CIRCULAR TOTAL COLORINGS OF CUBIC CIRCULANT GRAPHS

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Dedicated to Heiko Harborth on the occasion of his sixty-fifth birthday

ABSTRACT. A (k,d)-total coloring ($k,d \in \mathbb{N}$, $k \geq 2d$) of a graph G is an assignment c of colors $\{0,1,\ldots,k-1\}$ to the vertices and edges of G such that $d \leq |c(x_i) - c(x_j)| \leq k-d$ whenever x_i and x_j are two adjacent edges, two adjacent vertices or an edge incident to a vertex. The circular total chromatic number $\chi_c''(G)$ is defined by $\chi_c''(G) = \inf\{k/d : G$ has a (k,d)-total coloring$\}$. It was proved that $\chi''(G) - 1 < \chi_c''(G) \leq \chi''(G)$ — where $\chi''(G)$ is the total chromatic number of G — with equality for all type-1 graphs and most of the so far considered type-2 graphs. We determine an infinite class of graphs G such that $\chi_c''(G) < \chi''(G)$ and we list all graphs of order < 7 with this property.

1. INTRODUCTION

Given positive integers k and d with $k \geq 2d$, a (k,d)-total coloring of a graph G is an assignment c of colors $\{0,1,\ldots,k-1\}$ to the vertices and edges — together called the elements — of G such that $d \leq |c(x_i) - c(x_j)| \leq k-d$ for every two neighbored elements x_i and x_j. Two elements are called neighbored if they are either adjacent or incident. The circular total chromatic number $\chi_c''(G)$ of G is defined to be the infimum of fractions k/d such that G admits a (k,d)-total coloring.

Since a $(k,1)$-total coloring is an ordinary k-total coloring, the circular total chromatic number can be regarded as a refinement of the total chromatic number. For the total chromatic number $\chi''(G)$ it is conjectured that

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$\chi''(G) \leq \Delta(G) + 2$ [2, 18]. Since $\chi''(G) \geq \Delta(G) + 1$ is obvious, the truth of this so-called total coloring conjecture would imply that $\chi''(G)$ attains one of two values for every graph G. A graph G is called a type-1 graph if $\chi''(G) = \Delta(G) + 1$ and a type-2 graph if $\chi''(G) = \Delta(G) + 2$, respectively.

Circular vertex colorings were introduced by Vince. He used the name star chromatic number instead of circular chromatic number (see, e. g., [5, 17, 20] for results).

First properties and some exact values for the circular total chromatic number can be found in [10]. For example, it was proved that every (k, d)-total colorable graph has a (k', d')-total coloring whenever $k'/d' \geq k/d$. Moreover, the infimum in the definition of the circular total chromatic number can be replaced by the minimum and the number k of colors can be bounded from above by the number of elements of the graph. It also holds that $\chi''(G) - 1 < \chi''_c(G) \leq \chi''(G)$ which implies $\chi''(G) = \lceil \chi''_c(G) \rceil$. Moreover, if G is type 1 then $\chi''_c(G) = \chi''(G)$. Therefore, the task is to determine $\chi''_c(G)$ for type-2 graphs G. In [10] the circular total chromatic numbers are determined for cycles, complete graphs and some classes of complete multipartite graphs. It turns out that among these graphs $\chi''_c(G) < \chi''(G)$ only holds for cycles C_p with $p = 3n + 1$ or $p = 3n + 2$. Therefore, in [10] it was asked to determine further graphs or infinite classes of graphs such that their circular total chromatic number is less than their total chromatic number.

In this note, we list all graphs of order < 7 with this property. Moreover, we determine an infinite class of graphs G such that $\chi''_c(G) < \chi''(G)$, namely a class of cubic circulant graphs.

2. Small graphs

A graph of type 2 is called critical with respect to total coloring if deletion of any edge results in a type-1 subgraph. It is easy to see that each type-2 graph G of order $p \leq 7$ contains a critical subgraph H with $\Delta(H) = \Delta(G)$ since $\chi''(G) \leq \Delta(G) + 2$ is known for $\Delta(G) \leq 5$ [3,13–16] and for $\Delta(G) \geq 3|V(G)|/4$ [12]. Thus, all type-2 graphs of order p can be determined by inserting additional vertices and edges in all critical graphs of order $\leq p$ having the same maximum degree Δ in such a way that Δ remains unchanged. A list of all critical graphs of order ≤ 7 can be found in [11].

By this method, the 17 graphs shown in Figure 1 are determined to be a complete list of type-2 graphs with order $p \leq 7$.

