

Case 1. Suppose there is $y \notin C_{2m+1}, y \neq x$ such that there is a $w_i \in C_{2m+1}$ such that $w_ix \in E(G), w_iy \notin E(G)$.

Subcase 1. Suppose $w_2y \in E(G)$. By Lemma(a), $w_3y, w_1y \notin E(G)$ and at most one vertex of w_4, w_5 is adjacent to y. If $w_6y \in E(G)$, then G contains a C_{2m+2} : $w_6w_7 \cdots w_{2m+1}w_1xw_4w_3w_2yw_6$. This is a contradiction, thus $w_6y \notin E(G)$. Next, if $w_7y \in E(G)$, then by Lemma(a), $w_8y, w_6y \notin E(G)$. Since y is adjacent to m vertices of C_{2m+1}, Lemma(a) forces $w_9y, w_{11}y, \cdots, w_{2m+1}y \in E(G)$. Then G contains a C_{2m+2} : $w_{2m+1}y$ $w_2w_1xw_4w_5 \cdots w_{2m}w_{2m+1}$. This is a contradiction, thus $w_7y \notin E(G)$. Finally, suppose $w_6y, w_7y \notin E(G)$. Then, by Lemma(a), y at most is adjacent to $m-1$ vertices of C_{2m+1} - a contradiction.

Subcase 2. Suppose $w_3y \in E(G)$. By a similar method as Subcase 1 we can give a contradiction.

Subcase 3. Suppose $w_2y, w_3y \notin E(G)$. Lemma(a) forces y to be adjacent to the following m vertices of C_{2m+1}: $w_1, w_4, w_6, \cdots, w_{2m}$. This contradicts the supposition of case 1.

Case 2. Suppose for any $y \notin C_{2m+1}, y \neq x$, for any $w_i \in C_{2m+1}$, if $w_ix \in E(G)$, then $w_iy \in E(G)$. Then $w_1y, w_4y, w_6y, \cdots, w_{2m}y \in E(G)$.

Subcase 1. Suppose $w_2w_5 \in E(G)$. Then G contains *a* C_{2m+2} : $w_5w_2w_3w_4x\ w_1w_{2m+1}w_{2m} \cdots w_5$. This is a contradiction.

Subcase 2. Suppose $w_{2m+1}w_2 \in E(G)$. Then G contains a C_{2m+2} : $w_2w_{2m+1}w_1xw_{2m}w_{2m-1} \cdots w_2$. This is a contradiction.

Subcase 3. Suppose there is an $i(3 \leq i \leq m-1)$ such that $w_2w_{2i+1} \in E(G)$. Then G contains a C_{2m+2} : $w_{2i+1}w_2w_3w_4 \cdots w_{2i-2}xw_1w_{2m+1}w_{2m} \cdots$ $w_{2i+2}yw_{2i}\ w_{2i+1}$. This is a contradiction.

Subcase 4. Suppose $w_3w_5 \in E(G)$. Then G contains a C_{2m+2} : $w_3w_5w_4xw_6\ w_7 \cdots w_{2m+1}w_1w_2w_3$. This is a contradiction.

Subcase 5. Suppose $w_3w_{2m+1} \in E(G)$. Then G contains a C_{2m+2} : $w_{2m+1}w_3w_2w_1xw_4w_5 \cdots w_{2m}w_{2m+1}$. This is a contradiction.

Subcase 6. Suppose there is an $i(3 \leq i \leq m-1)$ such that $w_3w_{2i+1} \in E(G)$. Then G contains a C_{2m+2} : $w_{2i+1}w_3w_2w_1xw_4w_5 \cdots w_{2i}yw_{2m}w_{2m-1}$ $\cdots w_{2i+1}$. This is a contradiction.

Subcase 7. Suppose there is a $j(2 \leq j \leq m-1)$ such that $w_{2j+1}w_{2m+1} \in E(G)$. Then G contains a C_{2m+2} : $w_{2j+1}w_{2m+1}w_{2m} \cdots w_{2j+2}xw_1w_2 \cdots$

w_{2j+1}. This is a contradiction.

Subcase 8. Suppose there is a j and an $i(2 \leq j < i \leq m-1)$ such that $w_{2j+1}w_{2i+1} \in E(G)$. Then G contains a C_{2m+2} : $w_{2j+1}w_{2i+1}w_{2i} \cdots w_{2j+2}$ xw_{2i+2} $w_{2i+3} \cdots w_{2m+1}w_1w_2 \cdots w_{2j}w_{2j+1}$. This is a contradiction.

Subcase 9. Suppose for any $i(2 \leq i \leq m), w_2w_{2i+1}, w_3w_{2i+1} \notin E(G)$, and for any $i, j(2 \leq j < i \leq m), w_{2j+1}w_{2i+1} \notin E(G)$.