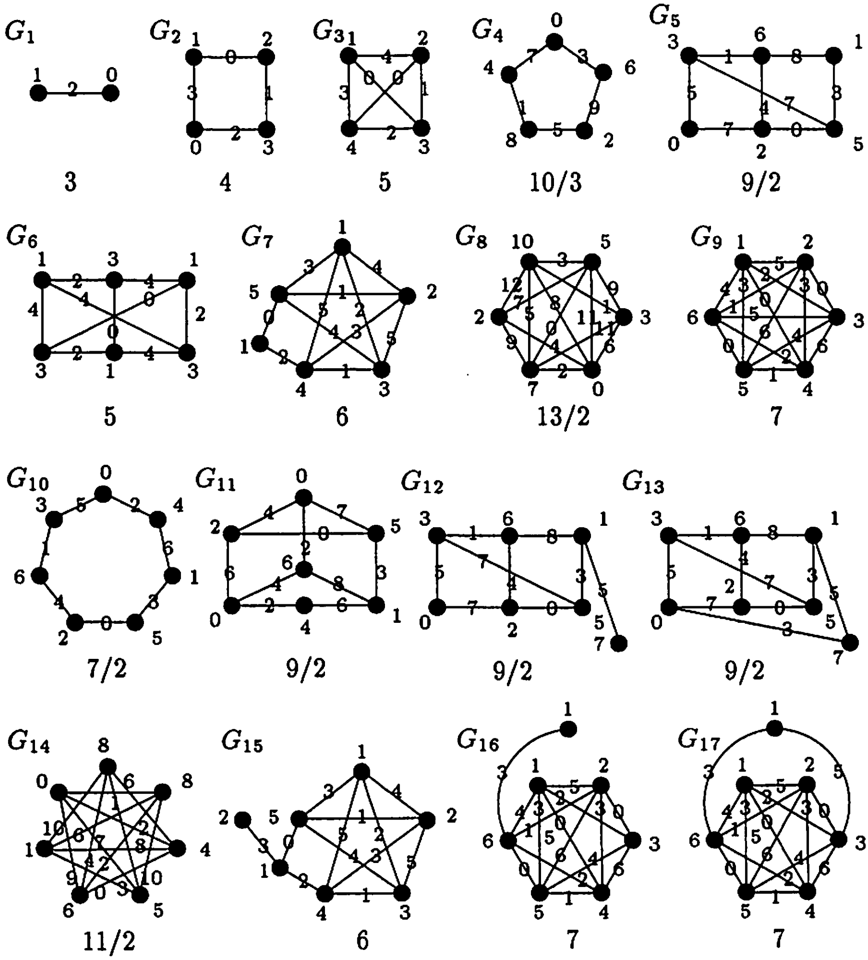

FIGURE 1. Circular total chromatic number of type-2 graphs with $p \leq 7$.

To calculate the circular total chromatic number of these graphs we used a backtracking algorithm which determines the circular chromatic number of a given graph [19]. Considering the total graph $T(G)$ of the graphs $G = G_1, \ldots, G_{17}$ of Figure 1 the algorithm determines $\chi''_c(G_i)$, $i = 1, \ldots, 17$.

Theorem 1. *There are exactly* 8 *connected graphs G with order $|V(G)| \leq 7$ such that $\chi''_c(G) < \chi''(G)$ (see graphs $G_4, G_5, G_8, G_{10}, G_{11}, G_{12}, G_{13}$ and G_{14} in Figure 1).*

3. Circulant graphs

If a_i, $i = 1, \ldots, r$, are positive integers such that $a_1 < a_2 < \ldots < a_r \leq p/2$ then the circulant graph $C_p(a_1, \ldots, a_r)$ is defined having vertices $v_1, v_2, \ldots, v_p$ and edges $v_i v_j$ whenever there exists an index t, $1 \leq t \leq r$, such that $i - j \equiv \pm a_t$ (modulo p).

Circulant graphs are $2r$-regular if $a_r < p/2$ and $(2r-1)$-regular if $a_r = p/2$. We consider cubic circulant graphs which implies that $r = 2$, p is even and $a_1 = a$, $a_2 = p/2$.

In the following we show that all cubic circulant graphs $C_p(a, p/2)$ are isomorphic to a number of copies of either $C_p(1, p/2)$ or $C_p(2, p/2)$.

For this characterization we use the following lemmas.

Lemma 1 ([6], see also [4]). *If l is the greatest common divisor of $a_1, \ldots, a_r, p$ then the circulant graph $C_p(a_1, \ldots, a_r)$ is isomorphic to l copies of the graph $C_{p/l}(a_1/l, \ldots, a_r/l)$.*

Lemma 2 ([1, 9]). *Let n, m and p be non-negative integers with $p \geq 1$, $m < p$, and $n \equiv m$ modulo p). Define $(n)_p = m$ if $m \leq p/2$ and $(n)_p = p - m$ otherwise. Let $b \leq p/2$ be an integer such that b and p are coprime. Then $C_p(a_1, \ldots, a_r)$ is isomorphic to $C_p((a_1 b)_p, \ldots, (a_r b)_p)$.*

With this, we can characterize cubic circulant graphs (see also [9]; a similar result has been proved just recently in [8]).

Theorem 2. *If l is the greatest common divisor of a and $p/2$ and $a = lm$, $p/2 = ln$, then $C_p(a, p/2)$ is isomorphic to l copies of $C_{2n}(1, n)$ if m is odd or of $C_{2n}(2, n)$ if m is even.*

Proof. Lemma 1 implies $C_p(a, p/2) \cong lC_{2n}(m, n)$. If m is odd then $C_{2n}(1, n) \cong C_{2n}(m, (mn)_{2n}) = C_{2n}(m, n)$ which follows from Lemma 2 since m and $2n$ are coprime and $m < n$.

If m is even, say $m = 2^t s$ where s and t are positive integers and s is odd, then $C_{2n}(2^t, n) \cong C_{2n}((2^t s)_{2n}, n) \cong C_{2n}(m, n)$ by Lemma 2 since s and $2n$ are coprime. If $t = 1$ we are done. If $t > 1$ then let $x = n + 2^{t-1}$. Since n and therefore x is odd which implies that x and $2n$ are coprime we obtain $C_{2n}(2, n) \cong C_{2n}((2x)_{2n}, n) \cong C_{2n}((2n + 2^t)_{2n}, n) \cong C_{2n}(2^t, n)$. Therefore, $C_{2n}(m, n) \cong C_{2n}(2, n)$. □

4. Cubic circulant graphs

In the following lemma the total chromatic number of cubic circulant graphs

Lemma 3. *Let l be the greatest common divisor of a and $p/2$, $a = lm$, $p/2 = ln$. Then $G = C_p(a, p/2)$ is a type-1 graph if and only if m is even and G is not isomorphic to $lC_{10}(2,5)$. Otherwise, G is a type-2 graph.*

For a proof of this lemma see [7] or [9].

For every type-1 graph its circular total chromatic number coincides with its total chromatic number (see [10]). We use Lemma 3 to determine upper bounds for the circular total chromatic number of type-2 graphs $G = C_p(a, p/2)$.

Theorem 3. *Every cubic circulant type-2 graph $C_{2ln}(lm, ln)$, where m and n are coprime and $n \geq 4$, is $(9,2)$-total colorable.*

Proof. Let $G = C_{2ln}(lm, ln)$ be a cubic circulant type-2 graph with coprime m and n and $n \geq 4$. Lemma 3 implies that either m is odd or $G \cong lC_{10}(2,5)$. In the first case, $G \cong lC_{2n}(1,n)$ by Theorem 2. Therefore, to prove Theorem 3 it suffices to show that every graph $C_{2n}(1,n)$ with $n \geq 4$ as well as $C_{10}(2,5)$ admits a $(9,2)$-total coloring.

In order to construct such a coloring for $C_{2n}(1,n)$ we draw the graph such that $v_1, v_2, \ldots, v_{2n}$ are the vertices of a regular $2n$-gon in clockwise order and vertices v_i and v_{i+n} are joined by a diagonal.

According to the residue class of n modulo 3 we distinguish three cases to construct a $(9,2)$-total coloring c of $C_{2n}(1,n)$.

$C_{6s}(1,3s)$

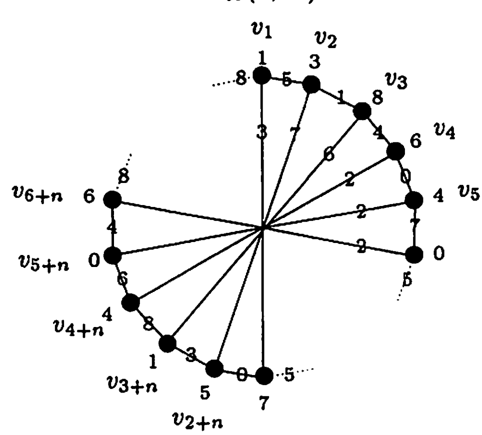

FIGURE 2. $(9,2)$-total coloring of $C_{6s}(1,3s)$.