Then

$$d(w_2),\ d(w_3) \leq m+1$$
$$d(w_5),\ d(w_7),\ \cdots,\ d(w_{2m+1}) \leq m$$

Since for any $y \notin C_{2m+1}$, $d(y) = m$. Hence

$$\begin{aligned}\sigma(S) &\leq m(n-2m-1) + d(w_2) + d(w_3) + d(w_5) + d(w_7) \\ &\quad + \cdots + d(w_{2m+1}) + d(w_1) + d(w_4) + d(w_6) + \cdots + d(w_{2m}) \\ &\leq m(n-2m-1) + 2(m+1) + m(m-1) + (n-1)m \\ &= (n-2m-1+2+m-1+n-1)m + 2 \\ &= m(2n-m-1) + 2.\end{aligned}$$

Theorem 3. Let $m \geq 2$. If $k = 2m+1,\ n \geq 3m$, then $\sigma(C_k, n) = m(2n-m-1)+2$; if $k = 2m+2, n \geq 3m$, then $\sigma(C_k, n) \leq m(2n-m-1)+2m+2$.

Proof. By [5] theorem 2 and 3, $\sigma(C_5, n) = 4n-4$ for $n \geq 5, \sigma(C_6, n) = 4n-2$ for $n \geq 7$. Clearly $\sigma(C_6, 6) = 24$. Hence for $m = 2$, if $k = 2m+1, n \geq 3m$, then $\sigma(C_k, n) \leq m(2n-m-1)+2$; if $k = 2m+2, n \geq 3m$, then $\sigma(C_k, n) \leq m(2n-m-1)+2m+2$.

Suppose for t, $2 \leq t < m$, if $k = 2t+1, n \geq 3t$, then $\sigma(C_k, n) \leq t(2n-t-1)+2$ and if $k = 2t+2, n \geq 3t$, then $\sigma(C_k, n) \leq t(2n-t-1)+2t+2$.

Case 1. If S is an n-term graphical sequence, with $k = 2m+1, n \geq 3m$,$\sigma(S) \geq m(2n-m-1)+2$. For $n = 3m$, $\sigma(S) \geq m(6m-m-1)+2 = 5m^2-m+2 = 2[\binom{k-1}{2} + \binom{n-k+2}{2} + 1]$, which by [1] (chapter III, theorem 5.9) implies that all realizations of S contain a C_k. Now assume that S_1 is a p-term graphical sequence, $3m \leq p < n$, $\sigma(S_1) \geq m(2p-m-1)+2$ and that there is a realization of S_1 containing a C_k. We will show that if $S = (d_1, d_2, ..., d_{p+1})$ is a $p+1$-term graphical sequence with realization G and $\sigma(S) \geq m(2(p+1)-m-1)+2$, then S has a realization containing a C_{2m+1}. Assume $d_1 \geq d_2 \geq \cdots \geq d_n \geq 0$. Let S' be the degree sequence of $G - v_{p+1}$ and suppose $d_{p+1} \leq m$. Then $\sigma(S') \geq m(2(p+1)-m-1)+2-2m =$

$m(2p-m-1)+2$. Therefore, by our assumption, S' has a realization containing a C_k. Hence S has a realization containing a C_k. Thus, we may assume that $d_{p+1} \geq m+1$. Since $\sigma(S) \geq m(2(p+1)-m-1)+2 \geq (m-1)(2(p+1)-(m-1)-1)+2(m-1)+2$, by our assumption, there is a realization of S containing a C_{2m}. Which by theorem 1 implies that S has a realization containing a C_{2m+1}.

Case 2. If $k=2m+2, n \geq 3m, S$ is an n-term graphical sequence with $\sigma(S) \geq m(2n-m-1)+2m+2$ then we can prove, via a similar method as Case 1, that S has a realization containing a C_{2m+2} .

Hence $\sigma(C_{2m+1}, n) \leq m(2n-m-1)+2$, $\sigma(C_{2m+2}, n) \leq m(2n-m-1)+2m+2$.

By [5], theorem 1 (Theorem A below), for $m \geq 2, k=2m+1, n \geq 2m+1, \sigma(C_k, n) \geq m(2n-m-1)+2$. Hence, for $m \geq 2$, if $k=2m+1, n \geq 3m$, then $\sigma(C_k, n) = m(2n-m-1)+2$.

Lemma 4. If $m \geq 3, n = 3m+t (t = 0, 1, 2, \cdots, 2m-2)$, then $\sigma(C_{2m+2}, n) \leq m(2n-m-1)+2m+2-2[\frac{t}{2}]$.

Proof. By theorem 3, Lemma 4 holds for $t=0,1$. Now assume that Lemma 4 holds for all $t-1, (1 \leq t \leq 2m-2)$. We now prove that Lemma holds for t.