If $n = 3s$, $s \geq 2$, then we color $c(v_1) = 1$, $c(v_1v_2) = c(v_{2+n}) = 5$, $c(v_2) = c(v_{2+n}v_{3+n}) = 3$, $c(v_2v_3) = c(v_{3+n}) = 1$, $c(v_3) = c(v_{3+n}v_{4+n}) = 8$,

$c(v_3v_4) = c(v_{4+n}) = 4$, $c(v_4) = c(v_{4+n}v_{5+n}) = 6$, $c(v_4v_5) = c(v_{5+n}) = 0$ and $c(v_5) = c(v_{5+n}v_{6+n}) = 4$, the $2(n-3)-1 = 6s-7$ elements $v_5v_6, v_6, \ldots, v_{1+n}v_{2+n}$ successively with colors $7,0,5,7,0,5,\ldots,7,0$ and the $2(n-4)-2 = 6s-10$ elements $v_{6+n}, v_{6+n}v_{7+n}, \ldots, v_{2n}v_1$ successively with colors $6,8,4,6,8,4,\ldots,6,8$. The diagonals are colored by $c(v_1v_{1+n}) = 3$, $c(v_2v_{2+n}) = 7$, $c(v_3v_{3+n}) = 6$, and $c(v_iv_{i+n}) = 2$ if $i \geq 4$ (see Figure 2). It can be easily checked that this is a $(9,2)$-total coloring of $C_{6s}(1,3s)$.

If $n = 3s+1$ or $n = 3s+2$ one can find similar constructions for $(9,2)$-total colorings of $C_{2n}(1,n)$ as shown in Figure 3.

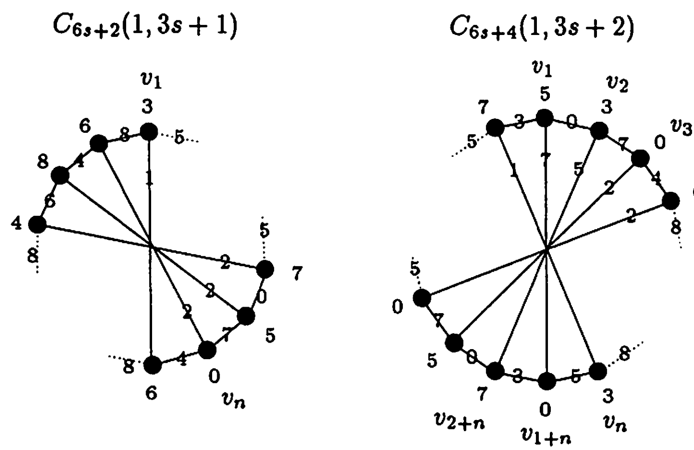

FIGURE 3. $(9,2)$-total colorings of $C_{6s+2}(1,3s+1)$ and $C_{6s+4}(1,3s+2)$.

Figure 4 provides a $(13,3)$-total coloring of $C_{10}(2,5)$. This concludes the proof since $13/3 < 9/2$.

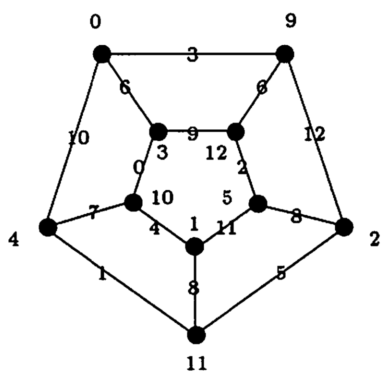

FIGURE 4. $(13,3)$-total coloring of $C_{10}(2,5)$. □

Theorem 3 yields another infinite class of graphs G such that their total chromatic number exceeds their circular total chromatic number.

The condition $n \geq 4$ in Theorem 3 is tight since the graph $G = lC_{2n}(1,n)$ is isomorphic to l copies of K_4 if $n = 2$ and to l copies of $K_{3,3}$ if $n = 3$. For those graphs $\chi_c''(G) = \chi''(G) = 5$ (see [10]).

Theorem 3 states that $\chi_c''(G) \leq 9/2$ for every type-2 graph $C_{2ln}(lm, ln)$ with m, n coprime and $n \geq 4$. Using the above mentioned algorithm we determined the exact values of $\chi_c''(G)$ for the three smallest cases: $\chi_c''(C_8(1,4)) = \chi_c''(C_{10}(1,5)) = 9/2$, $\chi_c''(C_{10}(2,5)) = 13/3$.

It would be an interesting task to determine further classes of graphs G such that $\chi_c''(G) < \chi''(G)$.

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