Let $S=(d_1, d_2, ..., d_n)$ be an n-term graphical sequence $(n=3m+t)$, G a realization of S and $\sigma(S) \geq m(2n-m-1)+2m+2-2[\frac{t}{2}]$. Assume $d_1 \geq d_2 \geq \cdots \geq d_n \geq 0$.

Let S' be the degree sequence of $G-v_n$. If $d_n \leq m-1$, then $\sigma(S') \geq m(2n-m-1)+2m+2-2[\frac{t}{2}]-2(m-1) \geq m(2(n-1)-m-1)+2m+2-2[\frac{t-1}{2}]$. By induction suppose, S' has a realization containing a C_{2m+2}. Hence S has a realization containing a C_{2m+2}. Thus, we may assume that $d_n \geq m$. Since $t \leq 2m-2$, one has $\sigma(S) \geq m(2n-m-1)+2m+2-2[\frac{t}{2}] > m(2n-m-1)+2$. This implies, by theorem 3, that S has a realization containing a C_{2m+1}. Let $w \in C_{2m+1}$, $x, y \notin C_{2m+1}$ and assume that every realization of S does not contain a C_{2m+2}. If $d(x) \geq m+1$, then since $d(w) \geq d_n \geq 3$, by theorem 1, S has a realization containing a C_{2m+2}. This is a contradiction. Hence for any $x \notin C_{2m+1}, d(x)=m$.

If for any $x, y \notin C_{2m+1}$, $xy \notin E(G)$, then, by theorem 2, $\sigma(S) \leq m(2n-m-1)+2 < m(2n-m-1)+2m+2-2[\frac{t}{2}] \leq \sigma(S)$. This is a contradiction. Thus, we may assume that there is $x, y \notin C_{2m+1}$ such that $xy \in E(G)$. Let S' be degree sequence of $G-\{x,y\}$. Since $d(x)=d(y)=m$, then $\sigma(S') \geq m(2n-m-1)+2m+2-2[\frac{t}{2}]-4m+2 = m(2(n-2)-m-1)+2m+2-2[\frac{t-2}{2}]$.

By induction suppose, S' has a realization containing a C_{2m+2}. Hence S has a realization containing a C_{2m+2}. This is a contradiction.

Therefore $\sigma(C_{2m+2}, n) \le m(2n - m - 1) + 2m + 2 - 2[\frac{t}{2}]$.

Theorem 5. $\sigma(C_{2m+2}, n) = m(2n-m-1)+4$, for $m \ge 3, n \ge 5m-2$.

Proof. By Lemma 4, for $m \ge 3, n = 5m - 2, \sigma(C_{2m+2}, n) \le m(2n - m - 1) + 2m + 2 - 2[\frac{2m-2}{2}] = m(2n - m - 1) + 4$.

Suppose for p, ($5m - 2 \le p < n), \sigma(C_{2m+2}, p) \le m(2p - m - 1) + 4$. Let $S = (d_1, d_2, ..., d_n)$ be an n-term graphical sequence with realization G and $\sigma(S) \ge m(2n - m - 1) + 4$. Assume $d_1 \ge d_2 \ge \cdots d_n \ge 0$.

If $d_n \le m$, then consider the degree sequence, S', formed by $G - v_n$. Then $\sigma(S') \ge m(2n - m - 1) + 4 - 2m = m(2(n - 1) - m - 1) + 4$. By the induction hypothesis, S' has a realization containing a C_{2m+2}. Hence S has a realization containing a C_{2m+2}. Thus, we may assume that $d_n \ge m + 1$. Since $\sigma(S) \ge m(2n - m - 1) + 4 \ge m(2n - m - 1) + 2$, theorem 3 implies that S has a realization containing a C_{2m+1}. Therefore, by theorem 1, S has a realization containing a C_{2m+2}.

Therefore $\sigma(C_{2m+2}, n) \le m(2n - m - 1) + 4$.

By [5] theorem 1 (Theorem A below), for $m \ge 2, n \ge 2m+2, \sigma(C_{2m+2}, n) \ge m(2n - m - 1) + 4$. Hence $\sigma(C_{2m+2}, n) = m(2n - m - 1) + 4$ for $m \ge 3, n \ge 5m - 2$.

For completeness, we give a short proofs of the lower bounds for $\sigma(C_{2m+1}, n)$ and $\sigma(C_{2m+2}, n)$ as following:

Theorem A. $\sigma(C_{2m+1}, n) \ge m(2n-m-1)+2$, for $n \ge 2m+1, m \ge 2$, $\sigma(C_{2m+2}, n) \ge m(2n - m - 1) + 4$, for $n \ge 2m + 2, m \ge 2$.

Proof. By noting that $G = K_m + \overline{K_{n-m}}$ gives a uniquely realizable degree sequence and G clearly does not contain C_{2m+1}, $H = K_m + (\overline{K_{n-m-2}} \bigcup K_2)$ gives a uniquely realizable degree sequence and H clearly does not contain C_{2m+2}, this result can easily be seen.

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